MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(04)**(2022), 292–321.
http://doi.org/10.26637/mjm1004/002

Generalized mixed higher order functional equation in various Banach spaces

E. SATHYA *[1] AND M. ARUNKUMAR[2]

[1] *Department of Mathematics, Shanmuga Industries Arts and Science College, Tiruvannamalai - 606 603, TamilNadu, India.*
[2] *Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.*

Received 04 June 2022; *Accepted* 07 September 2022

Abstract. In this article, the we establish the generalized Ulam-Hyers stability of a generalized mixed $n^{th}(n+1)^{th}$ Order Functional Equation in various Banach Spaces.
AMS Subject Classifications: 39B52, 32B72, 32B82.

Keywords: Mixed functional equations, Generalized Ulam - Hyers stability, Banach space, Modular Space, Fuzzy Banach space, Random Banach Space.

Contents

*Corresponding author. Email address: **sathya24mathematics@gmail.com** (E. Sathya), **drarun4maths@gmail.com** (M. Arunkumar)

1. Introduction

A inspiring and renowned talk presented by Ulam [50] in 1940, encouraged the study of stability problems for various functional equations. He gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems.

The first confident answer to celebrated Ulam's question concerning the problem of stability of functional equations was given by Hyers [17] for the case of additive mappings in Banach spaces. In development of time, the theorem conveyed by Hyers was generalized by Aoki [3], Rassias [39, 40], Gavruta [14] for additive mappings and Ravi [42] for quadratic mappings.

The general solution and generalized Ulam - Hyers stability of several types of functional equations in various normed spaces were discussed by many authors one can see [1, 9, 11, 19, 20, 41] and references there in.

The simplest functional equations are

$$f(-x) = -f(x); \qquad and \qquad g(-x) = g(x) \tag{1.1}$$

which are the well known odd and even functions.

Inspiring by the overhead idea, Arunkumar et. al., [4] introduced and established the general solution and generalized Ulam - Hyers stability of the simple additive-quadratic and simple cubic-quartic functional equations

$$f(2x) = 3f(x) + f(-x); \qquad and \qquad g(2x) = 12g(x) + 4g(-x); \tag{1.2}$$

having solutions

$$f(x) = ax + bx^2 \qquad and \qquad g(x) = cx^3 + dx^4. \tag{1.3}$$

Also, the generalized Ulam - Hyers stability of the functional equations (1.2) in Quasi-Beta Banach space, Intuitionistic fuzzy Banach space applying direct and fixed point methods were discussed in [5].

Infact, the generalized version of (1.2) was introduced and examined the generalized Ulam - Hyers stability of single variable generalized additive-quadratic and generalized cubic-quartic functional equations of the form

$$\phi(\lambda w) = \frac{\lambda}{2}\left(\phi(w) - \phi(-w)\right) + \frac{\lambda^2}{2}\left(\phi(w) + \phi(-w)\right); \tag{1.4}$$

$$\psi(\mu w) = \frac{\mu^3}{2}\left(\psi(w) - \psi(-w)\right) + \frac{\mu^4}{2}\left(\psi(w) + \psi(-w)\right) \tag{1.5}$$

having solutions

$$\phi(w) = aw + bw^2 \qquad and \qquad \phi(w) = cw^3 + dw^4, \tag{1.6}$$

was investigated by Arunkumar et. al., [6] .

Motivated from overhead ideas in this article, we establish the generalized Ulam-Hyers stability of a Generalized Mixed $n^{th}(n+1)^{th}$ Order Functional Equation

$$\mathcal{N}_{n;n+1}(\mathcal{T}v) = \frac{\mathcal{T}^n}{2}\left(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\right) + \frac{\mathcal{T}^{n+1}}{2}\left(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\right) \tag{1.7}$$

having solutions

$$\mathcal{N}_{n;n+1}(v) = av^n + bv^{n+1} \tag{1.8}$$

with $n \neq 0$ is an **odd positive integer** and $\mathcal{T} \geq 2$ in various Banach Spaces via Hyers Method.

The solution of the functional equation (1.7) are as follows. Assume A_1 and A_2 are vector spaces. Applying oddness and evenness of $\mathcal{N}_{n;n+1}$ the following lemmas are trivial.

Lemma 1.1. *An odd function $\mathcal{N}_{n;n+1} : A_1 \to A_2$ satisfying (1.7) and if we define*

$$\mathcal{N}_{n;n+1} = \mathcal{N}_n \tag{1.9}$$

then $\mathcal{N}_n$ is an n^{th} order function.

Lemma 1.2. *An even function $\mathcal{N}_{n;n+1} : A_1 \to A_2$ satisfying (1.7) and if we define*

$$\mathcal{N}_{n;n+1} = \mathcal{N}_{n+1} \tag{1.10}$$

then $\mathcal{N}_{n+1}$ is an $(n+1)^{th}$ order function.

2. Stability In Banach Space of (1.7)

In this section, we investigate the generalized Ulam - Hyers stability of the functional equations (1.7) in Banach space. To prove stability results, let us take $\mathcal{R}_1$ be an normed space and $\mathcal{R}_2$ be an Banach space.

2.1. Stability Results: Odd Case

Theorem 2.1. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\begin{aligned}\bigg\| \mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\bigg\| \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1,\end{aligned} \tag{2.1}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ with the condition

$$\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^{nmt}v)}{\mathcal{T}^{nmt}} = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{2.2}$$

Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\|\Gamma_n(v) - \mathcal{N}_n(v)\| \leq \frac{1}{\mathcal{T}^n} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{nrt}v)}{\mathcal{T}^{nrt}}; \quad \forall\, v \in \mathcal{R}_1; \tag{2.3}$$

with $t = \pm 1$. The mapping $\Gamma_n(v)$ is defined by

$$\Gamma_n(v) = \lim_{m\to\infty} \frac{\mathcal{N}_n(\mathcal{T}^{nmt}v)}{\mathcal{T}^{nmt}}; \quad \forall\, v \in \mathcal{R}_1. \tag{2.4}$$

Proof. Applying oddness of $\mathcal{N}_{n;n+1}$ in (2.1) and by (1.9), we observe that

$$\|\mathcal{N}_n(\mathcal{T}v) - \mathcal{T}^n\mathcal{N}_n(v)\| \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1. \tag{2.5}$$

The overhead inequality can be rewritten as

$$\left\| \frac{\mathcal{N}_n(\mathcal{T}v)}{\mathcal{T}^n} - \mathcal{N}_n(v) \right\| \leq \frac{\mathcal{M}(v)}{\mathcal{T}^n}; \quad \forall\, v \in \mathcal{R}_1. \tag{2.6}$$

