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ℵ Closed sets in intuitionistic fuzzy topological spaces

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Abstract. In this paper we have introduced family of intuitionistic fuzzy ℵ closed sets in intuitionistic fuzzy topological spaces. Some properties and several characterizations are investigated. Also we have obtained some interesting theorems.
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Contents

1. Introduction

Fuzzy set is introduced by Zadeh [9]. After that Atanassov [1] introduced the notion of intuitionistic fuzzy sets and Coker [3] introduced the notion of intuitionistic fuzzy topological spaces. In this paper we have introduced a family of intuitionistic fuzzy closed set called intuitionistic fuzzy ℵ closed sets. We investigated some of their properties. Additionally we obtain some interesting theorems.

2. Preliminaries

Definition 2.1. *[1] An* ***intuitionistic fuzzy set*** *(IFS for short) A is an object having the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non- membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$. Denote by $IFS(X)$, the set of all intuitionistic fuzzy sets in X.

An intuitionistic fuzzy set A in X is simply denoted by $A = \langle x, \mu_A, \nu_A\rangle$ instead of denoting $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$.

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Definition 2.2. *[1] Let A and B be two IFSs of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

and

$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$$

Then,

(a) $A \subseteq B$ *if and only if* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for all* $x \in X$*,*

(b) $A = B$ *if and only if* $A \subseteq B$ *and* $A \supseteq B$*,*

(c) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$*,*

(d) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$

(e) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$

The intuitionistic fuzzy sets $0_\sim = \langle x, 0, 1\rangle$ *and* $1_\sim = \langle x, 1, 0\rangle$ *are respectively the empty set and the whole set of* X.

Definition 2.3. *[3] An* ***intuitionistic fuzzy topology*** *(IFT in short) on* X *is a family* τ *of IFSs in* X *satisfying the following axioms:*

(i) $0_\sim, 1_\sim \in \tau$*,*

(ii) $G_1 \cap G_2 \in \tau$ *for any* $G_1, G_2 \in \tau$*,*

(iii) $\bigcup G_i \in \tau$ *for any family* $\{G_i : i \in J\} \in \tau$

In this case the pair (X, τ) *is called* ***intuitionistic fuzzy topological space*** *(IFTS in short) and any IFS in* τ *is known as an intuitionistic fuzzy open set (IFOS in short) in* X*. The complement* A^c *of an IFOS* A *in an IFTS* (X, τ) *is called an* ***intuitionistic fuzzy closed set*** *(IFCS in short) in* X*.*

Definition 2.4. *[3] Let* (X, τ) *be an IFTS and* $A = \langle x, \mu_A, \nu_A\rangle$ *be an IFS in* X*. Then the* ***intuitionistic fuzzy interior*** *and* ***intuitionistic fuzzy closure*** *are defined by*

$$int(A) = \bigcup\{G|G \text{ is an IFOS in } X \text{ and } G \subseteq A\}, cl(A) = \bigcap\{K|K \text{ is an IFCS in } X \text{ and } A \subseteq K\}.$$

Note that for any IFS A *in* (X, τ)*, we have* $cl(A^c) = (int(A))^c$ *and* $int(A^c) = (cl(A))^c$

Definition 2.5. *[8] Two IFSs* A *and* B *are said to be* ***q-coincident*** *(*$A_q B$ *in short) if and only if there exits an element* $x \in X$ *such that* $\mu_A(x) > \nu_B(x)$ *or* $\nu_A(x) < \mu_B(x)$*.*

Definition 2.6. *[8] Two IFSs* A *and* B *are said to be* ***not q-coincident*** *(*$A_{q^c} B$ *in short) if and only if* $A \subseteq B^c$*.*

Definition 2.7. *[4] An* ***intuitionistic fuzzy point*** *(IFP for short), written as* $p_{(\alpha,\beta)}$*, is defined to be an IFS of* X *given by*

$$p_{(\alpha\beta)}(x) = \begin{cases} (\alpha, \beta) \text{ if } x = p, \\ (0, 1) \text{ otherwise} \end{cases}$$

An IFP $p_{(\alpha,\beta)}$ *is said to belong to a set* A *if* $\alpha \leq \mu_A$ *and* $\beta \geq \nu_A$*.*

Definition 2.8. *[5] Let (X,τ) be an IFTS and $A=\langle x,\mu_A,\nu_A\rangle$ be an IFS in X. Then the IFS A is said to be **intuitionistic fuzzy dense** if there exists no IFCS B in (X,τ) such that $A\subseteq B\subseteq 1_\sim$.*

Definition 2.9. *[5] Let (X,τ) be an IFTS and $A=\langle x,\mu_A,\nu_A\rangle$ be an IFS in X. Then the IFS A is said to be **intuitionistic fuzzy no where dense** if A^c is intuitionistic fuzzy dense in X.*

Remark 2.10. *[5] If $int(A)=0_\sim$, then the IFS A is an **intuitionistic fuzzy no where dense subset in X**.*

Definition 2.11. *[4] Let (X,τ) be an IFTS and $A=\langle x,\mu_A,\nu_A\rangle$ be an IFS in X. Then **intuitionistic fuzzy kernel** of A is the intersection of all IFOSs containing A.*

Definition 2.12. *[7] Let $p_{(\alpha,\beta)}$ be an IFP of an IFTS (X,τ). An IFS A of X is called an **intuitionistic fuzzy neighborhood** (IFN for short) of $p_{(\alpha,\beta)}$ if there is an IFOS B in X such that $p_{(\alpha,\beta)}\in B\subseteq A$.*

3. Intuitionistic Fuzzy $\aleph$ Closed Sets

In this section we have introduced a new type of intuitionistic fuzzy closed set called intuitionistic fuzzy $\aleph$ closed sets and study some of its properties.

