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On dual π-endo Baer modules

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Abstract. We introduce the concept of dual π-endo Baer modules. We evolve several structural properties such as direct summands and direct sums. Moreover, we prove that the endomorphism ring of a dual π-endo Baer module is a π-Baer ring. The examples are presented to exhibit the results.

AMS Subject Classifications: 16D10, 16D80.

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Contents

1. Introduction

All rings are associate with unity and modules are unital right R-modules. R and M stand by such a ring and such a module, respectively. Throughout the paper, $\mathcal{H}$ denotes the endomorphism ring of M. A ring R is called *Baer* (*quasi-Baer*) [10], [8], if the right annihilator of each nonempty subset (ideal) of R is generated by an idempotent element of R. A kind of generalization of this condition is introduced in [4]. R is π-*Baer* [4], if the right annihilator of each projection invariant left ideal is generated by an idempotent of R. Observe that R is Baer implies that R is π-Baer implies that R is quasi-Baer.

A module M is *e.Baer* (*quasi-e.Baer*) [14], if for each $A_R \leq M_R$ ($A_R \trianglelefteq M_R$), $l_{\mathcal{H}}(A) = \mathcal{H}h$ for some $h = h^2 \in \mathcal{H}$. Recently, the authors in [5] have defined a module M is π-*endo Baer*, if for each for each $A_R \trianglelefteq_p M_R$, $l_{\mathcal{H}}(A) = \mathcal{H}p$ for some $p = p^2 \in \mathcal{H}$. In [12] and [2], the authors dualized the concept of e.Baer and quasi-e.Baer modules. M is called *dual Baer* (*quasi-dual Baer*), if for each ($A_R \trianglelefteq M_R$) $A_R \leq M_R$, $D_{\mathcal{H}}(A) = p\mathcal{H}$ for some $p = p^2 \in \mathcal{H}$, where $D_{\mathcal{H}}(A) = \{\psi \in \mathcal{H} \,|\, \psi(M) \subseteq A\}$. Following the ideas in [12] and [2], we explore the dual concept of π-e.Baer modules. We call a module M is *dual* π-*e.Baer*, if for each $A_R \trianglelefteq_p M_R$, $D_{\mathcal{H}}(A) = h\mathcal{H}$ for some $h = h^2 \in \mathcal{H}$. We indicate the fundamental results and connections between related notions in Section 2. Moreover, we study on the direct summands and direct sums properties for the former class of modules. In general, this class is neither closed under direct summands nor direct sums (see, Example 2.13 and Example 3.8). However, Proposition 2.11 and Corollary 2.12 explain some conditions when the dual π-e.Baer module property is inherited by direct summands. Further, we give a complete characterization for the direct sums of dual π-e.Baer modules in Theorem 2.14.

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In Section 3, we obtain the results related to the endomorphism rings. We prove that $\mathcal{H}$ is π-Baer if M is dual π-e.Baer in Proposition 3.1. Theorem 3.3 and Corollaries 3.4-3.5 provide some conditions which ensure the reverse of Proposition 3.1 fulfills. Finally, we represent the relations between e.Baer and dual-Baer modules when the module has a countable regular endomorphism ring in Proposition 3.7.

The notations $L_R \leq M_R$ ($L_R \leq R_R$), $L_R \trianglelefteq_p M_R$ ($L_R \trianglelefteq_p R_R$), $L_R \trianglelefteq M_R$ ($L \trianglelefteq R$) and $L_R \leq^{\oplus} M_R$ mean that L is a right R-submodule of M (L is a right ideal of R), L is a projection invariant right R-submodule of M (L is a projection invariant right ideal of R), L is a fully invariant submodule of M (L is an ideal of R), and L is a direct summand of M, respectively. $r_M(-)$ ($l_{\mathcal{H}}(-)$), **I** and $Mat_n(R)$ show the right (left) annihilator in M ($\mathcal{H}$), the subring of $\mathcal{H}$ generated by the idempotent elements of $\mathcal{H}$ and the n-by-n matrix ring over R, respectively. Recall that a right submodule A of M is called *projection (fully) invariant* in M, if $p(A) \subseteq A$ for all $p = p^2 \in \mathcal{H}$ ($p \in \mathcal{H}$). A ring R is *Abelian* if every idempotent of R is central. An idempotent $e \in R$ is called *left* (*right*) *semicentral* if $re = ere$ ($er = ere$) for each $r \in R$. $S_l(R)$ ($S_r(R)$) denotes the set of left (right) semicentral idempotents of R. A module M has *FI-SSSP*, if the sum of any number of fully invariant direct summands is a direct summand. For undefined notation or terminology, see [3, 6, 13].

