MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(01)**(2023), 25–42.
http://doi.org/10.26637/mjm1101/003

Approximation of solution for generalized Basset equation with finite delay using Rothe's approach

RAKSHA DEVI*[1] AND DWIJENDRA N. PANDEY[2]

[1,2] *Department of Mathematics, Indian Institute of Technology Roorkee, Uttarakhand, India.*

*R*eceived 27 August 2022; *A*ccepted 02 November 2022

Abstract. This study focuses on the use of the Riemann-Liouville fractional (R-L) derivative to address an initial boundary value problem for a fractional order differential equation with finite delay (FDDE). Rothe's methodology is used to prove the existence and uniqueness of the strong solution and classical solution to the restated abstract FDDE. Some examples based on abstract theory and numerical solutions of FDDEs arising in fluid dynamics are presented.

AMS Subject Classifications: 34G20, 34K37, 12H20.

Keywords: Accretive operator, strong solution, classical solution, delay differential equation, Rothe's method.

Contents

1. Introduction

Fractional differential equations are now being utilized to describe real-world issues in engineering, science, finance, and other fields. Numerous methods based on integer order derivatives do not adequately capture the complexity of real-world occurences[17, 18]. There are various definitions for fractional derivatives in contrast to integer order derivatives. The R-L derivative is dealt in this analysis as it is seen in the study by Li et al. [28] that the R-L derivative is more realistic compared to other derivatives and the Riemann derivative is quite helpful in characterizing anomalous diffusion, Levy flights, and traps [24, 29, 30].

It is seen in the literature that differential equations with R-L fractional derivatives are difficult to study because of initial conditions as there is a singularity at $t = 0$. While R-L FDEs with homogeneous initial conditions are treated similarly to FDEs with Caputo derivatives [17]. Heymans et al. [25] and Hristova et al.

*__Corresponding author.__ Email address: **raksha_d@ma.iitr.ac.in** (Raksha Devi), **dwijendra.pandey@ma.iitr.ac.in** (Dwijendra N. Pandey)

[26] provide an excellent summary of the study of initial conditions for fractional differential equations with R-L derivatives.

Furthermore, many real-world processes and phenomena are defined by the influence of the state variable's past values, which gives rise to delays in differential equations. Caini et al. [15] studied the effect of delay in Mars to Earth communications through orbiters. Kyrychko et al.[27] studied the models for high-speed milling: which is a very common cutting process in the industry, a moving conveyor belt loaded with two oscillating connected masses and digital control contains handles by finite delay differential equation. There is a vast application of delay differential equations in population dynamics [16]. This motivates us to study the existence of solutions to FDDEs. Apart from this, to the best of our knowledge in the literature, there is no study on the strong and classical solutions of delay differential equations with the R-L fractional derivative. This study is concerned with the following fractional differential equation with finite delay in a Banach space X having uniformly convex dual X^*

$$\begin{cases} \frac{du(t)}{dt} + D_\alpha u(t) + Au(t) = f(t, u(t), u(t-\tau)), \quad t \in (0, T] \\ u(t) = \phi(t), \quad t \in [-\tau, 0) \\ {}_0I_t^{1-\alpha} u(t)|_{t=0} = \phi(0). \end{cases} \tag{1.1}$$

where D_α and ${}_0I_t^{1-\alpha}$ denote the R-L derivative and R-L integral of fractional order, α, $0 < \alpha < 1$ respectively and $\phi \in \mathcal{C}_0 := C([-\tau, 0]; X)$ i.e ϕ is a continuous $X-$ valued function on $[-\tau, 0]$. $-A$ generates an analytic semigroup of contractions in X, and $\tau > 0$ and $T < \infty$ are constants. Here, the considered equation (1.1) is also known as a generalized Basset equation with finite delay. In particular, if we consider $\alpha = \frac{1}{2}$ and $f(t, u(t), u(t-\tau)) \equiv f(t)$, equation (1.1) becomes Basset equation. In [11], Ashyralyev proved the well-posedness of the Basset equation in a Banach space X.

This analysis uses Rothe's methodology to demonstrate the existence of the unique solution of FDDE since it may also be used to determine a numerical solution. E. Rothe proposed Rothe's approach in 1931 to solve a second-order scalar parabolic initial value problem [1]. In [1], a parabolic boundary value problem of second order in two variables was converted in the system of ordinary differential equations to get approximate solutions. Later the method of lines was used to solve various partial differential equations of higher orders. In 1956, Lady$\hat{z}$enskaja [2] applied the Rothe approach to equations higher than the second order. Then, a number of other authors see, e.g., J. Ne$\hat{c}$as [3], J. Ka$\hat{c}$ur [4] applied this technique to demonstrate a few a priori estimates, based on which questions about existence and convergence are easily resolved. Rothe's approach was used by Bahuguna and Raghavendra [5] to demonstrate that nonlinear Schrodinger-type problems have a strong solution.

Several authors, including Agarwal and Bahuguna [7], Bahuguna and Raghvendra [6], S. Abbas et al. [8], Shruti [9], and Darshana et al. [10], used Rothe's approach to demonstrate the existence of the unique strong and weak solutions to integer order differential equations.

In 2019, motivated by Ashyralyav [11], Bahuguna and Anjali [12] proved the existence of the unique, strong solution to the following initial value problem for an FDE

$$\begin{cases} \frac{du(t)}{dt} + D_{0+}^\alpha u(t) + Au(t) = f(t), \quad t \in (0, T), \quad \alpha \in (0, 1) \\ u(0) = 0 \end{cases} \tag{1.2}$$

in a Banach space, X whose dual X^* is uniformly convex, and $-A$ generates an analytic semigroup of contractions in X. Here D_{0+}^α is the R-L derivative.

Rothe's approach was used by Chaoui et al. [14] to demonstrate the existence of the one and only solution as well as a few regularity findings for fractional diffusion integrodifferential with the fractional integral condition. This approach was then used by Bahuguna and Anjali [13] to prove the existence of the unique strong solution to the abstract fractional integrodifferential equations.

The article is structured as follows; section 2 contains some fundamental definitions, notations, and presumptions. Section 3, outlines the Rothe's methodology based on which some apriori estimates are proved. The major result is stated and proven in section 4, and the application of previous section is shown in section 5.

2. Basics and Assumptions

This section contains some basic definitions, preliminary information, and assumptions that will be utilised to demonstrate the main theorem.

Throughout the work, assume that X is a Banach space with uniformly convex dual X^* and $\|.\|$, $\|.\|_{X^*}$ are the norms of X and X^*. Here $\mathcal{C}_t := C([-\tau, t]; X)$ for $t \in [0, T]$ is the Banach space of all continuous functions from $[-\tau, t]$ into X endowed with the supremum norm

$$\|\phi\|_t := \underset{-\tau \leq \zeta \leq t}{sup} \|\phi(\zeta)\|, \quad \phi \in \mathcal{C}_t$$

Definition 2.1. *[17] Let $I = (a, b)$ and $f(x) \in AC^n(a, b)$ and $n-1 < \alpha < n, n \in \mathbb{N}_0$. The R-L derivative of function f of order α is defined as*

$$\mathcal{D}_{a+}^{\alpha} f(x) = \left(\frac{d}{dx}\right)^n I_{a+}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t) dt$$

where $AC^n[a, b] = \{f : [a, b] \to \mathbb{R} : f^{(n-1)} \in AC[a, b]\}$ and $I_{a+}^{n-\alpha}$ is known as the R-L integral of fractional order.