Definition 3.1. *An IFS A of an IFTS (X,τ) is said to be an intuitionistic fuzzy $\aleph$ closed set (briefly $IF\aleph CS$) if $cl(A^c)\subseteq U$ whenever $A\subseteq U$ and U is an IFOS in (X,τ).*

The family of all $IF\aleph CS$s in (X,τ) is denoted by $IF\aleph C(X)$.

Definition 3.2. *An IFS A of an IFTS (X,τ) is said to be an intuitionistic fuzzy $\aleph$ preclosed set (briefly $IF\aleph PCS$) if $Pcl(A^c)\subseteq U$ whenever $A\subseteq U$ and U is an IFOS in (X,τ).*

Definition 3.3. *An IFS A of an IFTS (X,τ) is said to be an intuitionistic fuzzy $\aleph$ semiclosed set (briefly $IF\aleph SCS$) if $scl(A^c)\subseteq U$ whenever $A\subseteq U$ and U is an IFOS in (X,τ).*

Definition 3.4. *An IFS A of an IFTS (X,τ) is said to be an intuitionistic fuzzy $\aleph$ regularclosed set (briefly $IF\aleph RCS$) if $rcl(A^c)\subseteq U$ whenever $A\subseteq U$ and U is an IFOS in (X,τ).*

Definition 3.5. *An IFS A of an IFTS (X,τ) is said to be an intuitionistic fuzzy $\aleph$ alphaclosed set (briefly $IF\aleph\alpha CS$) if $\alpha cl(A^c)\subseteq U$ whenever $A\subseteq U$ and U is an IFOS in (X,τ).*

Definition 3.6. *An IFS A of an IFTS (X,τ) is said to be an intuitionistic fuzzy $\aleph$ betaclosed set (briefly $IF\aleph\beta CS$) if $\beta cl(A^c)\subseteq U$ whenever $A\subseteq U$ and U is an IFOS in (X,τ).*

Example 3.7. *Let $X=\{a,b\}$ and $\tau=\{0_\sim,G_1,G_2,1_\sim\}$ be an IFTS on X, where $G_1=\langle x,(0.3_a,0.2_b),(0.4_a,0.2_b)\rangle$ and $G_2=\langle x,(0.4_a,0.4_b),(0.2_a,0.2_b)\rangle$. Let $A=\langle x,(0.3_a,0.4_b),(0.4_a,0.2_b)\rangle$ be an IFS in (X,τ). Then $A^c=\langle x,(0.4_a,0.2_b),(0.3_a,0.4_b)\rangle$.*

We have $A\subseteq G_2$. Then the IFS A is an $IF\aleph CS$, an $IF\aleph PCS$, an $IF\aleph SCS$, an $IF\aleph\alpha CS$ and an $IF\aleph\beta CS$, as $cl(A^c)=G_1^c\subseteq G_2$, $A^c\cup cl(int(A^c))=A^c\cup G_1^c=G_1^c\subseteq G_2$, $A^c\cup int(cl(A^c))=A^c\cup G_1$ $=\langle x,(0.4_a,0.2_b),(0.3_a,0.2_b)\rangle\subseteq G_2$, $A^c\cup cl(int(cl(A^c)))=A^c\cup G_1^c=G_1^c=\langle x,(0.4_a,0.2_b),(0.3_a,0.2_b)\rangle$ $\subseteq G_2$, $A^c\cup int(cl(int(A^c)))=A^c\cup G_1=\langle x,(0.4_a,0.2_b),(0.3_a,0.2_b)\rangle\subseteq G_2$, where G_2 is an IFOS in X.

Theorem 3.8. *Every $IF\aleph$ closed set is an $IF\aleph P$ closed set in (X,τ) but not conversly.*

Proof. Let A be an $IF\aleph$ closed set in (X,τ). Therefore $cl(A^c)\subseteq U$. Let $A\subseteq U$ where U be an IFOS in X. Now $pcl(A^c)\subseteq cl(A^c)\subseteq U$, we have $pcl(A^c)\subseteq U$. Hence A is an $IF\aleph PCS$ in (X,τ). ■

Theorem 3.9. *Every $IF\aleph$ closed set is an $IF\aleph S$ closed set in (X,τ) but not conversely.*

Proof. Let A be an $IF\aleph$ closed set in (X,τ). Therefore $cl(A^c) \subseteq U$. Let $A \subseteq U$ where U be an IFOS in X. Now $scl(A^c) \subseteq cl(A^c) \subseteq U$, we have $scl(A^c) \subseteq U$. Hence A is an $IF\aleph SCS$ in (X,τ). ■

Theorem 3.10. *Every $IF\aleph$ closed set is an $IF\aleph\alpha$ closed set in (X,τ) but not conversely.*

Proof. Let A be an $IF\aleph$ closed set in (X,τ). Therefore $cl(A^c) \subseteq U$. Let $A \subseteq U$ where U be an IFOS in X. Now $\alpha cl(A^c) \subseteq cl(A^c) \subseteq U$, we have $\alpha cl(A^c) \subseteq U$. Hence A is an $IF\aleph\alpha CS$ in (X,τ). ■

Theorem 3.11. *Every $IF\aleph$ closed set is an $IF\aleph\beta$ closed set in (X,τ) but not conversely.*

Proof. Let A be an $IF\aleph$ closed set in (X,τ). Therefore $cl(A^c) \subseteq U$. Let $A \subseteq U$ where U be an IFOS in X. Now $\beta cl(A^c) \subseteq cl(A^c) \subseteq U$, we have $\beta cl(A^c) \subseteq U$. Hence A is an $IF\aleph\beta CS$ in (X,τ). ■

Theorem 3.12. *Every $IF\aleph\alpha$ closed set is an $IF\aleph S$ closed set in (X,τ) but not conversely.*