2. Structural Properties

We evolve principal results and relations between the dual π-e.Baer modules and connected notions. Recall that $D_{\mathcal{H}}(A) = \{\psi \in \mathcal{H} \,|\, \psi(M) \subseteq A\}$ for some $A_R \leq M_R$ and $E_M(\mathcal{Y}) = \sum_{\psi \in \mathcal{Y}} \psi(M)$ for some $\mathcal{Y}_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$.

Lemma 2.1. *Assume $\mathcal{I}$ is a right ideal of $\mathcal{H}$ and A is a right submodule of M. Then*

(i) $E_M(D_{\mathcal{H}}(E_M(\mathcal{I}))) = E_M(\mathcal{I})$.

(ii) $D_{\mathcal{H}}(E_M(D_{\mathcal{H}}(A))) = D_{\mathcal{H}}(A)$.

(iii) $D_{\mathcal{H}}(hM) = h\mathcal{H}$ *for some* $h = h^2 \in \mathcal{H}$.

(iv) $E_M(h\mathcal{H}) = hM$ *for some* $h = h^2 \in \mathcal{H}$.

Proof. (i) and (ii) These parts follow from [2, Lemma 1.3].

(iii) Let $g \in h\mathcal{H}$. Then $g = hg$ and $g(M) = hg(M) \subseteq h(M)$. Thus $g \in D_{\mathcal{H}}(hM)$, so $h\mathcal{H} \subseteq D_{\mathcal{H}}(hM)$. Conversely, assume $f \in D_{\mathcal{H}}(hM)$. Then $f(M) \subseteq hM$, so $(1-h)f = 0$ and hence $f = hf + (1-h)f = hf \in h\mathcal{H}$. Therefore $D_{\mathcal{H}}(hM) \subseteq h\mathcal{H}$. It follows that $D_{\mathcal{H}}(hM) = h\mathcal{H}$.

(iv) Observe that $hM \subseteq E_M(h\mathcal{H})$. Let $m \in E_M(h\mathcal{H})$. Then $m = \alpha_1(m_1) + \alpha_2(m_2) + \cdots + \alpha_n(m_n)$, where $\alpha_i \in h\mathcal{H}$ and $m_i \in M$. Note that $h\alpha_i = \alpha_i$, so $m \in hM$. Thus $E_M(h\mathcal{H}) \subseteq hM$. ∎

Lemma 2.2. (i) *$D_{\mathcal{H}}(A)$ is a projection invariant right ideal of $\mathcal{H}$, for each $A_R \trianglelefteq_p M_R$.*

(ii) *$E_M(\mathcal{Y})_R$ is a projection invariant submodule of M_R, for each $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$.*

Proof. (i) Let $A_R \trianglelefteq_p M_R$. Then $D_{\mathcal{H}}(A)$ is a right ideal of $\mathcal{H}$. Consider $e = e^2 \in \mathcal{H}$ and $\alpha \in D_{\mathcal{H}}(A)$. Then $e\alpha(M) = e(\alpha(M)) \subseteq e(A) \subseteq A$, as $A_R \trianglelefteq_p M_R$. It follows that $D_{\mathcal{H}}(A)_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$.

(ii) Note that $E_M(\mathcal{Y})$ is a submodule of M. Assume $f = f^2 \in \mathcal{H}$. Since $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$, $f(E_M(\mathcal{Y})) = f(\sum_{\psi \in Y} \psi(M)) = \sum_{\psi \in Y} (f\psi)M \subseteq \sum_{\beta \in \mathcal{Y}} \beta(M)$ for some $\beta \in \mathcal{Y}$. Thence $f(E_M(\mathcal{Y})) \subseteq E_M(\mathcal{Y})$, so $E_M(\mathcal{Y})_R \trianglelefteq_p M_R$. ∎

Definition 2.3. *We call a module M dual endo π-Baer (denoted, dual π-e.Baer), provided that for all $A_R \trianglelefteq_p M_R$, there exists $h = h^2 \in \mathcal{H}$ such that $D_{\mathcal{H}}(A) = h\mathcal{H}$.*

Proposition 2.4. *M is dual π-e.Baer if and only if there exists $h = h^2 \in \mathcal{H}$ such that $E_M(\mathcal{Y}) = hM$ for each $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$.*

Proof. Suppose M is dual π-e.Baer and $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Then $E_M(\mathcal{Y}) \trianglelefteq_p M_R$ by Lemma 2.2. Thus $D_{\mathcal{H}}(E_M(\mathcal{Y})) = h\mathcal{H}$ for some $h = h^2 \in \mathcal{H}$. It follows from Lemma 2.1 that $E_M(\mathcal{Y}) = E_M(D_{\mathcal{H}}(E_M(\mathcal{Y}))) = E_M(f\mathcal{H}) = hM$.