Definition 2.2. *[19] "Let $\Delta = \{z : \varphi_1 < \arg z < \varphi_2, \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$, let $T(z)$ be a bounded linear operator. The family $T(z)$, z $\in \Delta$ is an analytic semigroup in Δ if*

1. *$z \to T(z)$ is analytic in Δ.*
2. *$T(0) = I$ and $\lim_{\substack{z \to 0 \\ z \in \Delta}} T(z)x = x$ for every $x \in X$.*
3. *$T(z_1 + z_2) = T(z_1) T(z_2)$ for $z_1, z_2 \in \Delta$.*

A semigroup $T(t)$ will be called analytic in some sector Δ containing the nonnegative real axis.

Definition 2.3 (Chapter-3 [23])**.** *For a given 'Gauge function' (A continuous and strictly increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$ and $\underset{t \to \infty}{lim} \phi(t) = \infty$) the mapping $J_\phi : X \to 2^{X^*}$ defined by*

$$J_\phi x := \{u^* \in X^* : \langle x, u^* \rangle = \|x\| \|u^*\|_{X^*}; \|u^*\|_{X^*} = \phi(\|x\|)\}$$

is called the 'duality mapping' with Gauge function ϕ. This mapping is single-valued as X^* is uniformly convex.

Definition 2.4. *[19] An operator A is "m-accretive" if*

$$\langle Au, J(u) \rangle \geq 0, \quad \forall \quad u \in D(A),$$

where J is the duality mapping and $R(I + \lambda A) = X$ for $\lambda > 0$.

Remark 2.5.

1. *If $-A$ generates C_0 semigroup, then A is m-accretive [19].*
2. *For a linear operator A, its domain of definition is given by*

$$D(A) := \left\{ v \in X : \underset{t \to 0^+}{lim} \frac{T(t)v - v}{t} exists \right\}$$

where $T(t)$ is a semigroup of bounded linear operators.

Definition 2.6. *A function u is said to be strong solution of problem (1.1) if it satisfies following properties:*

1. *$u \in C(I; X)$ and $u \in D(A)$.*
2. *D_α exists and is continuous on I, where $0 < \alpha < 1$.*
3. *u satisfies given equation (1.1) a.e. on I with initial condition $u(t) = \phi(t)$.*

The form of strong solution [19] of problem (1.1) is given by

$$u(t) = T(t)\phi(0) - \int_0^t T(t-s)D_\alpha u(s)ds + \int_0^t T(t-s)f(s, u(s), u(s-\tau))ds \quad in \quad X$$

Lemma 2.7. *[13] If $(p_n), (q_n)$ and (r_n) are nonnegative sequences and*

$$p_n \leq q_n + \sum_{0\leq k<n} r_k p_k \text{ for } n \geq 0,$$

$$\& \text{ then } \quad p_n \leq q_n + \sum_{0\leq k<n} r_k q_k \exp\left(\sum_{k<j<n} r_j\right) \text{ for } n \geq 0.$$

This is known as 'Discrete Gronwall's' lemma.

Lemma 2.8. *[13] Let $y(t)$ is a non-negative continuous function on $(0, T]$ and $g(t) > 0$ be continuous, increasing function on $[0, T]$. If there are positive constants A, B such that*

$$y(t) \leq Ag(t) + B\int_0^t \frac{y(s)}{(t-s)^\alpha}ds, \quad 0 \leq t \leq T,$$

then there exists a constant C such that,

$$y(t) \leq Cg(t).$$

where $0 < \alpha < 1$ and this reult also holds if $y(t)$ is piecewise continuous function.

Lemma (3) is given to make us understand that the initial condition in the integral sense in equation (1.1) can also be written differently:

Lemma 2.9. *[18] Let $\alpha \in (0, 1)$ and $r > 0, u : [0, r] \to \mathbb{R}$ be a Lebesgue measurable function.*

1. *If $\exists$ a.e. a limit $\lim_{t\to 0+}\left[t^{1-\alpha}u(t)\right] = e \in \mathbb{R}$, then there also exists a limit*

$$\left.{}_0I_t^{1-\alpha}u(t)\right|_{t=0} \& := \lim_{t\to 0+}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u(s)}{(t-s)^\alpha}ds = e\Gamma(\alpha)$$
$$\& = \Gamma(\alpha)\lim_{t\to 0+}\left[t^{1-\alpha}u(t)\right].$$

2. *If $\exists$ a.e. a limit $\lim_{t\to 0+} {}_0I_t^{1-\alpha}u(t) = e \in \mathbb{R}$, and if $\exists$ the limit $\lim_{t\to 0+}\left[t^{1-\alpha}u(t)\right]$, then*

$$\lim_{t\to 0+}\left[t^{1-\alpha}u(t)\right] = \frac{e}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\lim_{t\to 0+} {}_0I_t^{1-\alpha}u(t)$$

Now we consider the following assumptions for proving the main result:

- (B1) $-A$ generates an analytic semigroup of contractions in X.

- **(B2)** The function f defined from $[0,T]\times X\times X\to X$ satisfies a local-Lipschitz like condition

$$\|f(t,u_1,\tilde{u_1})-f(s,u_2,\tilde{u_2}\|\le L_f(r)[|t-s|+\|u_1-u_2\|+\|\tilde{u_1}-\tilde{u_2}\|],$$

for all u_i, $\tilde{u_i}\in B_r(X,\phi(0))$ and $L_f(r)$ is a non-decreasing function and, for $r>0$,

$$B_r(X,\phi(0))=\{u\in X:\|u-\phi(0)\|\le r\}.$$

- **(B3)** $(I+A)^{-1}$ is compact.

3. Discretization and apriori estimates

We divide the interval $[0,T]$ into n-subintervals of lengths $h_n=\frac{T}{n}$ and at each of the division points $t_j^n=jh,\quad j=1,2,3,...,n$ in order to apply Rothe's approach of temporal discretization.

Discretization scheme for fractional derivative $D_\alpha u(t)$:
At $t=t_j^n$, the L_1 [20] approach is used to estimate the R-L fractional derivative:

$$D_\alpha u(t_j^n)\approx\frac{\phi(0)}{(t_j^n)^\alpha}+\frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j}b_{j-i}\frac{\left(u_i^n-u_{i-1}^n\right)}{h_n}h_n^{1-\alpha},\quad \text{where}\quad j=1,2,3,..,n \tag{3.1}$$

$$=\frac{\phi(0)}{(t_j^n)^\alpha}+\sum_{i=1}^{j}\left(u_i^n-u_{i-1}^n\right)d_i^{j,n},\quad \text{where}\quad j=1,2,3..,n \tag{3.2}$$

where $b_k=(k+1)^{1-\alpha}-k^{1-\alpha}$ and $d_i^{j,n}=b_{j-i}\frac{h_n^{-\alpha}}{\Gamma(2-\alpha)}$.
We discretize $\frac{du}{dt}$ by forward difference scheme

$$\frac{du}{dt}=\frac{u_i^n-u_{i-1}^n}{h_n}$$

We let that $\phi(0)\in D(A)$. For a fix $r>0$, we choose r_0 such that

$$M=\|f(0,\phi(0),\phi(0))\|+\|Au_0\|+r_0d$$

where d is a positive constant. We consider $u_0^n=\phi(0)\ge0\ \forall,\ n\in\mathbb{N}$ and now we establish a discretization of our problem in the direction of time-axis, at $t=t_j^n\quad j=1,2,3..,n$ problem (1.1) becomes

$$\begin{cases}\frac{u_j^n-u_{j-1}^n}{h_n}+Au_j^n+D_\alpha u_j^n=f_j^n,\\ u(t_j^n)=\phi(t_j^n)\\ {}_0I_{t_j^n}^{1-\alpha}u(t_j^n)|_{t_j^n\to0}=\phi(0)\quad\forall j=1,2,..,n\quad\text{whenever}\quad n\to\infty\end{cases} \tag{3.3}$$

where, $f_j^n=f(t_j^n,u_j^n,u(t_j^n-\tau))$.