Changing v by $\mathcal{T}v$ and multiplying by $\frac{1}{\mathcal{T}^n}$ in (2.6), we notice that

$$\left\| \frac{\mathcal{N}_n(\mathcal{T}^2v)}{\mathcal{T}^{2n}} - \frac{\mathcal{N}_n(\mathcal{T}v)}{\mathcal{T}^n} \right\| \leq \frac{\mathcal{M}(\mathcal{T}v)}{\mathcal{T}^{2n}}; \quad \forall\, v \in \mathcal{R}_1. \tag{2.7}$$

From (2.6) and (2.7), we obtain that

$$\left\|\frac{\mathcal{N}_n(\mathcal{T}^2 v)}{\mathcal{T}^{2n}} - \mathcal{N}_n(v)\right\| \leq \frac{1}{\mathcal{T}^n}\left(\mathcal{M}(v) + \frac{\mathcal{M}(\mathcal{T}v)}{\mathcal{T}^n}\right); \quad \forall\, v \in \mathcal{R}_1. \tag{2.8}$$

Generalizing for a positive integer m, we acquire that

$$\left\|\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(v)\right\| \leq \frac{1}{\mathcal{T}^n}\sum_{r=0}^{m-1}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}}; \quad \forall\, v \in \mathcal{R}_1. \tag{2.9}$$

Thus $\left\{\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}\right\}$ is a Cauchy sequence and it converges to a point $\Gamma_n(v) \in \mathcal{R}_2$.

Indeed, replacing v by $\mathcal{T}^\kappa w$ and divided by $\mathcal{T}^{n\kappa}$ in (2.9), we achieve that

$$\begin{aligned}\left\|\frac{\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)}{\mathcal{T}^{nm+n\kappa}} - \frac{\mathcal{N}_n(\mathcal{T}^\kappa v)}{\mathcal{T}^{n\kappa}}\right\| &= \frac{1}{\mathcal{T}^{n\kappa}}\left\|\frac{\mathcal{N}_n(\mathcal{T}^m\cdot\mathcal{T}^\kappa v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(\mathcal{T}^\kappa v)\right\| \\ &\leq \frac{1}{\mathcal{T}^n}\sum_{r=0}^{m-1}\frac{\mathcal{M}(\mathcal{T}^{r+\kappa}v)}{\mathcal{T}^{nr+n\kappa}} \\ &\to 0 \quad as \quad \kappa \quad \to \quad \infty\end{aligned} \tag{2.10}$$

for all $v \in \mathcal{R}_1$. Thus, we define mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ such that

$$\Gamma_n(v) = \lim_{m\to\infty}\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}; \quad \forall\, v \in \mathcal{R}_1.$$

Letting limit $m \to \infty$ in (2.9) and applying the definition of $\Gamma_n(v)$, we arrive that

$$\left\|\lim_{m\to\infty}\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(v)\right\| \leq \frac{1}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}} \Rightarrow \|\Gamma_n(v) - \mathcal{N}_n(v)\| \leq \frac{1}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}}$$

for all $v \in \mathcal{R}_1$. Thus (2.3) holds for $t = 1$. Now, to show that $\Gamma_n(v)$ satisfies (1.7), changing v by $\mathcal{T}^m v$ and divided by $\mathcal{T}^{nm}$ in (2.1), we observe that

$$\begin{aligned}\frac{1}{\mathcal{T}^{nm}}\Big\|\mathcal{N}_{n;n+1}(\mathcal{T}^m\cdot\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)& \\ -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\Big\| &\leq \frac{1}{\mathcal{T}^{nm}}\mathcal{M}(\mathcal{T}^m v)\end{aligned}$$

for all $v \in \mathcal{R}_1$. Approaching $m \to \infty$ and applying the definition of $\Gamma_n(v)$ and (2.2) in the overhead inequality, we identify that

$$\Gamma_n(\mathcal{T}v) = \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) + \frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big); \quad \forall\, v \in \mathcal{R}_1.$$

Hence $\Gamma_n(v)$ satisfies the functional equation (1.7) for all $v \in \mathcal{R}_1$. In order to prove the existence of $\Gamma_n(v)$ is unique, assume $\Gamma_B(v)$ be another n^{th} order mapping satisfying (1.7) and (2.3). Now,

$$\begin{aligned}\|\Gamma_n(v) - \Gamma_B(v)\| &= \frac{1}{\mathcal{T}^{n\kappa}}\|\Gamma_n(\mathcal{T}^\kappa v) - \Gamma_B(\mathcal{T}^\kappa v)\| \\ &= \frac{1}{\mathcal{T}^{n\kappa}}\|\Gamma_n(\mathcal{T}^\kappa v) - \mathcal{N}_n(\mathcal{T}^\kappa v) + \mathcal{N}_n(\mathcal{T}^\kappa v) - \Gamma_B(\mathcal{T}^\kappa v)\| \\ &\leq \frac{1}{\mathcal{T}^{n\kappa}}\{\|\Gamma_n(\mathcal{T}^\kappa v) - \mathcal{N}_n(\mathcal{T}^\kappa v)\| + \|\Gamma_B(\mathcal{T}^\kappa v) - \mathcal{N}_n(\mathcal{T}^\kappa v)\|\} \\ &\leq \frac{2}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^{r+\kappa}v)}{\mathcal{T}^{nr+n\kappa}} \\ &\to 0 \quad as \quad \kappa \quad \to \quad \infty\end{aligned}$$

for all $v \in \mathcal{R}_1$. This proves that $\Gamma_n(v) = \Gamma_B(v)$ for all $v \in \mathcal{R}_1$. Thus $\Gamma_n(v)$ is unique. Hence the theorem holds for $t = 1$.
Further, replacing v by $\dfrac{v}{\mathcal{T}}$ in (2.5), we find that

$$\left\|\mathcal{N}_n(v) - \mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right)\right\| \leq \mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{2.11}$$

for all $v \in \mathcal{R}_1$. Again replacing v by $\dfrac{v}{\mathcal{T}}$ and multiply by $\mathcal{T}^n$ in (2.11), we notice that

$$\left\|\mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^{2n}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right)\right\| \leq \mathcal{T}^n\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{2.12}$$

for all $v \in \mathcal{R}_1$. applying triangle inequality on (2.11) and (2.12), we obtain that

$$\left\|\mathcal{N}_n(v) - \mathcal{T}^{2n}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right)\right\| \leq \mathcal{M}\left(\frac{v}{\mathcal{T}}\right) + \mathcal{T}^n\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{2.13}$$

for all $v \in \mathcal{R}_1$. Generalizing for a positive integer m, we acquire that

$$\left\|\mathcal{N}_n(v) - \mathcal{T}^{nm}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right)\right\| \leq \sum_{r=1}^{m-1}\mathcal{T}^{nr-n}\mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) = \frac{1}{\mathcal{T}^n}\sum_{r=1}^{m-1}\mathcal{T}^{nr}\mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) \tag{2.14}$$

for all $v \in \mathcal{R}_1$. The rest of the proof is similar ideas to that of case $t = 1$. Thus the theorem is true for $t = -1$. Hence the proof is complete. ■

The following corollary is the immediate consequence of Theorem 2.1 concerning the stabilities of (1.7).