Conversely, let $A_R \trianglelefteq_p M_R$. Observe that $D_{\mathcal{H}}(A) \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ by Lemma 2.2. Then there exists $p = p^2 \in \mathcal{H}$ such that $E_M(D_{\mathcal{H}}(A)) = pM$. Therefore $D_{\mathcal{H}}(A) = D_{\mathcal{H}}(pM) = p\mathcal{H}$ by Lemma 2.1. Hence M is dual π-e.Baer. ■

Since $D_{\mathcal{H}}(A) \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $E_M(\mathcal{Y}) \trianglelefteq_p M_R$, $h, p \in S_l(\mathcal{H})$ in Proposition 2.4 by [7, Proposition 4.12].

Lemma 2.5. *Suppose M is a dual π-e.Baer module.*
(i) *If $\psi(M)_R \trianglelefteq_p M_R$ for some $\psi \in \mathcal{H}$, then $\psi(M)_R \leq^{\oplus} M_R$.*
(ii) *If $N_R \cong D_R \leq^{\oplus} M_R$ for each $N_R \trianglelefteq_p M_R$, then $N_R \leq^{\oplus} M_R$.*

Proof. (i) Assume $\psi(M)_R \trianglelefteq_p M_R$ for some $\psi \in \mathcal{H}$. Then $\mathbf{I}(\psi(M)) = \psi(M)$, and $\mathbf{I}\psi\mathcal{H} = \psi\mathcal{H}$. It follows that $(\mathbf{I}\psi\mathcal{H})_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $\psi(M) = E_M(\mathbf{I}\psi\mathcal{H})$. Thus there exists $h = h^2 \in \mathcal{H}$ such that $\psi(M) = hM$ by Proposition 2.4.

(ii) Let $N_R \trianglelefteq_p M_R$ and $N \cong hM$ for some $h = h^2 \in \mathcal{H}$. Then there exists an isomorphism $\alpha : hM \to N$. Consider the map $\psi = \iota\alpha\pi$, where $\pi : M \to hM$ is projection, and $\iota : N \to M$ is inclusion. Observe that $\psi \in \mathcal{H}$ and $\psi(M) = \iota\alpha\pi(M) = \alpha(hM) = N$. Since $N_R \trianglelefteq_p M_R$, part (i) yields that $N_R \leq^{\oplus} M_R$. ■

Theorem 2.6. *M is dual Baer implies that M is dual π-e.Baer implies that M is quasi-dual Baer.*

Proof. Suppose M is dual Baer and $A_R \trianglelefteq_p M_R$. Then there exists $h = h^2 \in \mathcal{H}$ such that $D_{\mathcal{H}}(A) = h\mathcal{H}$. Hence M is dual π-e.Baer. Observe that fully invariant submodules are projection invariant. Therefore the second part follows the similar arguments in the above. ■

At the end of the paper, we provide examples which shows that the implications in Theorem 2.6 are irreversible (see, Example 3.8).

Proposition 2.7. *Assume that $\psi(M)_R \trianglelefteq M_R$ for each $\psi \in \mathcal{H}$. Then M is dual π-e.Baer if and only if M has FI-SSSP and $\psi(M)_R \leq^{\oplus} M_R$ for all $\psi \in \mathcal{H}$.*

Proof. Suppose that $\psi(M)_R \trianglelefteq M_R$ for all $\psi \in \mathcal{H}$, and M is dual π-e.Baer. Then Lemma 2.5, Theorem 2.6, and [2, Lemma 2.2] complete the result. Conversely, assume M has the stated property. Let $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $E_M(\mathcal{Y}) = \sum_{\psi \in \mathcal{Y}} \psi(M)$. By hypothesis, $\psi(M) \trianglelefteq M$ and $\psi(M) \leq^{\oplus} M$ for all $\psi \in \mathcal{H}$. Then $E_M(\mathcal{Y}) \leq^{\oplus} M$ by the FI-SSSP condition. Therefore the proof is done. ■

Proposition 2.8. (i) *If $\psi(M)_R \trianglelefteq M_R$ for all $\psi \in \mathcal{H}$, then M is dual Baer $\Leftrightarrow$ M is dual π-e.Baer $\Leftrightarrow$ M is quasi-dual Baer.*
(ii) *If M is indecomposable, then M is dual Baer $\Leftrightarrow$ M is dual π-e.Baer.*
(iii) *Assume $\mathcal{H}$ is an Abelian ring. Then M is dual Baer $\Leftrightarrow$ M is dual π-e.Baer.*
(iv) *Assume $\mathcal{H} = \mathbf{I}$. Then M is dual π-e.Baer $\Leftrightarrow$ M is quasi-dual Baer.*

Proof. (i) [2, Theorem 2.3] and Theorem 2.6 complete the proof.

(ii) Observe that every submodule of an indecomposable module is projection invariant. Therefore Theorem 2.6 yields the result.