Lemma 3.1. *If conditions **(B1)-(B3)** hold then for $n\in N$, $j=1,2,...,n$,*

1. $\|u_j^n-u_0\|\le C_1$ *where* $u_0=\phi(0)$.
2. $\|\delta u_j^n\|\le C_2$.

where C_1, C_2 are positive constants, independent of j,h and n.

Proof. We may establish this claim by using two methods: first, discretizing the relevant fractional integral equation; and second, discretizing the fractional derivative in direct form. Here, the first half of the lemma is demonstrated by discretizing the fractional derivative, and the second part of the lemma is demonstrated by discretizing the fractional integral.

Firstly using definition of R-L derivative

$$D_\alpha u(t) = \frac{d}{dt}(I^{1-\alpha}u(t))$$

where $I^{1-\alpha}$ is R-L integral, then equation (3.3) becomes

$$\frac{1}{h_n}(u_j^n - u_{j-1}^n) + Au_j^n + \frac{1}{h_n}(I^{1-\alpha}u_j^n - I^{1-\alpha}u_{j-1}^n) = f_j^n \tag{3.4}$$

Putting $j = 1$ in (3.4) and subtracting Au_0 from both sides of obtained equation

$$\frac{1}{h_n}(u_1^n - u_0^n) + Au_1^n - Au_0^n + \frac{1}{h_n}(I^{1-\alpha}u_1^n - I^{1-\alpha}u_0^n) = f_1^n - Au_0^n \tag{3.5}$$

Here for simplicity, we write $u_j^n = u_j$

By applying $J(u_1 - u_0)$ on both sides and using the definition of J for gauge function $\phi(t) = t$, we obtain

$$\langle u_1 - u_0, J(u_1 - u_0)\rangle + h_n\langle A(u_1 - u_0), J(u_1 - u_0) + \langle I^{1-\alpha}(u_1 - u_0), J(u_1 - u_0)\rangle = h_n\langle f_1^n - Au_0, J(u_1 - u_0)\rangle \tag{3.6}$$

By using m-accretivity of A and the definition of duality map J

$$\|u_1 - u_0\| \le h_n\left[\|f(t_1, u_1, \tilde{u}_1) - f(0, u_0, \tilde{u}(0))\| + \|f(0, u_0, \tilde{u}(0))\| + \|Au_0\|\right]$$

considering $u(t-\tau) = \tilde{u}$ and using inequality $\langle I^{1-\alpha}(u_1 - u_0), (u_1 - u_0)\rangle \ge 0$ (see [14])

$$\|u_1 - u_0\| \le h_n\left[|t_1| + \|u_1 - \phi(0)\| + \|\tilde{u}_1 - \phi(0)\| + \|f(0, \phi(0), \phi(0))\| + \|Au_0\|\right] \le h_n M \le C$$

We will prove this result by induction; for this, we assume that

$$\|u_i^n - u_0\| \le C \quad \forall i < j. \tag{3.7}$$

Now we show that

$$\|u_j^n - u_0\| \le C \tag{3.8}$$

Subtracting Au_0 from both sides of (3.4)

$$\frac{1}{h_n}(u_j^n - u_{j-1}^n) + Au_j^n - Au_0 + \frac{1}{h_n}(I^{1-\alpha}u_j^n - I^{1-\alpha}u_{j-1}^n) = f_j^n - Au_0$$

$$\frac{1}{h_n}((u_j^n - u_0) - (u_{j-1}^n - u_0)) + A(u_j^n - u_0) + \frac{1}{h_n}(I^{1-\alpha}u_j^n - I^{1-\alpha}u_{j-1}^n) = f_j^n - Au_0$$

By applying $J(u_j^n - u_0)$ on both sides, we get

$$\begin{aligned}\langle u_j - u_0, J(u_j - u_0)\rangle + h_n\langle A(u_j - u_0), J(u_j - u_0)\rangle + \langle I^{1-\alpha}(u_j - u_{j-1}), J(u_j - u_0)\rangle \\ = \langle (u_{j-1} - u_0), J(u_j - u_0)\rangle + h_n\langle f_j - Au_0, J(u_j - u_0)\rangle\end{aligned}$$

Considering,

$$\langle I^{1-\alpha}(u_j - u_{j-1}), J(u_j - u_0)\rangle = \langle I^{1-\alpha}(u_j - u_{j-1} + u_0 - u_0), J(u_j - u_0)\rangle \ge 0$$

Accretivity of A and the previous condition implies that

$$\|u_j - u_0\| \leq \|u_{j-1} - u_0\| + Mh_n$$

Now using equation (3.7) we get required result i.e. $\|u_j - u_0\| \leq C_1 \quad \forall\ j = 1, 2, 3, ...$ Now, for proving inequality (2), we consider the discretized form of an equation (1.3) by discretizing the R-L integral of fractional order

$$\frac{1}{h_n}(u_j^n - u_{j-1}^n) + Au_j^n + \frac{\phi(0)}{(t_j^n)^\alpha} + \sum_{i=1}^{j}\left(u_i^n - u_{i-1}^n\right) d_i^{j,n} = f_j^n \tag{3.9}$$

where b_k and d_i are defined same as previously.
For $j = 1$

$$\frac{1}{h_n}(u_1^n - u_0^n) + Au_1^n + \frac{\phi(0)}{(t_j^n)^\alpha} + \frac{1}{h_n\Gamma(2-\alpha)}(u_1^n - u_0^n)h_n^{1-\alpha} = f_1^n$$

i.e.

$$\frac{1}{h_n}\left(1 + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)(u_1^n - u_0^n) + \frac{\phi(0)}{t^\alpha} + Au_1^n = f_1^n.$$

Due to accretivity of A and $(1 + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}) > 1$, we have

$$\|\delta u_1^n\| \leq \|f_1^n\|.$$

where $\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h_n}, \forall\ j = 1, 2, 3, ...$

For values of $j \geq 2$, subtracting the equation (3.3) $j-1$ from the equation (3.3) for j

$$\frac{1}{h_n}(u_j^n - u_{j-1}^n) + Au_j^n - Au_{j-1}^n + D_\alpha u_j^n - D_\alpha u_{j-1}^n = \frac{1}{h_n}(u_{j-1}^n - u_{j-2}^n) + f_j^n - f_{j-1}^n.$$

Using discretized value of D_α from equation (3.1), we obtain

$$\tfrac{1}{h_n}(u_j^n - u_{j-1}^n) + Au_j^n - Au_{j-1}^n + \tfrac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\delta u_j^n$$

$$= \delta u_{j-1}^n + \tfrac{1}{\Gamma(2-\alpha)}\textstyle\sum_{i=1}^{j-1} b_{j-1-i}\delta u_i^n h_n^{1-\alpha} - \tfrac{1}{\Gamma(2-\alpha)}\textstyle\sum_{i=1}^{j-1} b_{j-i}\delta u_i^n h_n^{1-\alpha} + f_j^n - f_{j-1}^n.$$

Hence

$$\left(1 + \tfrac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\delta u_j^n + Au_j^n - Au_{j-1}^n$$
$$= \left(1 + (b_0 - b_1)\tfrac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\delta u_{j-1}^n + f_j^n - f_{j-1}^n + \tfrac{1}{\Gamma(2-\alpha)}\textstyle\sum_{i=1}^{j-2}\left(b_{j-1-i} - b_{j-i}\right)\delta u_i^n h_n^{1-\alpha}$$