Corollary 2.2. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\begin{aligned}\Bigg\|\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)\\ -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg\| \leq \begin{cases} s; \\ s\|v\|^\mu;\ \mu \neq n\end{cases}\end{aligned} \tag{2.15}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\|\Gamma_n(v) - \mathcal{N}(v)\| \leq \begin{cases} \dfrac{s}{|\mathcal{T}^n - 1|}; \\ \dfrac{s\|v\|^\mu}{|\mathcal{T}^n - \mathcal{T}^\mu|}; \end{cases} \tag{2.16}$$

for all $v \in \mathcal{R}_1$.

2.2. Stability Results: Even Case

The proof of the following theorem and corollary is similar clues that of Theorem 2.1 and Corollary 2.2 with the help of (1.10). Hence the details of the proof are omitted.

Theorem 2.3. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function satisfies the inequality*

$$\begin{aligned}\Bigg\|\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)\\ -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg\| \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1,\end{aligned} \tag{2.17}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ *with the condition*

$$\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^{mt}v)}{\mathcal{T}^{2nmt}} = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{2.18}$$

Then there exists one and only $(n+1)^{th}$ *order mapping* $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ *satisfying the functional equation (1.7) and*

$$\|\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)\| \le \frac{1}{\mathcal{T}^{2n}} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{2nrt}}; \quad \forall\, v \in \mathcal{R}_1 \tag{2.19}$$

with $t = \pm 1$. *The mapping* $\Gamma_{n+1}(v)$ *is defined by*

$$\Gamma_{n+1}(v) = \lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^{mt}v)}{\mathcal{T}^{2nmt}}; \quad \forall\, v \in \mathcal{R}_1. \tag{2.20}$$

Corollary 2.4. *Assume* s *and* μ *be positive numbers. Let* $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ *be an even function fulfilling the inequality*

$$\begin{aligned} &\Bigg\|\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &\qquad - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg\| \le \begin{cases} s; \\ s||v||^{\mu};\ \mu \neq 2n \end{cases} \end{aligned} \tag{2.21}$$

for all $v \in \mathcal{R}_1$. *Then there exists one and only* $(n+1)^{th}$ *order mapping* $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ *satisfying the functional equation (1.7) and*

$$\|\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)\| \le \begin{cases} \dfrac{s}{|\mathcal{T}^{2n} - 1|}; \\ \dfrac{s||v||^{\mu}}{|\mathcal{T}^{2n} - \mathcal{T}^{\mu}|}; \end{cases} \tag{2.22}$$

for all $v \in \mathcal{R}_1$.

2.3. Stability Results: Odd-Even Case

Theorem 2.5. *Assume* $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ *be a function satisfies the inequality*

$$\begin{aligned} &\Bigg\|\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &\qquad - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg\| \le \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \end{aligned} \tag{2.23}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ *satisfying the conditions (2.2) and (2.18) for all* $v \in \mathcal{R}_1$. *Then there exists one and only* n^{th} *order mapping* $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ *and one and only* $(n+1)^{th}$ *order mapping* $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ *satisfying the functional equation (1.7) and*

$$\begin{aligned} &\|\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)\| \\ &\le \frac{1}{2}\left\{\frac{1}{\mathcal{T}^n}\sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{rnt}} + \frac{\mathcal{M}(-\mathcal{T}^{rt}v)}{\mathcal{T}^{rnt}} + \frac{1}{\mathcal{T}^{2n}}\sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{2rnt}} + \frac{\mathcal{M}(-\mathcal{T}^{rt}v)}{\mathcal{T}^{2rnt}}\right\} \end{aligned} \tag{2.24}$$

for all $v \in \mathcal{R}_1$ *with* $t = \pm 1$. *The mappings* $\Gamma_n(v)$ *and* $\Gamma_{n+1}(v)$ *are respectively defined in (2.4) and (2.20) for all* $v \in \mathcal{R}_1$.

Proof. Suppose define a function $\mathcal{N}_{odd}(v)$ by

$$\mathcal{N}_{odd}(v) = \frac{\mathcal{N}_n(v) - \mathcal{N}_n(-v)}{2}; \quad \forall\ v\ \in\ \mathcal{R}_1. \tag{2.25}$$

Then it is easy to verify from (2.25) that

$$\mathcal{N}_{odd}(0) = 0 \qquad and \qquad \mathcal{N}_{odd}(-v) = -\mathcal{N}_{odd}(v); \quad \forall\ v\ \in\ \mathcal{R}_1.$$

By Theorem 2.1 and (2.25), we notice that

$$\|\Gamma_n(v) - \mathcal{N}_{odd}(v)\| \leq \frac{1}{2\mathcal{T}^n} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{rnt}} + \frac{\mathcal{M}(-\mathcal{T}^{rt}v)}{\mathcal{T}^{rnt}}; \quad \forall\ v\ \in\ \mathcal{R}_1. \tag{2.26}$$

Again define a function $\mathcal{N}_{even}(v)$ by

$$\mathcal{N}_{even}(v) = \frac{\mathcal{N}_{n+1}(v) + \mathcal{N}_{n+1}(-v)}{2}; \quad \forall\ v\ \in\ \mathcal{R}_1. \tag{2.27}$$

Then it is easy to verify from (2.27) that

$$\mathcal{N}_{even}(0) = 0 \qquad and \qquad \mathcal{N}_{even}(-v) = \mathcal{N}_{even}(v); \quad \forall\ v\ \in\ \mathcal{R}_1.$$

By Theorem 2.3 and (2.27), we notice that

$$\|\Gamma_{n+1}(v) - \mathcal{N}_{even}(v)\| \leq \frac{1}{2\mathcal{T}^{2n}} \sum_{r=\frac{1-t}{2}}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{rt}v)}{\mathcal{T}^{2rnt}} + \frac{\mathcal{M}(-\mathcal{T}^{rt}v)}{\mathcal{T}^{2rnt}}; \ \forall\ v\ \in\ \mathcal{R}_1. \tag{2.28}$$