Proof. Let A be an $IF\aleph\alpha$ closed set in (X,τ). Therefore $\alpha cl(A^c) \subseteq U$. Let $A \subseteq U$ where U be an IFOS in X. Now $scl(A^c) \subseteq \alpha cl(A^c) \subseteq U$, we have $scl(A^c) \subseteq U$. Hence A is an $IF\aleph SCS$ in (X,τ). ■

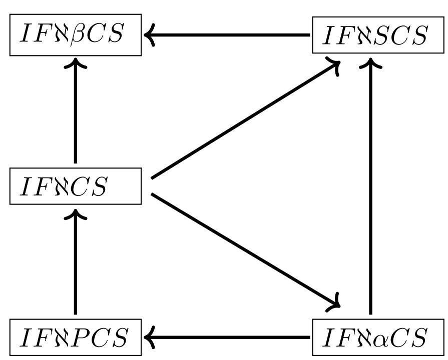
Theorem 3.13. *Every $IF\aleph\alpha$ closed set is an $IF\aleph P$ closed set in (X,τ) but not conversely.*

Proof. Let A be an $IF\aleph\alpha$ closed set in (X,τ). Therefore $\alpha cl(A^c) \subseteq U$. Let $A \subseteq U$ where U be an IFOS in X. Now $pcl(A^c) \subseteq \alpha cl(A^c) \subseteq U$, we have $pcl(A^c) \subseteq U$. Hence A is an $IF\aleph PCS$ in (X,τ). ■

Theorem 3.14. *Every $IF\aleph S$ closed set is an $IF\aleph\beta$ closed set in (X,τ) but not conversely.*

Proof. Let A be an $IF\aleph S$ closed set in (X,τ). Therefore $cl(A^c) \subseteq U$. Let $A \subseteq U$ where U be an IFOS in X. Now $\beta cl(A^c) \subseteq scl(A^c) \subseteq U$, we have $\beta cl(A^c) \subseteq U$. Hence A is an $IF\aleph\beta CS$ in (X,τ). ■

Remark 3.15. *In the following diagram, we have provided relations between various types of intuitionistic fuzzy $\aleph$closedness.*



In the above diagram the reverse implications are not true in general. This can be easily seen from the following examples.

Example 3.16. *Let $X = \{a,b\}$ and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.4_a, 0.3_b), (0.2_a, 0.1_b)\rangle$ and $G_2 = \langle x, (0.5_a, 0.4_b), (0.4_a, 0.4_b)\rangle$. Let $A = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.4_b)\rangle$ be an IFS in (X,τ). Then $A^c = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.4_b)\rangle$.*

We have $A \subseteq G_2$. Then the IFS A is an $IF\aleph PCS$, $IF\aleph SCS$, $IF\aleph\alpha CS$ and an $IF\aleph\beta CS$ as $A^c \cup cl(int(A^c))$ $= A^c \cup cl(G_1) \subseteq A^c \cup G_2^c = A^c \subseteq G_2$, $A^c \cup int(cl(A^c)) = A^c \cup int(G_1^c) \subseteq A^c \cup G_1 = A^c \subseteq G_2$, $A^c \cup cl(int(cl(A^c))) = A^c \cup cl(G_1) \subseteq A^c \cup G_2^c = G_2^c \subseteq G_2$ and $A^c \cup int(cl(int(A^c))) = A^c \cup int(G_2^c)$ $\subseteq A^c \cup G_1 = A^c \subseteq G_2$, where G_2 is an IFOS in X. But $cl(A^c) = G_1^c \nsubseteq G_2$, so A is not an $IF\aleph CS$.

Example 3.17. *Let $X = \{a, b\}$ and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.4_a, 0.3_b), (0.5_a, 0.4_b)\rangle$ and $G_2 = \langle x, (0.5_a, 0.4_b), (0.4_a, 0.4_b)\rangle$. Let $A = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.4_b)\rangle$ be an IFS in (X, τ). Then $A^c = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.4_b)\rangle$.*

Then the IFS A is an $IF\aleph SCS$ and $IF\aleph PCS$, as $A^c \cup int(cl(A^c)) = A^c \cup int(G_1^c) \subseteq A^c \cup G_2 = G_2 \subseteq G_2$ and $A^c \cup cl(int(A^c)) = A^c \cup cl(G_1) \subseteq A^c \cup G_2^c = A^c \subseteq G_2$ whenever $A \subseteq G_2$. But A is not an $IF\aleph\alpha CS$ as $A^c \cup cl(int(cl(A^c))) = A^c \cup cl(G_1) \subseteq A^c \cup G_1^c = G_1^c \nsubseteq G_2$.

Example 3.18. *Let $X = \{a, b\}$ and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.3_b)\rangle$ and $G_2 = \langle x, (0.5_a, 0.4_b), (0.4_a, 0.5_b)\rangle$. Let $A = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.4_b)\rangle$ be an IFS in (X, τ). Then $A^c = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.4_b)\rangle$.*

Then the IFS A is an $IF\aleph\beta CS$, as $A^c \cup int(cl(int(A^c))) = A^c \cup 0_\sim \subseteq A^c \subseteq G_1$, whenever $A \subseteq G_1$ where G_1 is an IFOS in X. But A is not an $IF\aleph SCS$ as $A^c \cup int(cl(A^c)) = A^c \cup 1_\sim \subseteq A^c \cup 1_\sim \nsubseteq G_1$

Remark 3.19. *IFCS [8], IFPCS [6], IFSCS [6], IFRCS [6], $IF\alpha CS$ [6], $IF\beta CS$ [6] and IFGCS [8] are independent to $IF\aleph CS$ in X. This can be seen from the following examples.*