Since $(1 + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}) > 1$,

$$\delta u_j^n + Au_j^n - Au_{j-1}^n \leq \left(1 + (b_0 - b_1)\frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\delta u_{j-1}^n + f_j^n - f_{j-1}^n + \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j-2}\left(b_{j-1-i} - b_{j-i}\right)\delta u_i^n h_n^{1-\alpha} \tag{3.10}$$

Taking the duality product with $J(u_j^n - u_{j-1}^n)$ owing Accretivity of A and hypotheses $(B1), (B2), (B3)$ and Lemma $3(i)$ we have

$$\left\|\delta u_j^n\right\| \leq \left(1 + (b_0 - b_1)\frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\left\|\delta u_{j-1}^n\right\| + \left\|f_j^n - f_{j-1}^n\right\| + \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j-2}\left(b_{j-1-i} - b_{j-i}\right)\left\|\delta u_i^n\right\| h_n^{1-\alpha} \tag{3.11}$$

Using Lipschitz condition on f

$$\|f(t_j^n, u_j^n, \tilde{u}_{j-1}^n) - f(t_{j-1}^n, u_{j-1}^n, \tilde{u}_{j-1}^n\| \leq L_f(r)[|t_j^n - t_{j-1}^n| + \|u_j^n - u_{j-1}^n\| + \|\tilde{u}_j^n - \tilde{u}_{j-1}^n\|],$$

$$\|f(t_j^n, u_j^n, \tilde{u}_{j-1}^n) - f(t_{j-1}^n, u_{j-1}^n, \tilde{u}_{j-1}^n\| \leq C\left[h_n + 3h_n\delta u_j^n\right]$$

Now equation (3.10) becomes,

$$\|\delta u_j^n\| \leq \left(1 + (b_0 - b_1)\frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\|\delta u_{j-1}^n\| + C\left[h_n + 3h_n\delta u_j^n\right] + \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j-2}(b_{j-1-i} - b_{j-i})\|\delta u_i^n\| h_n^{1-\alpha} \tag{3.12}$$

Rearranging equation (3.11)

$$\|\delta u_j^n\| \leq D_1\left(1 + (b_0 - b_1)\frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\|\delta u_{j-1}^n\| + D_2 h_n + \frac{D_3}{\Gamma(2-\alpha)}\sum_{i=1}^{j-2}(b_{j-1-i} - b_{j-i})\|\delta u_i^n\| h_n^{1-\alpha}$$

where D_1, D_2, D_3 are positive constants depending on C and h_n.
Now using Discrete Gronwall's inequality, we will get

$$\|\delta u_j^n\| \leq C_2.$$

■

Lemma 3.2. *If conditions **(B1)-(B3)** hold then for $n \in N$, $j = 1, 2, .., n$,*

$$\|D^\alpha u_j^n\| \leq C_3, \quad \|Au_j^n\| \leq C_4$$

where C_3, C_4 are positive constants, independent of j,h and n.

Proof. Since

$$D_\alpha u_j^n = \frac{\phi(0)}{t^\alpha} + \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j} b_{j-i}\delta u_i^n h_n^{1-\alpha}$$

Using Lemma1(ii) and

$$b_{j-i} = (j-i+1)^{1-\alpha} - (j-i)^{1-\alpha} = (j-i)^{1-\alpha}\left[\left(1 + \frac{1}{(j-i)}\right)^{1-\alpha} - 1\right]$$

Using binomial expansion

$$b_{j-i} = (j-i)^{1-\alpha}\left(\frac{1-\alpha}{j-i} - \frac{\alpha(1-\alpha)}{2!(j-i)^2} + \frac{\alpha(1-\alpha)(1+\alpha)}{3!(j-i)^3} + ...\right)$$

since $0 < \alpha < 1$, so

$$b_{j-i} = \frac{1-\alpha}{(j-i)^\alpha} - \frac{\alpha(1-\alpha)}{2!(j-i)^{1+\alpha}} + \frac{\alpha(1-\alpha)(1+\alpha)}{3!(j-i)^{2+\alpha}} + ...$$

using the fact that $0 < \alpha < 1$ so $\frac{1}{(j-i)^n} < \frac{1}{(j-i)^\alpha}$ for each $n \in N$ and here $i < j$.
We select d_α as the maximum of all the numerators in the previous expression of b_{j-i}.

$$b_{j-i} \leq d_\alpha\left(\frac{1}{j^\alpha} + \frac{1}{(j-1)^\alpha} + \frac{1}{(j-2)^\alpha} + \cdots\right)$$

Now, we can express b_{j-i} by the following inequality

$$b_{j-i} \leq d_\alpha \sum_{i=1}^{j} \frac{1}{(j-i+1)^\alpha}$$

Hence $D_\alpha u_j^n$ becomes

$$\|D_\alpha u_j^n\| \leq d_\alpha C_2 \sum_{i=1}^{j} \frac{h_n}{[(j-i+1)h_n]^\alpha}$$

Now proceeding similarly as [Lemma 7 [13]], we will get required result $\|D_\alpha u_j^n\| \leq C_3$.

Now using triangle inequality,

$$\|Au_j^n\| \leq \|f_j^n\| + \|D_\alpha u_j^n\| + \|\delta u_j^n\|$$
$$\|Au_j^n\| \leq \|f_j^n - f_0^n\| + \|f_0^n\| + \|D_\alpha u_j^n\| + \|\delta u_j^n\|$$

Using lemma(3.1), and the first part of the Lemma and local-Lipschitz condition on f, we get the required result

$$\|Au_j^n\| \leq C_4.$$

■

We next define a sequence of step functions. $X^n : [-h_n, T] \to D(A)$ by

$$X^n(t) = \begin{cases} \phi(0), & t \in [-h_n, 0], \\ u_j^n, & t \in \left[t_{j-1}^n, t_j^n\right]. \end{cases}$$

Further we introduce sequence U^n of polygonal functions from $[-\tau, T] \to D(A)$, given by

$$U^n(t) = \begin{cases} \phi(t), & t \in [-\tau, 0], \\ u_{j-1}^n + \frac{t-t_{j-1}^n}{h_n}\left(u_j^n - u_{j-1}^n\right), & t \in \left(t_{j-1}^n, t_j^n\right]. \end{cases}$$

Remark 3.3. *From Lemma* (3.1)*, we observe that the function* $U^n(t)$ *are Lipschitz continuous on* $[-\tau, T]$ *with a uniform Lipschitz constant. The sequence* $U^n(t) - X^n(t) \to 0$ *in* X *as* $n \to \infty$ *on* $(0, T]$*. Furthermore the sequences* $X^n(t)$ *and* $U^n(t)$ *are uniformly bounded in* X*. By definition of* $X^n(t)$*, the boundedness of this sequence is clear using lemma*(3.1)*.*

To prove boundedness of Rothe's sequence $U^n(t)$ in X.
Since,

$$0 \leq \frac{t - t_{j-1}^n}{h_n} \leq 1 \quad in \quad [t_j^n - t_{j-1}^n],$$

in order that by definition of $U^n(t)$, for an arbitrary $t \in [t_j^n, t_{j-1}^n]$,

$$\begin{aligned} \|U^n(t)\| &= \left\| u_{j-1}^n\left(1 - \frac{t - t_{j-1}^n}{h_n}\right) + u_j^n \frac{t - t_{j-1}^n}{h_n}\right\| \\ &\leq \left\| u_{j-1}^n\left(1 - \frac{t - t_{j-1}^n}{h_n}\right)\right\| + \left\| u_j^n \frac{t - t_{j-1}^n}{h_n}\right\| \\ &\leq \left(1 - \frac{t - t_{j-1}^n}{h_n}\right) C_1 + \frac{t - t_{j-1}^n}{h_n} C_1 \leq C_1. \end{aligned}$$

Now, we define a sequence of step functions $\tilde{D}_\alpha U^n : [0, T] \to X$ by

$$\tilde{D}_\alpha U^n(t) = \begin{cases} 0, & t = 0, \\ \frac{\phi(0)}{t^\alpha} + \sum_{i=1}^{j} a_{j-i} \frac{u_i^n - u_{i-1}^n}{h_n} \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}, & t \in \left(t_{j-1}^n, t_j^n\right]. \end{cases}$$

Lemma 3.4. $\|\tilde{D}_\alpha U^n(t) - D_\alpha U^n(t)\| \to 0$ *as* $n \to \infty$ *uniformly on* $(0,T)$.