Define a function $\mathcal{N}_{n;n+1}$ by

$$\mathcal{N}_{n;n+1}(v) = \mathcal{N}_{odd}(v) + \mathcal{N}_{even}(v); \quad \forall\ v\ \in\ \mathcal{R}_1. \tag{2.29}$$

Now, it follows from (2.26), (2.28) and (2.29), we achieve our desired result. ■

Corollary 2.6. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function fulfilling the inequality*

$$\begin{aligned}\Bigg\| \mathcal{N}_{n;n+1}(\mathcal{T}v) &- \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &- \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg\| \leq \begin{cases} s; \\ s||v||^{\mu};\ \mu \neq n; 2n \end{cases}\end{aligned} \tag{2.30}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\|\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)\| \leq \begin{cases} s\left(\dfrac{1}{|\mathcal{T}^n - 1|} + \dfrac{1}{|\mathcal{T}^{2n} - 1|}\right); \\ s||v||^{\mu}\left(\dfrac{1}{|\mathcal{T}^n - \mathcal{T}^{\mu}|} + \dfrac{1}{|\mathcal{T}^{2n} - \mathcal{T}^{\mu}|}\right); \end{cases} \tag{2.31}$$

for all $v \in \mathcal{R}_1$.

3. Stability In Modular Space of (1.7)

In this section, we investigate the generalized Ulam - Hyers stability of the functional equation (1.7) in Modular space. To prove stability results, let us take $\mathcal{R}_1$ be an linear space and $\mathcal{R}_{2\rho}$ be an $\rho-$ complete convex modular space.

3.1. Basic Concepts on Modular Spaces

Now, we introduce to adopt the usual terminologies, notations, definitions and properties of the theory of modular spaces given in [2, 22, 23, 25, 27–29, 34, 37, 38, 45, 49, 52, 55].

Definition 3.1. *Let X be a linear space over a field $K(R$ or $C)$. We say that a generalized functional $\rho : X \to [0,\infty]$ is a modular if for any $x, y \in X$,*

(MS1) $\rho(x) = 0$ *if and only if* $x = 0$*;*

(MS2) $\rho(\alpha x) = \rho(x)$ *for all scalar* α *with* $|\alpha| = 1$*;*

(MS3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *for all scalar* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$.

(MS4) If (MS3) is replaced by $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *for all scalar* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$*, then the functional* ρ *is called a convex modular.*

Definition 3.2. *A modular ρ defines the following vector space:*

$$X_\rho = \{x \in X : \rho(\lambda x) \ \to \ 0 \ as \ \lambda \ \to 0\},$$

and we say that X_ρ is a modular space.

Definition 3.3. *Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ then $\{x_n\}$ is $\rho-$convergent to a point $x \in X_\rho$and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.*

Definition 3.4. *Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ then $\{x_n\}$ is called $\rho-$Cauchy if for any $\epsilon > 0$one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in N$.*

Definition 3.5. *Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ. A subset $K \subseteq X_\rho$ is called $\rho-$complete if any $\rho-$Cauchy sequence is $\rho-$convergent to a point in K.*

Definition 3.6. *A modular space ρ has the Fatou property if and only if $\rho(x) \leq \liminf\limits_{n \to \infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is $\rho-$convergent to x in modular space X_ρ .*

Definition 3.7. *A modularρ is said to satisfy the Δ_2-condition if there exists $k > 0$ such that $\rho(\mathcal{T}^n x) \leq k\rho(x)$ for all $x \in X_\rho$.*

Remark 3.8. *Suppose that ρ is convex and satisfies Δ_2-condition with Δ_2- constant $k > 0$. If $k < \mathcal{T}^n$, then $\rho(x) \leq k\rho(x) \leq \frac{k}{\mathcal{T}^n}\rho(x)$, which implies $\rho = 0$. Therefore, we must have the Δ_2- constant $k \geq \mathcal{T}^n$ if ρ is convex modular.*

3.2. Stability Results: Odd Case : Without Applying Δ_2 Condition

Theorem 3.9. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \ \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\rho\ \Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg) \leq \mathcal{M}(v); \quad \forall\, v \,\in\, \mathcal{R}_1, \tag{3.1}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ *with the condition*

$$\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^{nm}v)}{\mathcal{T}^{nm}} = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.2}$$

Then there exists one and only n^{th} *order mapping* $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ *satisfying the functional equation (1.7) and*

$$\rho\, \left(\Gamma_n(v) - \mathcal{N}_n(v)\right) \le \frac{1}{\mathcal{T}^n} \sum_{r=0}^{\infty} \frac{\mathcal{M}(\mathcal{T}^{nr}v)}{\mathcal{T}^{nr}}; \quad \forall\, v \in \mathcal{R}_1. \tag{3.3}$$

The mapping $\Gamma_n(v)$ *is defined by*

$$\lim_{m\to\infty} \rho\left(\frac{\mathcal{N}_n(\mathcal{T}^{nm}v)}{\mathcal{T}^{nm}} - \Gamma_n(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.4}$$

Proof. Using oddness of $\mathcal{N}_{n;n+1}$ in (3.1) and by (1.9), we observe that

$$\rho\, \left(\mathcal{N}_n(\mathcal{T}v) - \mathcal{T}^n\mathcal{N}_n(v)\right) \le \mathcal{M}(v) \tag{3.5}$$

for all $v \in \mathcal{R}_1$. Without applying the Δ_2- condition it follows from (3.5), generalizing for a positive integer m, we acquire that

$$\begin{aligned}
\rho\, \left(\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(v)\right) &= \rho\, \left(\sum_{r=0}^{m-1} \frac{1}{\mathcal{T}^{n(r+1)}}\left[\mathcal{T}^{nr}\mathcal{N}_n(\mathcal{T}^r v) - \mathcal{N}_n(\mathcal{T}^{r+1}v)\right]\right) \\
&\le \sum_{r=0}^{m-1} \frac{1}{\mathcal{T}^{n(r+1)}} \rho\, \left(\mathcal{T}^{nr}\mathcal{N}_n(\mathcal{T}^r v) - \mathcal{N}_n(\mathcal{T}^{r+1}v)\right) \\
&\le \frac{1}{\mathcal{T}^n} \sum_{r=0}^{m-1} \frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}}
\end{aligned} \tag{3.6}$$

for all $v \in \mathcal{R}_1$. Thus $\left\{\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}\right\}$ is a $\rho-$ Cauchy sequence in $\mathcal{R}_{2\rho}$ and $\mathcal{R}_{2\rho}$ is $\rho-$complete there exists a $\rho-$ limit function $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ given by

$$\lim_{m\to\infty} \rho\left(\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \Gamma_n(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1.$$