Example 3.20. *In Example 3.16, the IFS $A = \langle x, (0.3_a, 0.4_b), (0.4_a, 0.2_b)\rangle$ is an $IF\aleph CS$ but not an IFCS, IFPCS, IFSCS, IFRCS, $IF\alpha CS$ and IFGCS in (X, τ) as $cl(A) = 1_\sim \neq A$, $cl(int(A)) = cl(G_1) = G_1^c \nsubseteq A$, $int(cl(A)) = 1_\sim \nsubseteq A$, $int(cl(A)) = 1_\sim \neq A$, $cl(int(cl(A))) = 1_\sim \nsubseteq A$ and $cl(A) \nsubseteq G_2$ but $A \subseteq G_2$.*

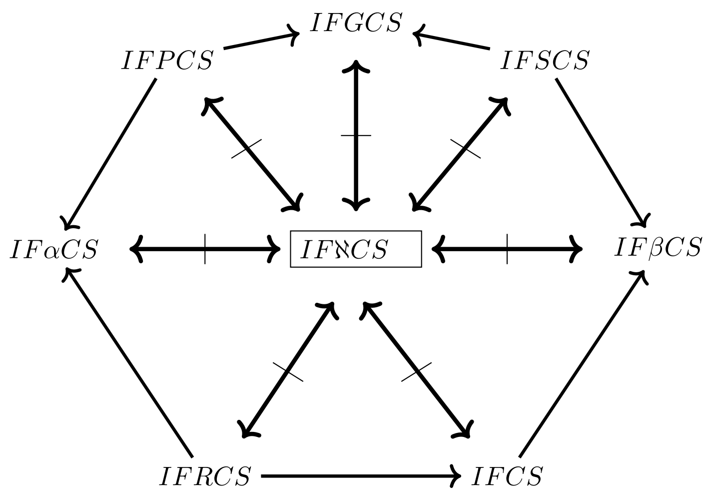
Example 3.21. *Let $X = \{a, b\}$ and $\tau = \{0_\sim, G_1, G_2, G_3, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.8_a, 0.9_b), (0.2_a, 0.1_b)\rangle$, $G_2 = \langle x, (0.3_a, 0.4_b), (0.7_a, 0.6_b)\rangle$ and $G_3 = \langle x, (0.3_a, 0.3_b), (0.7_a, 0.6_b)\rangle$. Let $A = \langle x, (0.8_a, 0.3_b), (0.2_a, 0.6_b)\rangle$ be an IFS in (X, τ). Then $A^c = \langle x, (0.2_a, 0.6_b), (0.8_a, 0.3_b)\rangle$.*

We have $A \subseteq G_1$. As $cl(A^c) = G_3^c \subseteq G_1$, where G_1 is an IFOS in X. This implies that A is an $IF\aleph CS$ in X. But $int(cl(int(A))) = int(cl(G_3)) = int(G_2^c) = G_2 \nsubseteq A$, so A is not an $IF\beta CS$.

Example 3.22. *Let $X = \{a, b\}$ and $\tau = \{0_\sim, G_1, G_2, G_3, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.4_b), (0.4_a, 0.4_b)\rangle$, $G_2 = \langle x, (0.4_a, 0.4_b), (0.5_a, 0.5_b)\rangle$ and $G_3 = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.4_b)\rangle$. Let $A = \langle x, (0.4_a, 0.4_b), (0.5_a, 0.4_b)\rangle$ be an IFS in (X, τ). Then $A^c = \langle x, (0.5_a, 0.4_b), (0.4_a, 0.4_b)\rangle$.*

Then the IFS A is an IFCS, an IFRCS, an IFSCS, an IFPCS, an $IF\alpha CS$, an $IF\beta CS$ and an IFGCS, as $cl(A) = A$, $cl(int(A)) = cl(G_2) = G_2^c = A$, $int(cl(A)) = int(G_1^c) = G_2 \subseteq A$, $cl(int(A)) = cl(G_2) = G_1^c \subseteq A$, $cl(int(cl(A))) = cl(int(G_1^c)) = cl(G_2) = G_1^c \subseteq A$, $int(cl(int(A))) = int(cl(G_2)) = int(G_1^c) = G_2 \subseteq A$ and $cl(A) = G_1^c \subseteq G_1, G_3$ whenever $A \subseteq G_1, G_3$. But A is not an $IF\aleph CS$ in X, as $cl(A^c) = G_2^c \nsubseteq G_3$ whenever $A \subseteq G_3$.

Remark 3.23. *From the above discussions and known results we have the following implications. In the following diagram $A \longleftrightarrow B$ represents A and B are independent of each other.*



Theorem 3.24. *The arbitrary union of $IF\aleph CSs$ in (X, τ) is an $IF\aleph CS$ in (X, τ).*

Proof. Let $\{A_k | k \in I\}$ be an arbitrary class of $IF\aleph CS$s in X. Let $A = \bigcup\limits_k A_k$, $k \in I$. Let U be an IFOS such that $A \subseteq U$. This implies $A_k \subseteq U$, for every $k \in I$. By assumption $cl(A_k^c) \subseteq U$ for every $k \in I$. Then $cl(A^c) = cl((\bigcup\limits_k A_k)^c) = cl(\bigcap\limits_k A_k^c) \subseteq \bigcap\limits_k (cl(A_k^c)) \subseteq U$. Therefore $cl(A^c) \subseteq U$.
Hence A is an $IF\aleph CS$. ■

Remark 3.25. *The intersection of two $IF\aleph CSs$ need not be an $IF\aleph CS$.*