(iii) Suppose M is dual π-e.Baer and $\mathcal{Y}_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$. Then $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ by [4, Lemma 2.3]. Thus $E_M(\mathcal{Y}) = hM$ for some $h = h^2 \in \mathcal{H}$ by Proposition 2.4. It follows from [12, Theorem 2.1] that M is dual Baer. Theorem 2.6 yields the converse.

(iv) Suppose $\mathcal{H} = \mathbf{I}$ and M is quasi-dual Baer. Let $A_R \trianglelefteq_p M_R$. Since $\mathcal{H} = \mathbf{I}$, $A_R \trianglelefteq M_R$. Thus $D_{\mathcal{H}}(A) = h\mathcal{H}$ for some $h = h^2 \in \mathcal{H}$, so M is dual π-e.Baer. Converse is clear from Theorem 2.6. ■

Corollary 2.9. *The free R-module F with a finite rank is dual π-e.Baer if and only if it is quasi-dual Baer.*

Proof. Suppose $F_R = \bigoplus_{t=1}^{n} R_t$ where $n > 1$ and $R_t \cong R$. Then $\mathcal{H} \cong Mat_n(R)$ and $\mathbf{I}(Mat_n(R)) = Mat_n(R)$. Therefore Proposition 2.8(iv) ensures the result. ■

Now, we study on the direct summands and directs sums properties for the former class of modules. A module M is *retractable*, if $Hom_R(M, A) \neq 0$ for all $0 \neq A \leq M$.

Lemma 2.10. *Assume M is a dual π-e.Baer and retractable module. Then every $0 \neq A_R \trianglelefteq_p M_R$ includes a nonzero direct summand of M.*

Proof. Suppose M satisfies the stated property. Let $0 \neq A_R \trianglelefteq_p M_R$. Then $D_{\mathcal{H}}(A) = h\mathcal{H}$ for some $h = h^2 \in \mathcal{H}$ by Proposition 2.4. Note that $h \in S_l(\mathcal{H})$. Since M is retractable, $\psi M \subseteq A$ for some $0 \neq \psi \in \mathcal{H}$. Thus $\psi \in D_{\mathcal{H}}(A)$, so $\psi = h\psi$. Observe that $(\psi h)^2 = \psi(h\psi h) = \psi h \in \mathcal{H}$, as $h \in S_l(\mathcal{H})$. Moreover $0 \neq \psi hM \subseteq \psi M \subseteq A$, so $\psi hM \leq^{\oplus} M$. ■

We mention in Example 3.8 that a direct summand of a dual π-e.Baer module need not to be dual π-e.Baer. To this end, we investigate when the direct summands fulfill the property.

Proposition 2.11. *Assume M is dual π-e.Baer and $(hM)_R \trianglelefteq M_R$ for all $h = h^2 \in \mathcal{H}$. Then $(hM)_R$ and $((1-h)M)_R$ are dual π-e.Baer.*

Proof. Let M be dual π-e.Baer, $(hM)_R \trianglelefteq M_R$ and $A_R \trianglelefteq_p (hM)_R$. Then $A_R \trianglelefteq_p M_R$ by [5, Lemma 3.1]. Hence $D_{\mathcal{H}}(A) = p\mathcal{H}$ for some $p \in S_l(\mathcal{H})$. Notice that $\mathcal{H}_{hM} \cong h\mathcal{H}h$ and $h \in S_l(\mathcal{H})$. Moreover, $(hph)^2 = hph \in h\mathcal{H}h$ and $(hph)M \subseteq hp(M) \subseteq h(A) \subseteq A$. Hence $hph \in D_{h\mathcal{H}h}(A)$. Thus $(hph)(h\mathcal{H}h) \subseteq D_{h\mathcal{H}h}(A)$. Let $\psi \in D_{h\mathcal{H}h}(A)$. Then $\psi(M) \subseteq A$ and $\psi \in h\mathcal{H}h$. It follows that $\psi \in D_{\mathcal{H}}(A) = f\mathcal{H}$, so $\psi = f\psi$. Since $\psi \in h\mathcal{H}h$ and $h \in S_l(\mathcal{H})$, we obtain that $\psi = fh\psi = (hfh)\psi \in (hfh)(h\mathcal{H}h)$. Therefore $D_{h\mathcal{H}h}(A) \subseteq (hfh)(h\mathcal{H}h)$. It follows that $D_{h\mathcal{H}h}(A) = (hfh)(h\mathcal{H}h)$, where $(hfh)^2 = hfh \in h\mathcal{H}h$. Consequently, $(hM)_R$ is dual π-e.Baer.