Indeed, replacing v by $\mathcal{T}^\kappa w$ and divided by $\mathcal{T}^{n\kappa}$ in (3.6), we achieve that

$$\begin{aligned}
\rho\, \left(\frac{\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)}{\mathcal{T}^{nm+n\kappa}} - \frac{\mathcal{N}_n(\mathcal{T}^\kappa v)}{\mathcal{T}^{n\kappa}}\right) &= \frac{1}{\mathcal{T}^{n\kappa}} \rho\, \left(\frac{\mathcal{N}_n(\mathcal{T}^m \cdot \mathcal{T}^\kappa v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(\mathcal{T}^\kappa v)\right) \\
&\le \frac{1}{\mathcal{T}^n} \sum_{r=0}^{m-1} \frac{\mathcal{M}(\mathcal{T}^{r+\kappa}v)}{\mathcal{T}^{nr+n\kappa}} \\
&\to 0 \quad as \quad \kappa \quad \to \quad \infty
\end{aligned} \tag{3.7}$$

for all $v \in \mathcal{R}_1$. Thus, we define mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ such that

$$\Gamma_n(v) = \lim_{m\to\infty} \frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}$$

for all $v \in \mathcal{R}_1$. It follows from the Fatou property that the inequality

$$\rho\ \left(\Gamma_n(v) - \mathcal{N}_n(v)\right) \le \liminf_{m\ \to \infty} \rho\ \left(\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(v)\right) \le \frac{1}{\mathcal{T}^n} \sum_{r=0}^{\infty} \frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{nr}}$$

for all $v \in \mathcal{R}_1$. Thus, we see (3.3) holds. Now, to show that $\Gamma_n(v)$ satisfies (1.7), changing v by $\mathcal{T}^m v$ and divided by $\mathcal{T}^{nm}$ in (3.1), we observe that

$$\begin{aligned}\rho\ \Bigg(\frac{1}{\mathcal{T}^{nm}} \Bigg\{\mathcal{N}_{n;n+1}(\mathcal{T}^m \cdot \mathcal{T} v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\\ -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\Bigg\}\Bigg) \le \frac{1}{\mathcal{T}^{nm}} \mathcal{M}(\mathcal{T}^m v)\end{aligned}$$

for all $v \in \mathcal{R}_1$. By convexity of ρ that

$$\begin{aligned}&\rho\left(\frac{1}{4}\Gamma_n(\mathcal{T} v) - \frac{1}{4}\,\frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{1}{4}\,\frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big)\right)\\
&\le \frac{1}{4}\rho\left(\Gamma_n(\mathcal{T} v) - \frac{1}{\mathcal{T}^{nm}}\,\mathcal{N}_{n;n+1}(\mathcal{T}^m \cdot \mathcal{T} v)\right)\\
&\quad + \frac{1}{4}\rho\left(-\frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) + \frac{1}{\mathcal{T}^{nm}}\frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\right)\\
&\quad + \frac{1}{4}\rho\left(-\frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big) + \frac{1}{\mathcal{T}^{nm}}\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\right)\\
&\quad + \frac{1}{4}\rho\ \Bigg(\frac{1}{\mathcal{T}^{nm}}\Bigg\{\mathcal{N}_{n;n+1}(\mathcal{T}^m \cdot \mathcal{T} v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\\
&\qquad\qquad -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\Bigg\}\Bigg)\end{aligned}$$

for all $v \in \mathcal{R}_1$. Approaching $m \to \infty$, we notice that

$$\rho\left(\frac{1}{4}\Gamma_n(\mathcal{T} v) - \frac{1}{4}\,\frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{1}{4}\,\frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big)\right) = 0$$

for all $v \in \mathcal{R}_1$. Hence $\Gamma_n(v)$ satisfies the functional equation (1.7) for all $v \in \mathcal{R}_1$. In order to prove the existence of $\Gamma_n(v)$ is unique, assume $\Gamma_B(v)$ be another n^{th} order mapping satisfying (1.7) and (3.3). Now,

$$\begin{aligned}\rho\ \left(\frac{1}{2}\Gamma_n(v) - \frac{1}{2}\Gamma_B(v)\right) &\le \frac{1}{2}\rho\ \left(\frac{1}{\mathcal{T}^{n\kappa}}\Gamma_n(\mathcal{T}^\kappa v) - \frac{1}{\mathcal{T}^{n\kappa}}\Gamma_B(\mathcal{T}^\kappa v)\right)\\
&\le \frac{1}{2}\frac{1}{\mathcal{T}^{n\kappa}}\rho\ \left(\Gamma_n(\mathcal{T}^\kappa v) - \mathcal{N}_n(\mathcal{T}^\kappa v) + \mathcal{N}_n(\mathcal{T}^\kappa v) - \Gamma_B(\mathcal{T}^\kappa v)\right)\\
&\le \frac{1}{2}\frac{1}{\mathcal{T}^{n\kappa}}\left\{\rho\ \left(\Gamma_n(\mathcal{T}^\kappa v) - \mathcal{N}_n(\mathcal{T}^\kappa v)\right) + \rho\ \left(\Gamma_B(\mathcal{T}^\kappa v) - \mathcal{N}_n(\mathcal{T}^\kappa v)\right)\right\}\\
&\le \frac{1}{2}\frac{2}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^{r+\kappa} v)}{\mathcal{T}^{nr+n\kappa}}\\
&\to 0 \quad as \quad \kappa \quad \to \quad \infty\end{aligned}$$

for all $v \in \mathcal{R}_1$. This proves that $\Gamma_n(v) = \Gamma_B(v)$ for all $v \in \mathcal{R}_1$. Thus $\Gamma_n(v)$ is unique. This completes the proof of the theorem. ■

The following corollary is the immediate consequence of Theorem 3.9 concerning the stabilities of (1.7).

Corollary 3.10. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)\right.$$
$$\left. - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \begin{cases} s; \\ s||v||^{\mu};\ \mu < n \end{cases} \tag{3.8}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\rho\ (\Gamma_n(v) - \mathcal{N}(v)) \leq \begin{cases} \dfrac{s}{(\mathcal{T}^n - 1)}; \\ \dfrac{s||v||^{\mu}}{(\mathcal{T}^n - \mathcal{T}^{\mu})}; \end{cases} \tag{3.9}$$

for all $v \in \mathcal{R}_1$.

3.3. Stability Results: Even Case : Without Applying Δ_2 Condition

The proof of the following theorem and corollary is similar clues that of Theorem 3.9 and Corollary 3.10 with the help of (1.10). Hence the details of the proof are omitted.