Example 3.26. *Let $X = \{a, b\}$ and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFTS on X, where $G_1 = \langle x, (0.5_a, 0.8_b), (0.2_a, 0.2_b)\rangle$, $G_2 = \langle x, (0.4_a, 0.2_b), (0.4_a, 0.8_b)\rangle$. Let $A = \langle x, (0.5_a, 0.2_b), (0.3_a, 0.8_b)\rangle$ and $B = \langle x, (0.4_a, 0.7_b), (0.4_a, 0.3_b)\rangle$ are IFSs in (X, τ). Then the IFS A and B are $IF\aleph CS$, as $cl(A^c) = G_2^c \subseteq G_1$ whenever $A \subseteq G_1$ and $cl(B^c) = G_2^c \subseteq G_1$ whenever $B \subseteq G_1$. But $A \cap B = \langle x, (0.4_a, 0.2_b), (0.4_a, 0.8_b)\rangle$ is not an $IF\aleph CS$, as $cl((A \cap B)^c) = G_2^c \nsubseteq G_2$, where $A \cap B \subseteq \{G_1, G_2\}$.*

Theorem 3.27. *Every superset of an $IF\aleph CS$ is an $IF\aleph CS$ in X.*

Proof. Let A and B be IFSs in X such that $B \supseteq A$ and A is an $IF\aleph CS$ in X. Let U be an IFOS in (X, τ) such that $B \subseteq U$. Then $A \subseteq U$. Since A is an $IF\aleph CS$, we have $cl(A^c) \subseteq U$. Now, by hypothesis $B^c \subseteq A^c$, $cl(B^c) \subseteq cl(A^c) \subseteq U$. Then B is an $IF\aleph CS$ in (X, τ). ■

Theorem 3.28. *If A is an $IF\aleph CS$ in X such that $A \subseteq B$ and $B^c \subseteq cl(A^c)$, where B is an IFS in X, then B is an $IF\aleph CS$ in (X, τ).*

Proof. Let U be an IFOS in (X, τ) such that $B \subseteq U$. Then $A \subseteq U$ as $A \subseteq B$. Since A is an $IF\aleph CS$, we have $cl(A^c) \subseteq U$. Now, by hypothesis $cl(B^c) \subseteq cl(A^c)$ and $cl(A^c) \subseteq U$. So $cl(B^c) \subseteq U$. Hence B is an $IF\aleph CS$ in (X, τ). ■

Theorem 3.29. *An IFS A is an $IF\aleph CS$ in an IFTS (X, τ) if and only if $A_{q^c} F$ implies $cl(A^c)_{q^c} F$ for every IFCS F of (X, τ).*

Proof. **Necessity:** Assume that A is an $IF\aleph CS$ in (X, τ). Let F be an IFCS and $A_{q^c} F$. Then $A \subseteq F^c$, where F^c is an IFOS in (X, τ). Then by hypothesis, $cl(A^c) \subseteq F^c$. Hence $cl(A^c)_{q^c} F$.

Sufficiency: Let U be an IFOS in (X, τ) such that $A \subseteq U$. Then $A \subseteq (U^c)^c$ and $A_{q^c} U^c$. By hypothesis, $cl(A^c)_{q^c} U^c$. This implies $cl(A^c) \subseteq (U^c)^c = U$. Hence A is an $IF\aleph CS$ in (X, τ). ■

Theorem 3.30. *Let A be an $IF\aleph CS$ in (X,τ) and $p_{(\alpha,\beta)}$ be an IFP in X such that $cl(A^c)_q(p_{(\alpha,\beta)})$. Then $A_q cl(p_{(\alpha,\beta)})$.*

Proof. Assume that A is an $IF\aleph CS$ in (X,τ) and $cl(A^c)_q(p_{(\alpha,\beta)})$. Suppose that $A^c_q cl(p_{(\alpha,\beta)})$, then $A \subseteq (cl(p_{(\alpha,\beta)}))^c$ where $(cl(p_{(\alpha,\beta)}))^c$ is an IFOS in (X,τ). Then by hypothesis, $cl(A^c) \subseteq (cl(p_{(\alpha,\beta)}))^c = int(p_{(\alpha,\beta)})^c \subseteq (p_{(\alpha,\beta)})^c$. Therefore $cl(A^c)_{q^c}(p_{(\alpha,\beta)})$, which is contradiction to the hypothesis. Hence $A_q cl(p_{(\alpha,\beta)})$. ■

Theorem 3.31. *Let $F \subseteq A \subseteq X$, where A is an IFOS in X. If F is an $IF\aleph CS$ in X, then F is an $IF\aleph CS$ in A.*

Proof. Let $U = V \cap A$ be an IFOS in A such that $F \subseteq U$ for some IFOS V of X. Then $F \subseteq V$ and $F \subseteq A$. Since F is an $IF\aleph CS$ in X, $cl(F^c) \subseteq V$ whenever $F \subseteq V$. Then $cl(F^c) \cap A \subseteq V \cap A = U$.
This implies $cl_A(F^c) \subseteq U$. Hence F is an $IF\aleph CS$ in A. ■

Theorem 3.32. *Let $A \subseteq Y \subseteq X$ and suppose that A is an $IF\aleph CS$ in X. Then A is an $IF\aleph CS$ relative to Y.*

Proof. Let $A \subseteq U$, where U be any $IFOS$ in Y such that $U = V \cap Y$, for some $IFOS$ V in X. Then $A \subseteq V$ and $A \subseteq Y$. Since A is an $IF\aleph CS$ in X, $cl(A^c) \subseteq V$. Therefore $Y \cap cl(A^c) \subseteq Y \cap V$. That is $cl_Y(A^c) \subseteq U$. Hence A is an $IF\aleph CS$ relative to Y. ■

Theorem 3.33. *For an $IFOS$ A in (X,τ), if A and A^c are $IF\aleph CS$ in X then A is an $IFGCS$ in X.*

Proof. Since $A \subseteq A$ and A is an $IFOS$, by hypothesis $cl(A^c) \subseteq A$ where A^c is an $IFCS$. This implies $cl(A^c) = A^c$. Then $A^c \subseteq A$, where A is an $IFOS$. Now by definition of $IF\aleph CS$ for A^c, $cl(A) \subseteq A$. Therefore $cl(A) \subseteq A$ whenever $A \subseteq A$, where A is an $IFOS$. Hence A is an $IFGCS$. ■