Now, let $B \trianglelefteq_p ((1-h)M)_R$. Then $(hM \oplus B)_R \trianglelefteq_p M_R$ by [7, Lemma 4.13]. Then $J = D_{\mathcal{H}}(hM \oplus B) = g\mathcal{H}$ for some $g \in S_l(\mathcal{H})$. Note that $\mathcal{H}_{(1-h)M} \cong (1-h)\mathcal{H}(1-h)$ and $(1-h)J(1-h) = J \cap (1-h)\mathcal{H}(1-h)$. Since $1-h \in S_r(\mathcal{H})$, $(1-h)J(1-h) = (1-h)g\mathcal{H}(1-h) = (1-h)g(1-h)\mathcal{H}(1-h) = (1-h)g(1-h)((1-h)\mathcal{H}(1-h))$. Further, $(1-h)g(1-h) = ((1-h)g(1-h))^2 \in (1-h)\mathcal{H}(1-h)$. Our claim is $(1-h)J(1-h) = D_{(1-h)\mathcal{H}(1-h)}(B)$. Let $\alpha \in J$. Then $(1-h)\alpha(1-h)(M) \subseteq (1-h)\alpha(M) \subseteq (1-h)(hM \oplus B) = (1-h)B \subseteq B$, as $B_R \trianglelefteq_p (1-h)M_R$. It follows that $(1-h)J(1-h) \subseteq D_{(1-h)\mathcal{H}(1-h)}(B)$. Assume that $(1-h)\beta(1-h) \in (1-h)\mathcal{H}(1-h)$ such that $(1-h)\beta(1-h)(M) \subseteq B$. Hence $(1-h)\beta(1-h) \in J$. But $(1-h)\beta(1-h) \in (1-h)\mathcal{H}(1-h)$, so $(1-h)\beta(1-h) \in J \cap (1-h)\mathcal{H}(1-h) = (1-h)J(1-h)$. It follows that $D_{(1-h)\mathcal{H}(1-h)}(B) \subseteq (1-h)J(1-h)$, so $((1-h)M)_R$ is dual π-e.Baer. ■

Corollary 2.12. *Suppose M is dual π-e.Baer and $\mathcal{H}$ is Abelian. Then $(hM)_R$ and $((1-h)M)_R$ are dual π-e.Baer for all $h = h^2 \in \mathcal{H}$.*

Proof. Since $\mathcal{H}$ is Abelian, $(hM)_R \trianglelefteq M_R$ for all $h = h^2 \in \mathcal{H}$. Hence Proposition 2.11 completes the proof. ■

The following example illustrates the direct sums of dual π-e.Baer modules.

Example 2.13. *For any prime p, consider $M_{\mathbb{Z}} = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p$. Then $\mathbb{Z}(p^\infty)$ and $\mathbb{Z}_p$ are dual π-e.Baer modules. On the other hand, $M_{\mathbb{Z}}$ is not dual π-e.Baer by [2, Example 2.3] and Theorem 2.6.*

Theorem 2.14. *Suppose $M = \bigoplus_{\kappa \in \mathcal{K}} M_\kappa$ such that $(M_\kappa)_R \trianglelefteq M_R$ for all $\kappa \in \mathcal{K}$. Then M is dual π-e.Baer if and only if M_κ is dual π-e.Baer for all $\kappa \in \mathcal{K}$.*

Proof. Assume that for each $\kappa \in \mathcal{K}$, M_κ is dual π-e.Baer. Since $(M_\kappa)_R \trianglelefteq M_R$, $Hom_R(M_\kappa, M_\chi) = 0$ for all $\kappa \neq \chi \in \mathcal{K}$. Observe that $\mathcal{H} = \prod_{\kappa \in \mathcal{K}} \mathcal{H}_\kappa$, where $\mathcal{H}_\kappa = \mathcal{H}_{M_\kappa}$. Let $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Then $\mathcal{Y} = \prod_{\kappa \in \mathcal{K}} (\mathcal{Y} \cap \mathcal{H}_\kappa) = \prod_{\kappa \in \mathcal{K}} \mathcal{Y}_\kappa$, where $\mathcal{Y}_\kappa = \mathcal{Y} \cap \mathcal{H}_\kappa$ for $\kappa \in \mathcal{K}$. Notice that $(\mathcal{Y}_\kappa)_{\mathcal{H}_\kappa} \trianglelefteq_p (\mathcal{H}_\kappa)_{\mathcal{H}_\kappa}$. Since M_κ is dual π-e.Baer, $E_{M_\kappa}(\mathcal{Y}_\kappa) = h_\kappa M_\kappa$ for some $h_\kappa = h_\kappa^2 \in \mathcal{H}_\kappa$. Note that $E_M(\mathcal{Y}) = \sum_{\psi \in \mathcal{Y}} \psi(M) = \sum_{\kappa \in \mathcal{K}} E_{M_\kappa}(\mathcal{Y}_\kappa) = \bigoplus_{\kappa=1} h_\kappa M_\kappa$, as $h_\kappa M_\kappa \cap h_\chi M_\chi = 0$ for all $\kappa \neq \chi \in \mathcal{K}$. It gives that $E_M(\mathcal{Y}) \leq^{\oplus} M$, so M is dual π-e.Baer. Converse is a consequence of Proposition 2.11. ■

3. Endomorphism Rings of Dual π-e.Baer Modules

Our goal is to analyze the properties of the endomorphism ring of a dual π-endo Baer module.