Theorem 3.11. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ be an even function satisfies the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)\right.$$
$$\left. - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.10}$$

where $\mathcal{M} : \mathcal{R}_1 \rightarrow [0, \infty)$ with the condition

$$\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^m v)}{\mathcal{T}^{2nm}} = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.11}$$

Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\rho\ (\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)) \leq \frac{1}{\mathcal{T}^{2n}} \sum_{r=0}^{\infty} \frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{2nr}}; \quad \forall\, v \in \mathcal{R}_1. \tag{3.12}$$

The mapping $\Gamma_{n+1}(v)$ is defined by

$$\rho\left(\lim_{m\to\infty} \frac{\mathcal{M}(\mathcal{T}^m v)}{\mathcal{T}^{2nm}} - \Gamma_{n+1}(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.13}$$

Corollary 3.12. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ be an even function fulfilling the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)\right.$$
$$\left. - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \begin{cases} s; \\ s||v||^{\mu};\ \mu < 2n \end{cases} \tag{3.14}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\rho\ \left(\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)\right) \leq \begin{cases} \dfrac{s}{(\mathcal{T}^{2n}-1)|}; \\ \dfrac{s||v||^{\mu}}{(\mathcal{T}^{2n}-\mathcal{T}^{\mu})}; \end{cases} \tag{3.15}$$

for all $v \in \mathcal{R}_1$.

3.4. Stability Results: Odd- Even Case: Without Applying Δ_2 Condition

Theorem 3.13. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function satisfies the inequality*

$$\rho\ \left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.16}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ satisfying the conditions (3.2) and (3.11) for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\begin{aligned}&\rho\ \left(\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)\right) \\ &\leq \frac{1}{2}\left\{\frac{1}{\mathcal{T}^n}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{rn}} + \frac{\mathcal{M}(-\mathcal{T}^r v)}{\mathcal{T}^{rn}} + \frac{1}{\mathcal{T}^{2n}}\sum_{r=0}^{\infty}\frac{\mathcal{M}(\mathcal{T}^r v)}{\mathcal{T}^{2rn}} + \frac{\mathcal{M}(-\mathcal{T}^r v)}{\mathcal{T}^{2rn}}\right\}\end{aligned} \tag{3.17}$$

for all $v \in \mathcal{R}_1$ with $t = \pm 1$. The mappings $\Gamma_n(v)$ and $\Gamma_{n+1}(v)$ are respectively defined in (3.4) and (3.13) for all $v \in \mathcal{R}_1$.

Proof. The proof is similar lines to that of Theorem 2.5. ■

Corollary 3.14. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function fulfilling the inequality*

$$\rho\ \left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \begin{cases} s; \\ s||v||^{\mu};\ \mu < n; 2n \end{cases} \tag{3.18}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\rho\ \left(\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)\right) \leq \begin{cases} s\left(\dfrac{1}{(\mathcal{T}^n-1)} + \dfrac{1}{(\mathcal{T}^{2n}-1)}\right); \\ s||v||^{\mu}\left(\dfrac{1}{(\mathcal{T}^n-\mathcal{T}^{\mu})} + \dfrac{1}{(\mathcal{T}^{2n}-\mathcal{T}^{\mu})}\right); \end{cases} \tag{3.19}$$

for all $v \in \mathcal{R}_1$.

3.5. Stability Results: Odd Case: Applying Δ_2 Condition

Theorem 3.15. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be an odd function fulfilling the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \le \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.20}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ with the condition

$$\lim_{m\to\infty}\left(\frac{k^2}{\mathcal{T}^n}\right)^m \mathcal{M}\left(\frac{v}{\mathcal{T}^m}\right) = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.21}$$

Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\left(\Gamma_n(v) - \mathcal{N}_n(v)\right) \le \frac{1}{k}\sum_{r=1}^{\infty}\left(\frac{k^2}{\mathcal{T}^n}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right); \quad \forall\, v \in \mathcal{R}_1. \tag{3.22}$$

The mapping $\Gamma_n(v)$ is defined by

$$\lim_{m\to\infty}\rho\left(\frac{\mathcal{N}_n(\mathcal{T}^{nm}v)}{\mathcal{T}^{nm}} - \Gamma_n(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.23}$$

Proof. Applying oddness of $\mathcal{N}_{n;n+1}$ in (3.20) and by (1.9), we observe that

$$\rho\left(\mathcal{N}_n(\mathcal{T}v) - \mathcal{T}^n\mathcal{N}_n(v)\right) \le \mathcal{M}(v) \tag{3.24}$$

for all $v \in \mathcal{R}_1$. Further, replacing v by $\frac{v}{\mathcal{T}}$ in (3.24), we find that

$$\rho\left(\mathcal{N}_n(v) - \mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right)\right) \le \mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{3.25}$$

for all $v \in \mathcal{R}_1$. Applying the Δ_2 condition it follows from (3.25) and the convexity of the modular ρ that,

$$\rho\left(\mathcal{N}_n(v) - \mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right)\right) \le \frac{k}{\mathcal{T}^n}\,\mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{3.26}$$

for all $v \in \mathcal{R}_1$. Again, replacing v by $\frac{v}{\mathcal{T}}$ in (3.26), we notice that

$$\rho\left(\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right)\right) \le \frac{k}{\mathcal{T}^n}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.27}$$

for all $v \in \mathcal{R}_1$. Applying the Δ_2 condition it follows from (3.27) and the convexity of the modular ρ that,

$$\rho\left(\mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^{2n}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right)\right) \le \frac{k^3}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.28}$$

for all $v \in \mathcal{R}_1$. From (3.26) and (3.28), we obtain that

$$\rho\left(\mathcal{N}_n(v) - \mathcal{T}^{2n}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^2}\right)\right) \le \frac{k}{\mathcal{T}^n}\,\mathcal{M}\left(\frac{v}{\mathcal{T}}\right) + \frac{k^3}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.29}$$

for all $v \in \mathcal{R}_1$. Generalizing for a positive integer m, we acquire that

$$\rho\left(\mathcal{N}_n(v) - \mathcal{T}^{nm}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right)\right) \leq \frac{1}{k}\sum_{r=1}^{m}\left(\frac{k^2}{\mathcal{T}^n}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) \tag{3.30}$$

for all $v \in \mathcal{R}_1$. Thus $\left\{\mathcal{T}^{nm}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right)\right\}$ is a $\rho-$ Cauchy sequence in $\mathcal{R}_{2\rho}$ and $\mathcal{R}_{2\rho}$ is $\rho-$complete there exists a $\rho-$ limit function $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ given by

$$\lim_{m\to\infty} \rho\left(\mathcal{T}^{nm}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right) - \Gamma_n(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1.$$