Proof. See the proof of Lemma 8 in [13] and the calculation for $D_\alpha U^n(t)$ in [12]. ■

Lemma 3.5. *There exists a subsequence* $\{U^{n_k}\}$ *of* $\{U^n\}$ *and a function* $u : [0,T] \to D(A)$ *such that* $U^{n_k} \to u$ *in* $C([0,T],X)$ *and* $AX^{n_k}(t) \rightharpoonup Au(t)$ *uniformly in* X *as* $n \to \infty$*. Furthermore,* $Au(t)$ *is weakly continuous.*

Proof. [21] For proving this define $Y^n = (I + A)X^n$. $\{Y^n(t)\}$ is uniformly bounded by using Remark 1.
Since $X^n = (I + A)^{-1}Y^n$, assumption $(B2)$ implies that a subsequence $\{X^{n_k}(t)\}$ of $\{X^n(t)\}$ converges strongly in X (using compact criterion). Let $u(t)$ be the limit of $X^{n_k}(t)$. $U^m(t) \to u(t)$ as $n \to \infty$ by using Remark 1.The sequence $U^{n_k} \in C(I,X)$ is equicontinuous and for $t \in I$, $\{U^{n_k}(t)\}$ is relatively compact in X. Therefore, the Ascoli-Arzela theorem implies that $U^{n_k} \to u$ in $C(I,X)$ i.e. $U^n(t)$ converges to $u(t)$ uniformly on every compact subinterval. Since each U^{n_k} is Lipschitz continuous with uniform Lipschitz constant M_0, so

$$\|u(t_1) - u(t_2)\| = lim_{n\to\infty}\|U^n(t_1) - U^n(t_2)\| \le M_0|t_1 - t_2|$$

and

$$\|u(t)\| = lim_{n\to\infty}\|U^n(t)\| \le C, \quad u \in C([0,T],X).$$

u is Lipschitz continuous.
We use Lemma 4 the uniform boundedness of $\{AX^n\}$. For each $t \in I$, $X^{n_k} \to u(t)$ as $n \to \infty$, uniformly on I. Since X^* is uniformly convex, using Lemma (2.5) from [22] we can say that $u(t) \in D(A)$ for $t \in I$ and $AX^{n_k} \rightharpoonup Au(t)$ uniformly on I as $n \to \infty$. To prove the weak continuity of $Au(t)$, let $t_k \in I$ be such that $t_k \to t$ as $t_k \to t$ as $k \to \infty$. As u is Lipschitz continuous so $u(t_k) \to u(t)$ as $k \to \infty$. Since $\{Au(t_k)\}$ is uniformly bounded so $Au(t_k) \rightharpoonup Au(t)$ as $k \to \infty$ ([22], using Lemma (2.5)). ■

Lemma 3.6. *There exists a subsequence* $\{U^{n_k}\}$ *of* $\{U^n\}$ *such that* $\frac{d^-U^{n_k}}{dt} \rightharpoonup \frac{du}{dt}$ *and* $D_\alpha U^{n_k} \rightharpoonup D_\alpha u$ *in* $L^2([0,T],X)$*, as* $n \to \infty$.

Proof. The proof of this Lemma is based on Lemma (10) from [13].
Since $D_\alpha U^n(t)$ is uniformly bounded in $L^2([0,T],X)$. Every bounded sequence in L^2 has a weakly convergent subsequence so there exist a subsequence $D_\alpha U^{n_k}(t)$ and a function $\zeta \in L^2([0,T],X)$ such that

$$D_\alpha U^{n_k}(t) \rightharpoonup \zeta \in L^2([0,T],X).$$

Define $Y^n : [0,T] \to X$ by

$$Y^n(t) = I^{1-\alpha}U^n(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{U^n(s)}{(t-s)^\alpha}ds$$

For $t \in (0, t_1^n)$
Using Rothe's sequence $U^n(t) = u_0^n + \frac{(t-t_0^n)}{h_n}(u_1^n - u_0^n) = \phi(0) + \frac{t}{h_n}(u_1^n - \phi(0))$
As ϕ is continuous function on closed interval $[-\tau, 0]$ so $|\phi(0)| \le a$, where a is positive constant. Hence

$$Y^n(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\phi(0) + \frac{s(u_1^n - \phi(0))}{h_n}}{(t-s)^\alpha}ds$$

$$= \frac{\phi(0)}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-s)^\alpha}ds + \frac{(u_1^n - \phi(0))}{h_n\Gamma(1-\alpha)}\int_0^t \frac{s}{(t-s)^\alpha}ds$$

$$= \frac{-t^{1-\alpha}}{\Gamma(2-\alpha)}\phi(0) + \frac{t^{2-\alpha}}{h_n}\frac{(\delta u_1^n - \phi(0))}{\Gamma(3-\alpha)} \tag{3.13}$$

Since δu_1^n, $\phi(0)$ are bounded and $Y^n(0_+) = 0$ for every n.
For $x^* \in X^*$, we can see

$$\langle Y^{n_k}(t), x^* \rangle = \int_0^t \langle D_\alpha U^{n_k}(s), x^* \rangle ds.$$

Since $U^{n_k} \to u \in C([0,T], X)$, $Y^{n_k}(t) \to \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(s)}{(t-s)^\alpha} ds$ as $k \to \infty$. Thus passing the limit $k \to \infty$, we get

$$\left\langle \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(s)}{(t-s)^\alpha} ds, x^* \right\rangle = \int_0^t \langle \zeta(s), x^* \rangle ds$$

Hence

$$\zeta(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^\alpha} ds.$$

Similarly, we can show $\frac{d^- U^{n_k}}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2([0,T], X)$. ■

Remark 3.7.

$$\tilde{D}_\alpha U^n \rightharpoonup D_\alpha u \in L^2([0,T], X), as \quad n \to \infty.$$

Lemma 3.8. *$Au(t)$ is Bochner integrable on $[0,T]$.*

Proof. The proof of Bochner's integrability of $Au(t)$ is based on the Lemma 4.6 in [22]. ■

Remark 3.9. *Equation (1.3) can be written as*

$$\frac{d^- U^n(t)}{dt} + AX^n(t) + \tilde{D}_\alpha U^n(t) = f^n(t), t \in [0,T] \tag{3.14}$$

4. Main Results

Before proving the main theorem, let us recall well-known results from real analysis, which can be found in any standard analysis book.