Proposition 3.1. *The endomorphism ring of a dual π-e.Baer module is a π-Baer ring.*

Proof. Suppose M is dual π-e.Baer and $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Then $E_M(\mathcal{Y}) = \sum_{\psi\in\mathcal{Y}} \psi(M) = hM$ for some $h = h^2 \in \mathcal{H}$ by Proposition 2.4. Observe that $\psi(M) \subseteq hM$, so $(1-h)\psi(M) = 0$. Thus $(1-h)\psi = 0$ which gives that $1-h \in l_{\mathcal{H}}(\mathcal{Y})$. Hence $\mathcal{H}(1-h) \subseteq l_{\mathcal{H}}(\mathcal{Y})$. Let $\alpha \in l_{\mathcal{H}}(\mathcal{Y})$. Then $\alpha\mathcal{Y} = 0$, so $\alpha\psi(M) = 0$ for all $\psi \in \mathcal{Y}$. Thence $\alpha(E_M(\mathcal{Y})) = 0$, hence $(\alpha h)M = 0$, so $\alpha h = 0$. Therefore $\alpha = \alpha h + \alpha(1-h) = \alpha(1-h) \in \mathcal{H}(1-h)$, so $l_{\mathcal{H}}(\mathcal{Y}) \subseteq \mathcal{H}(1-h)$. Thus $\mathcal{H}$ is π-Baer. ■

The next example validates the reverse of Proposition 3.1 may not be true, in general.

Example 3.2. (i) *Assume $M_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}}$. Then $\mathcal{H} \cong \mathbb{Z}$ is a π-Baer ring, but $M_{\mathbb{Z}}$ is not dual π-e.Baer.*

(ii) *Let $R = \prod_{\iota=1}^{\infty} \mathcal{F}_\iota$, where $\mathcal{F}$ is a field and $\mathcal{F}_\iota = \mathcal{F}$ for $\iota = 1, 2, \cdots$. Then $M_R = R_R$ is not dual Baer by [12, Corollary 2.9]. Since R is a commutative ring, M_R is not dual π-e.Baer. However, $\mathcal{H} \cong R$ and R is a π-Baer ring by [4, Proposition 2.10].*

A module M_R is called *coretractable* (*quasi-coretractable*) [1], [11], provided that $Hom_R(M/A, M) \neq 0$ ($Hom_R(M/\sum_{\psi\in I}\psi(M), M) \neq 0$) for all proper $A \leq M$ ($I_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$ with $\sum_{\psi\in I}\psi(M) \neq M$). Notice that every coretractable module is quasi-coretractable. In the following result, we characterize a dual π-e.Baer (resp., quasi-dual Baer) module and its endomorphism ring being π-Baer (resp., quasi-Baer).

Theorem 3.3. *Assume M is quasi-coretractable. Then M is dual π-e.Baer (resp., quasi-dual Baer) if and only if $\mathcal{H}$ is π-Baer (resp., quasi-Baer).*

Proof. Assume M is dual π-e.Baer. By Proposition 3.1, $\mathcal{H}$ is π-Baer. Let $\mathcal{H}$ is π-Baer and $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. We claim that $E_M(\mathcal{Y}) = \sum_{\psi\in\mathcal{Y}} \psi(M) \leq^{\oplus} M_R$. Since $\mathcal{H}$ is π-Baer, there is $h = h^2 \in \mathcal{H}$ such that $l_{\mathcal{H}}(\mathcal{Y}) = \mathcal{H}h$. Observe $\mathcal{Y} \subseteq r_{\mathcal{H}}(l_{\mathcal{H}}(\mathcal{Y})) = (1-h)\mathcal{H}$. Consider the right ideal $\mathcal{A} = \mathcal{Y} + h\mathcal{H}$. Notice that $l_{\mathcal{H}}(\mathcal{A}) = l_{\mathcal{H}}(\mathcal{Y}) \cap l_{\mathcal{H}}(h\mathcal{H}) = \mathcal{H}h \cap \mathcal{H}(1-h) = 0$. Thus, $l_{\mathcal{H}}(\mathcal{A}) = 0$. By [11, Lemma 3.3], $\sum_{\psi\in\mathcal{A}} \psi(M) = M$. Furthermore, $M = \sum_{\psi\in\mathcal{A}} \psi(M) = \sum_{\psi\in I} \psi(M) \oplus \sum_{\psi\in h\mathcal{H}} \psi(M)$ which gives that $M = E_M(\mathcal{Y}) \oplus \sum_{\psi\in h\mathcal{H}} \psi(M)$. Hence M is dual π-e.Baer. The quasi-dual Baer case follows from the similar arguments and [2, Proposition 3.1]. ■