Indeed, replacing v by $\mathcal{T}^{\kappa}w$ and divided by $\mathcal{T}^{n\kappa}$ in (3.29), we achieve that

$$\begin{aligned}\rho\left(\mathcal{T}^{n\kappa}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^\kappa}\right) - \mathcal{T}^{nm}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right)\right) &\leq k^{\kappa}\,\rho\left(\mathcal{N}_n\left(\frac{v}{\mathcal{T}^\kappa}\right) - \mathcal{T}^{nm-n\kappa}\mathcal{N}_n\left(\frac{v}{\mathcal{T}^m}\right)\right)\\ &\leq k^{\kappa-1}\sum_{r=1}^{m-\kappa}\left(\frac{k^2}{\mathcal{T}^n}\right)^r\mathcal{M}\left(\frac{v}{\mathcal{T}^{r+\kappa}}\right)\\ &= k^{\kappa-1}\left(\frac{\mathcal{T}^n}{k^2}\right)^{\kappa}\sum_{r=\kappa+1}^{m}\left(\frac{k^2}{\mathcal{T}^n}\right)^r\mathcal{M}\left(\frac{v}{\mathcal{T}^{r+\kappa}}\right)\\ &\to 0 \quad as \quad \kappa \to \infty\end{aligned} \tag{3.31}$$

for all $v \in \mathcal{R}_1$. It follows from (3.30) and the Fatou property that

$$\rho\left(\Gamma_n(v) - \mathcal{N}_n(v)\right) \leq \liminf_{m\to\infty}\rho\left(\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \mathcal{N}_n(v)\right) \leq \frac{1}{k}\sum_{r=1}^{\infty}\left(\frac{k^2}{\mathcal{T}^n}\right)^r\mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right)$$

for all $v \in \mathcal{R}_1$. Thus, we see that (3.22) holds. The rest of the proof is similar to that of Theorem 3.9. ■

The following corollary is the immediate consequence of Theorem 3.15 concerning the stabilities of (1.7).

Corollary 3.16. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be an odd function fulfilling the inequality*

$$\begin{aligned}\rho\Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) &- \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big)\\ &-\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg) \leq \begin{cases} s;\\ s||v||^{\mu};\ \mu > \log_2\frac{k^2}{\mathcal{T}^n}\end{cases}\end{aligned} \tag{3.32}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\left(\Gamma_n(v) - \mathcal{N}(v)\right) \leq \begin{cases}\dfrac{sk}{\mathcal{T}^n - k^2};\\ \dfrac{sk||v||^{\mu}}{\mathcal{T}^{n+\mu} - k^2};\end{cases} \tag{3.33}$$

for all $v \in \mathcal{R}_1$.

3.6. Stability Results: Even Case : Applying Δ_2 Condition

Theorem 3.17. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be an even function satisfies the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.34}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ with the condition

$$\lim_{m\to\infty}\left(\frac{k^2}{\mathcal{T}^{2n}}\right)^m \mathcal{M}\left(\frac{v}{\mathcal{T}^m}\right) = 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.35}$$

Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\,\left(\Gamma_n(v) - \mathcal{N}_n(v)\right) \leq \frac{1}{k^2}\sum_{r=1}^{\infty}\left(\frac{k^3}{\mathcal{T}^{2n}}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right); \quad \forall\, v \in \mathcal{R}_1.; \quad \forall\, v \in \mathcal{R}_1. \tag{3.36}$$

The mapping $\Gamma_{n+1}(v)$ is defined by

$$\lim_{m\to\infty}\rho\left(\frac{\mathcal{N}_n(\mathcal{T}^{2nm}v)}{\mathcal{T}^{nm}} - \Gamma_n(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1. \tag{3.37}$$

Proof. Applying even of $\mathcal{N}_{n;n+1}$ in (3.34) and by (1.10), we observe that

$$\rho\,\left(\mathcal{N}_{n+1}(\mathcal{T}v) - \mathcal{T}^{2n}\mathcal{N}_{n+1}(v)\right) \leq \mathcal{M}(v) \tag{3.38}$$

for all $v \in \mathcal{R}_1$. Further, replacing v by $\frac{v}{\mathcal{T}}$ in (3.38), we find that

$$\rho\,\left(\mathcal{N}_{n+1}(v) - \mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}}\right)\right) \leq \mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{3.39}$$

for all $v \in \mathcal{R}_1$. Applying the Δ_2 condition it follows from (3.39) and the convexity of the modular ρ that,

$$\rho\,\left(\mathcal{N}_{n+1}(v) - \mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}}\right)\right) \leq \frac{k}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}}\right) \tag{3.40}$$

for all $v \in \mathcal{R}_1$. Again, replacing v by $\frac{v}{\mathcal{T}}$ in (3.40), we notice that

$$\rho\,\left(\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^2}\right)\right) \leq \frac{k}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.41}$$

for all $v \in \mathcal{R}_1$. Applying the Δ_2 condition it follows from (3.41) and the convexity of the modular ρ that,

$$\rho\,\left(\mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}}\right) - \mathcal{T}^{4n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^2}\right)\right) \leq \frac{k^4}{\mathcal{T}^{4n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.42}$$

for all $v \in \mathcal{R}_1$. From (3.40) and (3.42), we obtain that

$$\rho\,\left(\mathcal{N}_{n+1}(v) - \mathcal{T}^{2n}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^2}\right)\right) \leq \frac{k}{\mathcal{T}^{2n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}}\right) + \frac{k^4}{\mathcal{T}^{4n}}\,\mathcal{M}\left(\frac{v}{\mathcal{T}^2}\right) \tag{3.43}$$

for all $v \in \mathcal{R}_1$. Generalizing for a positive integer m, we acquire that

$$\rho\ \left(\mathcal{N}_{n+1}(v) - \mathcal{T}^{2nm}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^m}\right)\right) \leq \frac{1}{k^2}\sum_{r=1}^{m}\left(\frac{k^3}{\mathcal{T}^{2n}}\right)^r \mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) \tag{3.44}$$

for all $v \in \mathcal{R}_1$. Thus $\left\{\mathcal{T}^{2nm}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^m}\right)\right\}$ is a $\rho-$ Cauchy sequence in $\mathcal{R}_{2\rho}$ and $\mathcal{R}_{2\rho}$ is $\rho-$complete there exists a $\rho-$ limit function $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ given by

$$\lim_{m\to\infty} \rho\left(\mathcal{T}^{2nm}\mathcal{N}_{n+1}\left(\frac{v}{\mathcal{T}^m}\right) - \Gamma_{n+1}(v)\right) \to 0; \quad \forall\, v \in \mathcal{R}_1.$$

The rest of the proof is similar to that of Theorem 3.15. ■

The following corollary is the immediate consequence of Theorem 3.17 concerning the stabilities of (1.7).