Theorem 3.34. *If A is both an $IFOS$ and an $IFGCS$ in (X,τ) and if $A^c \subseteq A$, then A is an $IF\aleph CS$ in X.*

Proof. Let A be an $IFOS$ and $A \subseteq A$, then by hypothesis $cl(A) \subseteq A$. As $A^c \subseteq A$, $cl(A^c) \subseteq cl(A) \subseteq A$, where A is an $IFOS$ in X. Hence A is an $IF\aleph CS$ in X. ■

Theorem 3.35. *For an $IFOS$ A in (X,τ), if A and A^c are $IF\aleph CS$ in X then A is an $IFCS$ in X.*

Proof. Since $A \subseteq A$, by hypothesis $cl(A^c) \subseteq A$. This implies $(int(A))^c \subseteq A$. Since A is an $IFOS$ $int(A) = A$. Therefore $A^c \subseteq A$ and $cl(A) \subseteq A$ by hypothesis. But $A \subseteq cl(A)$. Therefore $cl(A) = A$ in X. Hence A is an $IFCS$ in X. ■

Theorem 3.36. *An IFS A of X is an $IF\aleph CS$ if $cl(A^c) \subseteq Ker(A)$.*

Proof. Let U be any openset such that $A \subseteq U$. By hypothesis $cl(A^c) \subseteq ker(A)$ and since $A \subseteq U$, $Ker(A) \subseteq U$. Therefore $cl(A^c) \subseteq U$ and hence A is an $IF\aleph CS$. ■

Theorem 3.37. *For an $IF\aleph CS$ A in an $IFTS$ (X,τ), then the following conditions hold.*

(1) If A is an $IFROS$ then $pint(A)$ and $scl(A)$ are $IF\aleph CS$

(2) If A is an $IFRCS$ then $pcl(A)$ and $sint(A)$ are $IF\aleph CS$

Proof. (1) Let A be an $IFROS$ in (X,τ). Then $int(cl(A)) = A$. By definition we have $scl(A) = A \cup int(cl(A)) = A$ and $pint(A) = A \cap int(cl(A)) = A$. Since A is an $IF\aleph CS$ in X, $scl(A)$ and $pint(A)$ are $IF\aleph CSs$ in X.

(2) Let A be an $IFRCS$ in (X,τ). Then $cl(int(A)) = A$. By definition we have $pcl(A) = A \cup cl(int(A)) = A$ and $sint(A) = A \cap cl(int(A)) = A$. Since A is an $IF\aleph CS$ in X, $pcl(A)$ and $sint(A)$ are $IF\aleph CSs$ in X. ■

Definition 3.38. *Let (X, τ) be an $IFTS$ and $A = \langle x, \mu_A, \nu_A \rangle$ be an IFS in X. Then the **intuitionistic fuzzy ℵ closure** is defined by*

$$\aleph cl(A) = \bigcap\{K | K \text{ is an } IF\aleph CS \text{ in } X \text{ and } A \subseteq K\}.$$

Theorem 3.39. *Let A and B be $IFSs$ of an $IFTS$ X. If $A \subseteq B$ is any $IF\aleph CS$, then $\aleph cl(A) \subseteq B$.*

Proof. Since $\aleph cl(A)$ is the intersection of all $IF\aleph CS$ containing A, by Definition 3.38 $\aleph cl(A)$ is contained in every ℵ closed set containing A. By hypothesis $A \subseteq B$ and B is an $IF\aleph CS$. Hence $\aleph cl(A) \subseteq B$. ■

Theorem 3.40. *Let A and B be $IFSs$ of an $IFTS$ X. If $A \subseteq B$, then $\aleph cl(A) \subseteq \aleph cl(B)$.*

Proof. $\aleph cl(B) = \bigcap\{K|K$ is an $IF\aleph CS$ in X and $B \subseteq K\}$, by Definition 3.38. Since $B \subseteq K$ and K is an $IF\aleph CS$, $\aleph cl(B) \subseteq K$, by Theorem 3.39. Since $A \subseteq B \subseteq K$, $\aleph cl(A) \subseteq K$. Therefore $\aleph cl(A) \subseteq \bigcap\{K|K$ is an $IF\aleph CS$ in X and $B \subseteq K\} = \aleph cl(B)$. Hence the theorem. ■

Theorem 3.41. *If $A \subseteq X$ is an $IF\aleph CS$, then $A = \aleph cl(A)$*

Proof. Obviously $A \subseteq \aleph cl(A)$. Since A is any ℵ-closed set containing A, then $\aleph cl(A) \subseteq A$ by Definition 3.38. Hence $A = \aleph cl(A)$. ■

Theorem 3.42. *Let A and B be IFS of X, then $\aleph cl(A \cap B) \subseteq \aleph cl(A) \cap \aleph cl(B)$.*

Proof. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $\aleph cl(A \cap B) \subseteq \aleph cl(A)$ and $\aleph cl(A \cap B) \subseteq \aleph cl(B)$ by Theorem 3.40. Thus, $\aleph cl(A \cap B) \subseteq \aleph cl(A) \cap \aleph cl(B)$. ■

Remark 3.43. *For any two $IFSs$ A and B in X,$\aleph cl(A) \cap \aleph cl(B) \nsubseteq \aleph cl(A \cap B)$*

Theorem 3.44. *If A and B are $IF\aleph CSs$ in X, then $\aleph cl(A \cup B) = \aleph cl(A) \cup \aleph cl(B)$.*

Proof. Let A and B are $IF\aleph CS$ in X, then $A \cup B$ is also an $IF\aleph CS$ in X. Hence $\aleph cl(A \cup B) = A \cup B = \aleph cl(A) \cup \aleph cl(B)$. ■

4. ℵ Open Sets In Intuitionistic Fuzzy Topological Spaces

In this section we have introduced a new type of intuitionistic fuzzy open set called intuitionistic fuzzy ℵ open sets and study some of its properties.