Corollary 3.4. *M is dual π-e.Baer if and only if $E_M(\mathcal{Y}) = r_M(l_{\mathcal{H}}(\mathcal{Y}))$ is a direct summand of M_R for all $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $\mathcal{H}$ is π-Baer.*

Proof. Suppose M is dual π-e.Baer. By Proposition 3.1, $\mathcal{H}$ is π-Baer. Let $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Then $E_M(\mathcal{Y}) = pM$ for some $p \in S_l(\mathcal{H})$. Thus $(1-p)\psi(M) = 0$ for all $\psi \in \mathcal{Y}$ by Proposition 2.4. Then $1-p \in l_{\mathcal{H}}(\mathcal{Y})$, so $\mathcal{H}(1-p) \subseteq l_{\mathcal{H}}(\mathcal{Y})$. It follows that $r_M(l_{\mathcal{H}}(\mathcal{Y})) \subseteq r_M(\mathcal{H}(1-p)) = pM = E_M(\mathcal{Y})$. We claim that $l_{\mathcal{H}}(\mathcal{Y})pM = 0$. Observe that $g\mathcal{Y} = 0$ for all $g \in l_{\mathcal{H}}(\mathcal{Y})$. Then $0 = g(\sum_{\psi\in\mathcal{Y}} \psi(M)) = g(E_M(\mathcal{Y})) = g(pM)$. Therefore $l_{\mathcal{H}}(\mathcal{Y})pM = 0$, so $pM \subseteq r_M(l_{\mathcal{H}}(\mathcal{Y}))$. It follows that $E_M(\mathcal{Y}) = r_M(l_{\mathcal{H}}(\mathcal{Y}) = pM$. Conversely, let $E_M(\mathcal{Y}) = r_M(l_{\mathcal{H}}(\mathcal{Y})) \leq^{\oplus} M_R$ for all $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$ and $\mathcal{H}$ be π-Baer. Thus $l_{\mathcal{H}}(\mathcal{Y}) = \mathcal{H}q$ for some $q \in S_r(\mathcal{H})$ by [4, Proposition 2.1] Hence $q\nu = 0$ for all $\nu \in \mathcal{Y}$. Thus $\nu = q\nu + (1-q)\nu = (1-q)\nu$ and $\nu(M) \subseteq (1-q)M$. Thence $E_M(\mathcal{Y}) \subseteq (1-q)M$. However, $(1-q)M = r_M(\mathcal{H}q) = r_M(l_{\mathcal{H}}(\mathcal{Y}))$. By hypothesis, $(1-q)M = E_M(\mathcal{Y})$, so M is dual π-e.Baer. ■

A ring R is called *right Kasch* [13], if every simple right R-module can be embedded in R_R.

Corollary 3.5. (i) *Suppose $\mathcal{H}$ is right Kasch. Then M is dual π-e.Baer if and only if $\mathcal{H}$ is π-Baer.*

(ii) *If M is an indecomposable dual π-e.Baer module with finite uniform dimension, then $\mathcal{H}$ is semilocal.*

Proof. (i) Since $\mathcal{H}$ is a right Kasch ring, $\mathcal{H}_{\mathcal{H}}$ is coretractable by [1, Theorem 2.14]. Then $\mathcal{H}_{\mathcal{H}}$ is quasi-coretractable. Therefore Theorem 3.3 yields the result.

(ii) Proposition 2.8(ii) and [12, Proposition 2.17] complete the proof. ■

Proposition 3.6. *The followings are equivalent.*

(i) *M is an indecomposable dual π-e.Baer module.*

(ii) *M is a quasi-coretractable module and $\mathcal{H}$ is a domain.*

(iii) *Every $0 \neq \tau \in \mathcal{H}$ is an epimorphism.*

(iv) *$E_M(\mathcal{Y}) = M$ for all $0 \neq \mathcal{Y}_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$.*

(v) *$D_{\mathcal{H}}(A) = \mathcal{H}$ for all $0 \neq A_R \leq M_R$.*

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ Proposition 2.8(ii), [11, Corollary 2.7] and [12, Corollary 2.2] yield the implications.

$(i) \Rightarrow (iv)$ Let $0 \neq \mathcal{Y}_{\mathcal{H}} \leq \mathcal{H}_{\mathcal{H}}$. Since M is indecomposable, $\mathcal{Y}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. Hence $E_M(\mathcal{Y}) = pM$ for some $p = p^2 \in \mathcal{H}$. Thence $E_M(\mathcal{Y}) = 0$ or $E_M(\mathcal{Y}) = M$. If $E_M(\mathcal{Y}) = 0$, then $\mathcal{Y} \subseteq D_{\mathcal{H}}(E_M(\mathcal{Y})) = 0$, a contradiction. Therefore $E_M(\mathcal{Y}) = M$.