Remark 4.1. ***Lebesgue bounded convergence theorem:*** *Let g_n be a sequence of measurable functions on a set of finite measure ω. Suppose g_n is uniformly pointwise bounded on ω, that is , there is a number $m \geq 0$ for which $|g_n| \leq m$ for all n. If $g_n \to g$ pointwise on ω, then $\lim\limits_{n\to\infty} \int_\omega g_n = \int_\omega g$.*

Theorem 4.2. *Let $-A$ generates an analytic semigroup of contractions in X such that $(B1)-(B3)$ hold. Then the fractional differential equation with finite delay (1.1) has a unique strong and classical solution.*

Proof. **Existence of strong solution-** Integrating equation (3.14) from 0 to t, we obtain

$$U^n(t) - \phi(0) + \int_0^t AX^n(s) ds + \int_0^t \tilde{D}_\alpha U^n(s) ds = \int_0^t f^n(s) ds \quad in \quad X$$

For each $\psi \in X^*$, we get

$$\langle U^n(t), \psi \rangle - \langle \phi(0), \psi \rangle + \int_0^t \langle AX^n(s), \psi \rangle ds + \int_0^t \langle \tilde{D}_\alpha U^n(s), \psi \rangle ds = \int_0^t \langle f^n(s), \psi \rangle ds \quad in \quad X^*$$

Rewriting the above equation for the subsequence n_k of n, we have

$$\langle U^{n_k}(t), \psi \rangle - \langle \phi(0), \psi \rangle + \int_0^t \langle AX^{n_k}(s), \psi \rangle ds + \int_0^t \langle \tilde{D}_\alpha U^{n_k}(s), \psi \rangle ds = \int_0^t \langle f^{n_k}(s), \psi \rangle ds$$

Owing to Lebesgue bounded convergence theorem, apriori estimates and remarks, as $k \to \infty$

$$\langle u(t), \psi\rangle - \langle \phi(0), \psi\rangle + \int_0^t \langle Au(s), \psi\rangle ds + \int_0^t \langle D_\alpha u(s), \psi\rangle ds = \int_0^t \langle f(s, u(s), u(s-\tau)), \psi\rangle ds \tag{4.1}$$

Using $\int_0^t \langle D_\alpha u(s), \psi\rangle ds = \langle I_{0+}^{1-\alpha} u(t), \psi\rangle$ in equation (4.1), we obtain

$$\langle u(t) + I_{0+}^{1-\alpha} u(t), \psi\rangle - \langle \phi(0), \psi\rangle = -\int_0^t \langle Au(s), \psi\rangle ds + \int_0^t \langle f(s, u(s), u(s-\tau)), \psi\rangle ds$$

The continuous differentiability of $\langle u(t) + I_{0+}^{1-\alpha} u(t), \psi\rangle$ is provided by the continuity of the integrands on the RHS. Now, owing to Bochner integrability of $Au(t)$, the strong derivative of $u(t) + I_{0+}^{1-\alpha} u(t)$ exists a.e. on $[0, T]$, implies that

$$\frac{du(t)}{dt} + D_\alpha u(t) + Au(t) = f(t, u(t), u(t-\tau)), \quad for \quad a.e. \quad t \in (0, T]$$

Clearly, u is Lipschitz continuous on $[0, T]$ and $u(t) \in D(A)$ for $t \in [0, T]$.

Uniqueness of strong solution- If there are two strong solutions, u_1 and u_2, they will both satisfy the differential equation (1.1), then

$$\begin{cases} \frac{dy(t)}{dt} + D_\alpha y(t) + Ay(t) = f(t, u_1(t), u_1(t-\tau)) - f(t, u_2(t), u_2(t-\tau)) \quad t > 0 \\ y(t) = 0, \quad t \in [-\tau, 0) \\ {}_{0t}I^{1-\alpha} y(t)|_{t=0} = 0; \end{cases} \tag{4.2}$$

where $y(t) = u_1(t) - u_2(t)$ and assume $H(t) = f(t, u_1(t), u_1(t-\tau)) - f(t, u_2(t), u_2(t-\tau))$

Now equation (4.2) becomes

$$\frac{dy(t)}{dt} + D_\alpha y(t) + Ay(t) = H(t)$$

$S(t)$ is the semigroup generated by $-A$. Then

$$y(t) = -\int_0^t S(t-s) D_\alpha y(s) ds + \int_0^t S(t-s) H(s) ds \tag{4.3}$$

Since $y(t)$ is differentiable almost everywhere and $-A$ generates an analytic semigroup, differentiating equation (4.3), we get

$$y'(t) = -D_\alpha y(t) + H(t) + \int_0^t AS(t-s) D_\alpha y(s) ds - \int_0^t AS(t-s) H(s) ds \tag{4.4}$$

For homogeneous initial condition, [17], Caputo fractional derivative and R-L derivative becomes the same. Hence for $0 < \alpha < 1$

$$D_\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{y(s)}{(t-s)^\alpha} ds = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^\alpha} ds \tag{4.5}$$

Using (4.4) in (4.5), we obtain

$$\begin{cases} D_\alpha y(t) = & \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(-D_\alpha y(s) + H(s))}{(t-s)^\alpha} ds \\ & + \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_s^t \frac{1}{(t-r)^\alpha} AS(r-s) dr D_\alpha y(s) ds \\ & - \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_s^t \frac{1}{(t-r)^\alpha} AS(r-s) dr H(s) ds. \end{cases} \tag{4.6}$$

We can easily show the following estimate, using properties of analytic semigroup [Chapter-2 [19]] and using estimate derived in [11].

$$\left\| \frac{1}{\Gamma(1-\alpha)} \int_r^t \frac{1}{(t-r)^\alpha} AS(r-s)dr \right\| \leq \frac{C}{(t-r)^\alpha} \tag{4.7}$$

where C is a positive constant. Hence equation (4.6) can be written as

$$\|D_\alpha y(t)\| \leq C \int_0^t \frac{\|D_\alpha y(s)\|}{(t-s)^\alpha} ds + C \int_0^t \frac{\|H(s)\|}{(t-s)^\alpha} ds$$

Now we find the estimate of $\|H(t)\|$ using hypothesis $(B2)$

$$\|H(t)\| = \|f(t, u_1(t), u_1(t-\tau)) - f(t, u_2(t), u_2(t-\tau))\| \tag{4.8}$$

$$\leq L_f \left[\|u_1(t) - u_2(t)\| + \|\tilde{u}_1(t) - \tilde{u}_2(t)\|\right] = L_f \left[\|y(t)\| + \|\tilde{y}(t)\|\right] \tag{4.9}$$

For second term in R.H.S, when $-\tau \leq t-\tau \leq 0$, then $y(t-\tau) = 0$ using condition (4.2) and when $0 \leq t-\tau \leq T-\tau \leq T$ then we can write $\|y(t-\tau)\| \leq \|y(t)\|_t$
Hence, equation(4.9) can be written as

$$\|H(t)\| \leq g_f \|y\|_t \tag{4.10}$$

where g_f is dependent on L_f and $\|y(s)\|_t = sup_{0\leq s\leq t}\|y(s)\|$
Now following similarly as in [13], we get

$$\|y\|_t^2 \leq 2C \int_0^t \|y(s)\|_s^2 ds$$

using Grownwall's inequality, we have

$$\|y(t)\|_t = 0 \quad t \in [0, T]$$

This implies $y(t) = u_1(t) - u_2(t) = 0$. This proves the uniqueness of strong solution.

Existence, uniqueness of classical solution:
The function $\bar{f}(t) = f(t, u(t), u(t-\tau))$ is local-Lipschitz continuous and X is reflexive Banach space so $\bar{f}(t)$ is differentiable a.e. and $\bar{f}'$ is in $L^1((0,T), X)$ by using Generalized Rademacher's theorem in reflexive Banach space.
Given that $-A$ is an analytical semigroup in X, the corollary 3.3 in [19] states that the existence of the unique strong solution u entails the existence of the unique classical solution to (1.1). ■

5. Application

The conclusion established in the previous section is applied in this section, and the Rothe's temporal discretization is used to determine a numerical solution.