Definition 4.1. *An IFS A of an IFTS (X, τ) is said to be an **intuitionistic fuzzy** ℵ **open set** (briefly $IF\aleph OS$) if the complement A^c is an $IF\aleph CS$ in (X, τ).*

The family of all $IF\aleph OS$ in (X, τ) is denoted by $IF\aleph O(X)$.

Example 4.2. *Let $X = \{a, b\}$ and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFTS on X, where $G_1 = \langle x, (0.3_a, 0.2_b), (0.4_a, 0.2_b)\rangle$ and $G_2 = \langle x, (0.4_a, 0.4_b), (0.2_a, 0.2_b)\rangle$. Let $A = \langle x, (0.4_a, 0.2_b), (0.3_a, 0.4_b)\rangle$ be an IFS in (X, τ). Then $A^c = \langle x, (0.3_a, 0.4_b), (0.4_a, 0.2_b)\rangle$.*

We have $A^c \subseteq G_2$. Now $cl(A) = G_1^c \subseteq G_2$, where G_2 is an IFOS in X. This implies that A^c is an $IF\aleph CS$ in X. Hence A is an $IF\aleph OS$ in X.

Theorem 4.3. *An IFS A of an IFTS (X, τ) is an $IF\aleph OS$ if and only if $F \subseteq int(A^c)$ whenever F is an IFCS and $F \subseteq A$.*

Proof. **Necessity:** Suppose A is an $IF\aleph OS$ in X. Let F be an IFCS such that $F \subseteq A$. Then F^c is an IFOS and $A^c \subseteq F^c$. By hypothesis A^c is an $IF\aleph CS$, we have $cl(A) \subseteq F^c$. Therefore $F \subseteq int(A^c)$.

Sufficiency: Let U be an IFOS such that $A^c \subseteq U$ and so $U^c \subseteq A$ where U^c is an IFCS, then by hypothesis, $U^c \subseteq int(A^c)$. Therefore $cl(A) \subseteq U$ and so A^c is an $IF\aleph CS$. Hence A is an $IF\aleph OS$ in X.

■

Theorem 4.4. *The arbitrary intersection of $IF\aleph OSs$ in (X, τ) is an $IF\aleph OS$ in (X, τ).*

Proof. Let $\{A_k | k \in I\}$ be an arbitrary class of $IF\aleph OSs$ in X. Let $A = \bigcap_k A_k$. Let F be an IFCS such that $F \subseteq A$. This implies $F \subseteq A_k$, for every $k \in I$. By assumption $F \subseteq int(A_k^c)$ for every $k \in I$. Then $int(A^c) = int((\bigcap_k A_k)^c) = int(\bigcup_k A_k^c) \subseteq \bigcup_k (int(A_k)^c) \supseteq F$. Therefore $int(A^c) \supseteq U$. Hence $F \subseteq int(A^c)$. Therefore A is an $IF\aleph OS$ in X.

■

Theorem 4.5. *Every subset of an $IF\aleph OS$ is an $IF\aleph OS$ in X.*

Proof. Let A and B be IFSs in X such that $B \subseteq A$ and A is an $IF\aleph OS$ in X. Let F be an IFCS in (X, τ) such that $F \subseteq B$. Then $F \subseteq A$. Since A is an $IF\aleph OS$, we have $F \subseteq int(A^c)$. Now, by hypothesis $B^c \supseteq A^c$. This implies $int(B^c) \supseteq int(A^c)$. Hence $F \subseteq int(B^c)$. Then B is an $IF\aleph OS$ in (X, τ). ■

Theorem 4.6. *An IFS A is an $IF\aleph OS$ in an IFTS (X, τ) if and only if $F_{q^c} A^c$ implies $F_{q^c} cl(A)$ for every IFCS F of (X, τ).*

Proof. **Necessity:** Let F be an IFCS and $F_{q^c} A^c$. Then $F \subseteq A$, where A is an $IF\aleph OS$ in (X, τ). Then by Theorem 4.3, $F \subseteq int(A^c)$". This implies $F_{q^c}(int(A^c))^c$. Hence $F_{q^c} cl(A)$.

Sufficiency: Let F be an IFCS in (X, τ) such that $F \subseteq A$. So $F_{q^c} A^c$. Then by assumption, $F_{q^c} cl(A)$. Hence by Definition 2.6, $F \subseteq (cl(A))^c$. Therefore $F \subseteq int(A^c)$. Hence A is an $IF\aleph OS$ in (X, τ).

■

Theorem 4.7. *If an IFOS A of an IFTS X is nowhere dense, then A is an $IF\aleph OS$ in X.*

Proof. If A is an intuitionistic fuzzy no where dense subset, then by remark 2.10, $int(A) = 0_\sim$. Let $A^c \subseteq U$ where U is an IFOS. Since A is an IFOS, $A = int(A)$. Then $cl(A) = cl(int(A)) = cl(0_\sim) = 0_\sim \subseteq U$ and hence A^c is an $IF\aleph CS$ in X. Therefore A is an $IF\aleph OS$ in X. ■

Theorem 4.8. *If A is both an IFOS and an $IF\aleph OS$ in (X, τ) and if $A^c \subseteq A$, then A is an IFGCS in X.*

Proof. Let A be an $IF\aleph OS$ in X, then A^c is an $IF\aleph CS$ in X. Therefore $cl(A) \subseteq A$, as $A^c \subseteq A$ and $A \subseteq A$, where A is an IFOS in X by hypothesis. Hence A is an IFGCS in X. ■