$(iv) \Rightarrow (i)$ Suppose $\mathcal{X}_{\mathcal{H}} \trianglelefteq_p \mathcal{H}_{\mathcal{H}}$. If $\mathcal{X} = 0$, then we are done. Let $0 \neq \mathcal{X}$. By part (iv), $E_M(\mathcal{X}) = M$ so M_R is dual π-e.Baer. Moreover, $E_M(h\mathcal{H}) = M$ for some $0 \neq h = h^2 \in \mathcal{H}$ by part (iv). Hence $M = E_M(h\mathcal{H}) = hM$, so $h = 1$. Therefore M is indecomposable.

$(i) \Leftrightarrow (v)$ This part follows from the similar steps in part $(i) \Rightarrow (iv)$ and part $(iv) \Rightarrow (i)$. ■

Assume T is the $\mathbb{Z}_2$-subalgebra of $\prod_{\varpi=1}^{\infty} F_{\varpi}$ generated by $\bigoplus_{\varpi=1}^{\infty} F_{\varpi}$ and 1, where $F_{\varpi} = \mathbb{Z}_2$. Then T is a countable von Neumann regular ring [6]. In the following result, we make connections between the related notions when the module has a countable regular endomorphism ring.

Proposition 3.7. *Assume $\mathcal{H}$ is countable regular. Then the following statements are equivalent.*

(i) *$\mathcal{H}$ is a Baer ring.*

(ii) *M_R is a dual Baer module.*

(iii) *$\mathcal{H}_{\mathcal{H}}$ is a dual Baer module.*

(iv) *M_R is an e.Baer module.*

Proof. $(i) \Rightarrow (2)$ $\mathcal{H}$ is a semisimple Artinian ring by [6, Corollary 3.1.13]. Then $D_{\mathcal{H}}(X) \leq^{\oplus} \mathcal{H}_{\mathcal{H}}$ for any $\emptyset \neq X \subseteq M$, so M is dual Baer.

$(ii) \Rightarrow (iii)$ By [11, Theorem 3.6], $\mathcal{H}$ is Baer. Thence $\mathcal{H}_{\mathcal{H}}$ is a dual Baer module by [6, Corollary 3.1.13] and [12, Corollary 2.9].

$(iii) \Rightarrow (iv)$ Observe that ${}_{\mathcal{H}}\mathcal{H}$ is semisimple by [12, Corollary 2.9]. Hence ${}_{\mathcal{H}}(l_{\mathcal{H}}(B)) \leq^{\oplus} {}_{\mathcal{H}}\mathcal{H}$ for all $\emptyset \neq B \subseteq M$. Thus M is e.Baer.

$(iv) \Rightarrow (i)$ This part follows from [14, Theorem 4.1]. ■

The following example explains dual Baer, dual π-e.Baer and quasi-dual Baer modules are strictly different from each other. Furthermore, it gives an answer to the question: is the dual π-e.Baer module property inherited by direct summands?

Example 3.8. *Assume that R be a simple Noetherian ring with $\{0, 1\}$ as its only idempotents and not Morita equivalent to a domain [9]. Observe from [4, Theorem 2.1], R is quasi-Baer but not π-Baer. Then consider the following examples:*

(1) *Let $M_R = R_R$. Observe that R is a quasi-Baer ring, and R_R is coretractable. Hence R_R is quasi-dual Baer by Theorem 3.3. Since R is not a π-Baer ring by [4, Theorem 2.1], R_R is not dual π-e.Baer.*

(2) *Let* $T_R = \bigoplus_{\kappa=1}^{n} R_\kappa$ *where* $R_\kappa \cong R$. *Hence* T_R *is dual* π*-e.Baer, but not dual Baer. To see this, observe that* T_R *is a coretractable module by [1, Proposition 2.6]. Notice that* T_R *is quasi-e.Baer by [14, Proposition 3.19], and hence* $\mathcal{H} \cong Mat_n(R)$ *is also a quasi-Baer ring by [14, Theorem 4.1]. It follows from Theorem 3.3 that* T_R *is quasi-dual Baer. Moreover,* T_R *is dual* π*-e.Baer by Corollary 2.9*(i). *However,* T_R *is not dual Baer. Because* $\mathcal{H} \cong Mat_n(R)$ *is not a Baer ring by [10, Exercise 3].*

(3) *Note that* $T_R = \bigoplus_{\kappa=1}^{n} R_\kappa$ *in part* (2) *includes a direct summand,* R_R, *which is not dual* π*-e.Baer.*

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