Example 5.1. *Consider the following fractional differential equation with finite delay in $X = L^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary*

$$\frac{\partial u(t,x)}{\partial t} + \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} + A(x,D)u(t,x) = F(t,x,u(t,x),u(t-\tau,x)) \quad t \in (0,T] \tag{5.1}$$

with conditions

$$u(t,x)=0, \quad x\in\partial\Omega, \quad t\in(0,T] \tag{5.2}$$

$$I^{1-\alpha}u(0)|_{t=0}=\phi(x), \quad in \quad \Omega \tag{5.3}$$

$$u(t,x)=\phi(t,x), \quad t\in[-\tau,0) \quad x\in\Omega \tag{5.4}$$

where $0<\alpha<1$ *and* $\frac{\partial^\alpha}{\partial t^\alpha}$ *is R-L derivative of order* α *and* $x\in\Omega$.
$\phi \in \mathcal{C}_0 := C([-\tau,0];X)$, *so there are many choices for* $\phi(t,x)$. *Here we consider* $\phi(t,x)=exp(x)t^{2+\alpha}$,$0<\alpha<1$,$x\in\Omega$ *and* $t\in[-\tau,0]$, *where* $\tau>0$. *It is easy to check that* $\phi(t,x)$ *is a continuous function.*

Let $A(x,D)u=\sum_{|\beta|\le 2m}a_\beta(x)D^\beta u$ *be a strongly elliptic operator in* Ω. *The coefficients* $a_\beta(x)$ *of* $A(x,D)$ *are assumed to be smooth enough. Let* $A=A(x,D)$ *be a strongly elliptic operator of order* $2m$ *on a bounded domain* Ω *in* $\mathbb{R}^n$.
Set $D(A_2)=W^{2m,2}(\Omega)\cap W_0^{m,2}(\Omega)$ *and* $A_2(u)=A(x,D)u \quad for \quad u\in D(A_2)$. *By using theorem [[19], Chapter 7], we say that the operator* $-A_2$ *generates an analytic semigroup of contractions on* $X=L^2(\Omega)$ *and then by using theorem* $(I+A_2)^{-1}$ *is a compact operator.*
Let $F(t,x,u(t,x),u(t-\tau,x))=f(t,x)+g(u(t,x),u(t-\tau,x))$, *we assume that* f *and* g *satisfy local-Lipschitz like condition such that*

$$\|f(t_1,x_1)-f(t_2,x_2)\|\le q[|t_1-t_2|+|x_1-x_2|]$$

$$\|g(u(t_1),\tilde{u}(t_1))-g(u(t_2),\tilde{u}(t_2))\|\le L_g(r)[\|u(t_1)-u(t_2)\|+\|\tilde{u}(t_1)-\tilde{u}(t_2)\|]$$

where $\tilde{u}=u(t-\tau)$, q *is a positive constant and* $L_g(r)$ *is a non-decreasing function.*
In particular, we may consider:

1. $f(t,x)=exp(-t)\sqrt(t)+x$ *satisfies Lipschitz condition for* $t\in(0,T]$ *and* $x\in X$.
2. $g(u(t),\tilde{u}(t))=sin(u(t))\,(\tilde{u}(t))+sin(\tilde{u}(t)))$ *and clearly,* g *satisfies local-Lipschitz like condition for* $L_g(r)=2(r+a)$, $r>0$ *and* $|\phi(0)|\le a$ *as* $u(t),\tilde{u}(t)\in B_r(X,\phi(0))$ *, which is defined in section* 2.

Now, we can rewrite equations $(5.1)-(5.4)$ *as*

$$\frac{du(t)}{dt}+D_\alpha u(t)+A_2u(t)=F(t,u(t),u(t-\tau)) \quad t\in(0,T] \tag{5.5}$$

with conditions

$$\begin{cases} u(t)=\phi(t), & t\in[-\tau,0) \\ {}_{0t}I^{1-\alpha}u(t)|_{t=0}=\phi(0) \end{cases} \tag{5.6}$$

The analysis met all of theorem 4.2's assumptions, hence the existence of the unique strong and classical solutions to $(5.5)-(5.6)$ *implies the existence of the unique strong and classical solutions to* $(5.1)-(5.4)$.

Numerical Example: If we know the existence of a solution for FDDEs then we can go for the numerical solution of FDDEs.

Example 5.2. *Using Rothe's methodology, we find out numerical solution of following FDDE.*

$$\frac{\partial}{\partial t}u(t,x)+\frac{\partial^\alpha}{\partial t^\alpha}u(t,x)-\frac{\partial^2}{\partial x^2}u(t,x)=-u(x,t-s)+h(x,t) \quad t\in(0,T] \quad x\in[0,2] \tag{5.7}$$

with conditions

$$\begin{cases} u(x,t) = \phi(x,t), \quad t \in [-s,0], x \in [0,2] \\ u(0,t) = u(2,t) = 0 \quad t \in [0,1] \end{cases} \tag{5.8}$$

assume $\phi(x,t) = t^2(2x - x^2) = u(x,t)$ *for this exact solution* $h(x,t) = \frac{\Gamma(3)}{\Gamma(3-\alpha)}(2x-x^2)t^{2-\alpha} + 2t(t+2x-x^2) + x(2-x)(t-s)^2$.
As we have checked the existence of the unique classical solution for the previous example similarly we can do this here by converting the differential equation into abstract form.
Let $X = L^2([0,2],\mathbb{R})$ *and the operator* A *defined on* X *by* $Au = -u''$ *with domain*

$$D(A) = \{u \in L^2([0,2]); u', u'' \in L^2[0,2], u(0) = u(2) = 0\}$$

It is well known that A *generates an analytic semigroup on* $L^2[0,2]$ *and it satisfies all the assumptions of the theorem (4.2). So there exist classical solution for considered example by using theorem(4.2).*
Since the classical solution for a given fractional differential equation is not always known, in this case we apply Rothe's approach to get the numerical solution for a given FDDEs.
To solve numerically for the delay s*=0.5 and order of the fractional derivative* $alpha$*=0.2, we follow a few steps.*

- *The article uses a time-stepping strategy to solve the delay problem* $(5.7) - (5.8)$.
- *The relevant difference quotients from the theory are used to replace the time derivatives.*
- *Now, at* $t = t_n$, *build a discretization of our problem* (5.7) *along the time axis.*
- *We get a set of differential equations in the variable* x.
- *There are other additional ways [31–34] to solve the system of ordinary differential equations, however this article uses further discretization in space variables.*

Figures 1 *and* 2 *are solution surfaces for numerical and analytic solutions and figure* 3 *shows the parabolic behavior of the solution for different values of the time. If we increase the time steps, we get a numerical solution approaching the classical solution.*

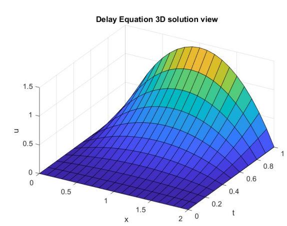

Figure 1: Numerical Solution surface for dx=dt=1/10

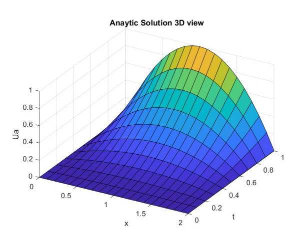

Figure 2: Analytic Solution surface for dx=dt=1/10

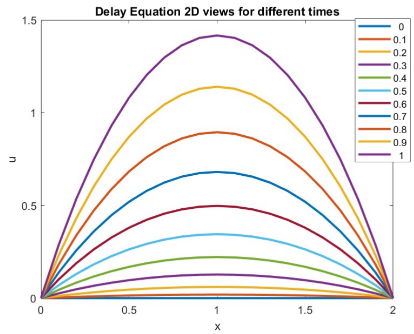

Figure 3: Numerical solution view for different values of time

6. Conclusion

In this study, we present some theoretical conclusions and numerical solution concerning the existence of the unique classical solution to the initial boundary value problem of fractional order differential equations with finite delay. The considered problem is a generalisation of the Basset problem with a finite delay, which occurs in fluid dynamics when an unstable particle accelerates in a viscous fluid due to the force of gravity. To demonstrate the existence of the unique considered problem, we calculated certain apriori results using semigroup theory and some hypotheses on the source function.