Definition 4.9. *Let X be an IFTS and let an IFP $p_{(\alpha,\beta)} \in X$. A subset N of X is said to be* ***intuitionistic fuzzy $\aleph$ neighborhood*** *($IF\aleph N$ in short) of $p_{(\alpha,\beta)}$ if there exists an $IF\aleph OS$ G such that $p_{(\alpha,\beta)} \in G \subseteq N$.*

Definition 4.10. *Let (X, τ) be an IFTS and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle\}$ be an IFS in X. Then the* ***intuitionistic fuzzy $\aleph$ interior*** *is defined by $\aleph int(A) = \bigcup\{G | G$ is an $IF\aleph OS$ in X and $G \subseteq A\}$.*

Theorem 4.11. *If B is any $IF\aleph OS$ contained in A, then $B \subseteq \aleph int(A)$.*

Proof. Let B be any $IF\aleph OS$ such that $B \subseteq A$. Since $\aleph int(A)$ is the union of all $IF\aleph OS$ contained in A, by Definition 4.10 $\aleph int(A)$ is containing every $\aleph$ open set contained in A. By hypothesis $B \subseteq A$ and B is an $IF\aleph OS$. Hence $B \subseteq \aleph int(A)$. ■

Theorem 4.12. *If $A \subseteq B$, then $\aleph int(A) \subseteq \aleph int(B)$.*

Proof. Let A and B be IFSs of X such that $A \subseteq B$. $\aleph int(A) = \bigcup\{G|G$ is an $IF\aleph OS$ in X and $G \subseteq A\}$ by Defintion 4.10. Since $A \subseteq B$ and $G \subseteq A$, $G \subseteq B$. Therefore $G \subseteq \aleph int(B)$, by Theorem 4.11. Hence $\aleph int(A) = \bigcup\{G|G$ is an $IF\aleph OS$ in X and $G \subseteq A\} \subseteq \aleph int(B)$. ■

Theorem 4.13. *Let A be an IFS of X, then $\aleph int(A) \subseteq A$.*

Proof. By Definition 4.10, it is obvious that $\aleph int(A) \subseteq A$. ■

Theorem 4.14. *For any $IF\aleph OS$ A in X, $\aleph int(A) = A$.*

Proof. Let A be an $IF\aleph OS$ of X. By Theorem 4.14 $\aleph int(A) \subseteq A$. As $A \subseteq A$ and A is an $IF\aleph OS$, we have $A \subseteq \aleph int(A)$ by Definition 4.10. Hence $\aleph int(A) = A$. ■

Theorem 4.15. *If A and B are IFS of an IFTS X, then $\aleph int(A) \cup \aleph int(B) \subseteq \aleph int(A \cup B)$.*

Proof. We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then $\aleph int(A) \subseteq \aleph int(A \cup B)$ and $\aleph int(B) \subseteq \aleph int(A \cup B)$, by Theorem 4.14.

This implies that $\aleph int(A) \cup \aleph int(B) \subseteq \aleph int(A \cup B)$. ■

Theorem 4.16. *If A and B are $IF\aleph CSs$ in X, then $\aleph int(A \cap B) = \aleph int(A) \cap \aleph int(B)$.*

Proof. Let A and B are $IF\aleph OS$ in X, then $A \cap B$ is also an $IF\aleph OS$ in X. Hence $\aleph int(A \cap B) = A \cap B = \aleph int(A) \cap \aleph int(B)$. ■

Theorem 4.17. *Let A be any subset of X. Then*

(1) $(\aleph cl(A))^c = \aleph int(A^c)$

(2) $(\aleph int(A))^c = \aleph cl(A^c)$

Proof. (1) We have $\aleph cl(A) = \bigcap\{K|K$ is an $IF\aleph CS$ in X and $A \subseteq K\}$.

Therefore $(\aleph cl(A))^c = \bigcup\{K^c|K^c$ is an $IF\aleph OS$ in X and $K^c \subseteq A^c\} = \aleph int(A^c)$.

Thus, $(\aleph cl(A))^c = \aleph int(A^c)$.

(2) We have $\aleph int(A) = \bigcup\{K|K$ is an $IF\aleph OS$ in X and $K \subseteq A\}$.

Therefore $(\aleph int(A))^c = \bigcap\{K^c|K^c$ is an $IF\aleph CS$ in X and $A^c \subseteq K^c\} = \aleph cl(A^c)$.

Thus, $(\aleph int(A))^c = \aleph cl(A^c)$.

■

5. Conclusion

$IF\aleph CS$ can be used to derive new separation axioms, new forms of continuity and new decompositions of continuity. This new concept can be extended to Bitopological, Ditopological, Soft topological spaces.

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References

[1] Atanassov K., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, (1986), 87-96.

[2] Chang C., Fuzzy topological spaces, *J. Math. Anal. Appl.*, (1968), 182–190.

[3] Coker D., An introduction to intuitionistic fuzzy topological space, *Fuzzy sets and systems*, (1997), 81–89.

[4] Coker D. and Demirci M., On Intuitionistic Fuzzy Points, *Notes on Intuitionistic Fuzzy Sets*, **1**(1995), 79-84.

[5] Dhavaseelan., Intuitionistic Baire Spaces, *Annals of Fuzzy Mathematics*, (2000), 1–10.

[6] Gurcay H., Coker, D and Haydar Es. A., On fuzzy continuity in intuitionistic fuzzy topological spaces, *The Journal of Fuzzy Mathematics*, **5**(1997), 365–378.

[7] Seok Jong Lee and Eun Pyo Lee, The category of intuitionistic fuzzy topological spaces, *Bull. Korean Math. Soc.* **37** (2000), 63-76.

[8] Thakur S.S. and Rekha Chaturvedi, Generalized closed set in intuitionistic fuzzy topology, *The Journal of Fuzzy Mathematics*, **16(3)**(2008), 559-572.

[9] Zadeh L.A, Fuzzy sets, *Information and control*, (1965), 338–353.