7. Acknowledgement

The author is thankful to the Ministry of Human Resource Development, Government of India for its financial support.

References

[1] E. Rothe, Zweidimensional parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben, *Math. Ann.*, ***102**(1930), 650-670 (in German)*

[2] Ladyẑenskaja, O. A, On the Solutions of Nonstationary operator Equations, *Mat. Sbornik **39**(1956) (In Russian).*

[3] Neĉas, J, Application of Rothe's Method to abstract Parabolic Equations, *Czech. Math. J. 24(1974), 496-500.*

[4] Kaĉur, J, Application of Rothe's method to Nonlinear Equations, *Nonlinear Evolution Equations and Potential Theory, 89–93.*

[5] D. Bahuguna, V. Raghvendra, Application of Rothe's method to nonlinear Schrodinger type equations, *Applicable Analysis, **31**(1994), 149-160.*

[6] D. Bahuguna, V. Raghvendra, Application of Rothe's method to nonlinear integrodifferential equations in Hilbert spaces, *Nonlinear Analysis: Theory, Methods and Applications, **23**(1)(1994), 75–81.*

[7] S. Agarwal and D. Bahuguna, *Method of semidiscretization in time to nonlinear retarded differential equations with nonlocal history conditions, IJMMS 2004:**37**, 1943-1956.*

[8] D. Bahuguna, S Abbas, J Dabas, *Partial functional differential equation with an integral condition and applications to population dynamics, Nonlinear Analysis: Theory, Methods, and Applications Volume **69**, Issue 8, Pages 2623-2635*

[9] SHRUTI A. DUBEY, The method of lines applied to nonlinear nonlocal functional differential equations, *J. Math. Anal. Appl., **376**(2011), 275-281.*

[10] DARSHANA DEVI, DURANTA, RAJIB HALOI, Rothe's Method For Solving Semi-linear Differential Equations With Deviating Arguments*; Electronic Journal of Differential Equations, 2020(2020), No. 120, 1-10.*

[11] A. ASHYRALYEV, Well-posedness of the Basset problem in space of smooth functions, *Applied Mathematics Letters, **24**(2011), 1176-1180.*

[12] D. BAHUGUNA AND ANJALI JAISWAL, Application of Rothe's Method to fractional differential equations, *Malaya Journal of Matemik, **7**(3)(2019), 399-407.*

[13] D. BAHUGUNA AND ANJALI JAISWAL, Rothe time discretization method for fractional integro-differential equations, *International Journal for Computational Methods in Engineering Science and Mechanics,* **20***(6)(2019), 540–547.*

[14] ABDERRAZEK CHAOUI AND AHMED HALLACI, On the solution of a fractional diffusion integrodifferential equation with Rothe time discretization, *Numerical Functional Analysis and Optimization,* **39***(6)(2018), 643-654.*

[15] *C. Caini, R. Firrincieli1, T. de Cola, I. Bisio3, M. Cello and G. Acar,* Mars to Earth communications through orbiters: Delay-Tolerrant/Disruption-Tolerant Networkin performance analysis, *Int. J. Satell. Commun. Network, **32**(2014), 127–140.*

[16] YANG KUANG, *Delay Differential Equations: With Application in Population Dynamics, Mathematica in Science and Engineering, Volume **191**.*

[17] I. PODLUBNY, *Fractional Differential Equations, Academic Press, New York, 1999.*

[18] A. KILBAS, H. SRIVASTAVA, J. TRUJILLO, *Theory and Applications of Fractional Differential Equations, Elsevier Science and Technology, 2006, J. Phys. Chem. 1964, **68**, 5, 1084–1091.*

[19] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations, New York: Springer Verlag, 1983.*

[20] C. LI AND Z. FANHAI, Finite difference methods for fractional differential equations, *Int. J. Bifurcation Chaos, **22**(04)(2012), 1–28.*

[21] D. BAHUGUNA AND V. RAGHAVENDRA, Rothe's method to parabolic integral-differential equations via abstract integra-differential equations, *Applicable Analysis,* **33***(3-4)(1989), 153–167.*

[22] TOSIO KATO, Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan. Vol. **19**, No 4, 1967.*

[23] C. CHIDUME, Geometric Properties of Banach Spaces and Nonlinear Iterations, *Lecture Notes in Mathematics 1965, Springer-Verlag London Limited 2009.*

[24] C. LI, D. CHEN, Y.Q, On Riemann-Liouville and Caputo Derivatives, *Discret. Dyn. Nat. Soc., 2011, 562494.*

[25] NICOLE HEYMANS AND IGOR PODLUBNY, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, *Rheol Acta, **45**(2006), 765–771.*

[26] SNEZHANA HRISTOVA, RAVI AGARWAL, AND DONAL O'REGAN, Explicit Solutions of Initial Value Problems for Linear Scalar Riemann-Liouville Fractional Differential Equations With a Constant Delay, *Advances in Difference Equations 2020(1).*

[27] YULIYA KYRYCHKO AND STEPHEN JOHN HOGAN, On the Use of Delay Equations in Engineering Applications, *Journal of Vibration and Control, **16**(7-8).*

[28] CHANGPIN LI AND WEIHUA DENG, Remarks on fractional derivatives, *Applied Mathematics and Computation, **187**(2)(2007), 777–784.*

[29] V. J. ERVIN, N. HEUER, AND J. P. ROOP, Numerical approximation of a time-dependent, nonlinear, space fractional diffusion equation, *SIAM Journal on Numerical Analysis, **45**(2)(2007), 572–591.*

[30] P. ZHUANG, F. LIU, V. ANH, AND I. TURNER, New solution and analytical techniques of the implicit numerical method for the anomalous subdiffusion equation, *SIAM Journal on Numerical Analysis, **46**(2)(2008), 1079–1095.*

[31] AKBAR MOHEBBI, Finite difference and spectral collocation methods for the solution of semilinear time fractional convection-reaction-diffusion equations with time delay, *Journal of Applied Mathematics and Computing, **61**(2019), 635–656.*

[32] SARITA NANDAL AND DWIJENDRA N. PANDEY, Numerical solution of non-linear fourth order fractional sub-diffusion wave equation with time delay, *Applied Mathematics and Computation, **369**(3)(2019), 124900.*

[33] DEVENDRA KUMAR, PARVIN KUMARI, A parameter-uniform collocation scheme for singularly perturbed delay problems with integral boundary condition, *Journal of Applied Mathematics and Computing, **63**(2020), 813–828.*

[34] MAHMOUD SHERIF, IBRAHIM ABOUELFARAG, TAREK AMER, Numerical solution of Fractional delay differential equations using Spline functions, *International Journal of Pure and Applied Mathematics **90**(1)(2014), 73–83.*