Corollary 3.18. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be an even function fulfilling the inequality*

$$\rho\ \Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg) \leq \begin{cases} s; \\ s\|v\|^{\mu};\ \mu > \log_2 \frac{k^3}{\mathcal{T}^{2n}} \end{cases} \tag{3.45}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\ \left(\Gamma_{n+1}(v) - \mathcal{N}_{n+1}(v)\right) \leq \begin{cases} \dfrac{sk}{\mathcal{T}^{2n} - k^3}; \\ \dfrac{sk\|v\|^{\mu}}{\mathcal{T}^{2n+\mu} - k^3}; \end{cases} \tag{3.46}$$

for all $v \in \mathcal{R}_1$.

3.7. Stability Results: Odd-Even Case: Applying Δ_2 Condition

Theorem 3.19. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be a function satisfies the inequality*

$$\rho\ \Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\Bigg) \leq \mathcal{M}(v); \quad \forall\, v \in \mathcal{R}_1, \tag{3.47}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0, \infty)$ satisfying the conditions (3.21) and (3.35) for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\ \left(\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)\right\| \leq \frac{1}{2}\left\{\frac{1}{k}\sum_{r=1}^{\infty}\left(\frac{k^2}{\mathcal{T}^n}\right)^r\left[\mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) + \mathcal{M}\left(\frac{-v}{\mathcal{T}^r}\right)\right] + \frac{1}{k^2}\sum_{r=1}^{\infty}\left(\frac{k^3}{\mathcal{T}^{2n}}\right)^r\left[\mathcal{M}\left(\frac{v}{\mathcal{T}^r}\right) + \mathcal{M}\left(\frac{-v}{\mathcal{T}^r}\right)\right]\right\} \tag{3.48}$$

for all $v \in \mathcal{R}_1$. The mappings $\Gamma_n(v)$ and $\Gamma_{n+1}(v)$ are respectively defined in (3.23) and (3.37) for all $v \in \mathcal{R}_1$.

Proof. The proof is similar lines to that of Theorem 2.5. ■

Corollary 3.20. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ be a function fulfilling the inequality*

$$\rho\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)\right) \le \begin{cases} s; \\ s||v||^\mu;\ \mu > \log_2 \frac{k^2}{\mathcal{T}^n}; \log_2 \frac{k^3}{\mathcal{T}^{2n}} \end{cases} \tag{3.49}$$

for all $v \in \mathcal{R}_1$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_{2\rho}$ satisfying the functional equation (1.7) and

$$\rho\ (\mathcal{N}_{n;n+1}(v) - \Gamma_n(v) - \Gamma_{n+1}(v)) \le \begin{cases} sk\left(\dfrac{1}{\mathcal{T}^n - k^2} + \dfrac{1}{\mathcal{T}^{2n} - k^2}\right); \\ sk||v||^\mu\left(\dfrac{1}{\mathcal{T}^{n+\mu} - k^2} + \dfrac{1}{\mathcal{T}^{2n+\mu} - k^3}\right); \end{cases} \tag{3.50}$$

for all $v \in \mathcal{R}_1$.

4. Stability In Fuzzy Banach Space of (1.7)

In this section, we investigate the generalized Ulam - Hyers stability of the functional equation (1.7) in Fuzzy Banach space. To prove stability results, let us take $\mathcal{R}_3, (\mathcal{R}_1, F)$ and $(\mathcal{R}_2, F')$ are linear space, fuzzy normed space and fuzzy Banach space.

4.1. Definitions on Fuzzy Banach Spaces

In this section, we present the definitions and notations on fuzzy normed spaces given in [7, 30–33].

Definition 4.1. *Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*
$(FNS1)$ $N(x,c) = 0$ *for* $c \le 0$;
$(FNS2)$ $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;
$(FNS3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;
$(FNS4)$ $N(x+y, s+t) \ge min\{N(x,s), N(y,t)\}$;
$(FNS5)$ $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x,t) = 1$;
$(FNS6)$ *for* $x \neq 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.
The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 4.2. *Let $(X, ||\cdot||)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, \quad x \in X, \\ 0, & t \le 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 4.3. *Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 4.4. *A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we obtain that $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

Definition 4.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

4.2. Stability Results: Odd Case

Theorem 4.6. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd mapping fullfilling the inequality*

$$F\left(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big), c\right) \geq F'\left(\mathcal{M}(v), c\right) \tag{4.1}$$

where $\mathcal{M} : \mathcal{R}_1 \to \mathcal{R}_3$ with the conditions

$$\lim_{m\to\infty} F'\left(\mathcal{M}\left(\mathcal{T}^{nmt}v\right), \mathcal{T}^{nmt}c\right) = 1; \tag{4.2}$$

$$F'\left(\mathcal{M}\left(\mathcal{T}^{t}v\right), c\right) \geq F'\left(O^t\mathcal{M}(v), c\right); \tag{4.3}$$

with $0 < \left(\frac{c}{\mathcal{T}^n}\right)^t < 1$. Then there exists a unique n^{th} order mapping $\Gamma_n : \mathcal{R}_1 \to \mathcal{R}_2$ which satisfies (1.7) and

$$F\left(\mathcal{N}_n(v) - \Gamma_n(v), c\right) \geq F'\left(\mathcal{M}(v),\ c\,|\mathcal{T}^n - O|\right). \tag{4.4}$$

The mapping $\Gamma_n(v)$ is defined by

$$\lim_{m\to\infty} F\left(\Gamma_n(v) - \frac{\mathcal{N}_n(\mathcal{T}^{nmt}v)}{\mathcal{T}^{nmt}}, c\right) = 1 \tag{4.5}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$ with $t \pm 1$.

Proof. Applying oddness of $\mathcal{N}_{n;n+1}$ in (4.1) and by (1.9), we observe that

$$F\left(\mathcal{N}_n(\mathcal{T}v) - \mathcal{T}^n\,\mathcal{N}_n(v), c\right) \geq F'\left(\mathcal{M}(v), c\right); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.6}$$

Applying $(FNS3)$ in (4.6), we obtain that

$$F\left(\frac{1}{\mathcal{T}^n}\,\mathcal{N}_n(\mathcal{T}v) - \mathcal{N}_n(v), \frac{c}{\mathcal{T}^n}\right) \geq F'\left(\mathcal{M}(v), c\right); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.7}$$

Replacing v by $\mathcal{T}^m v$ in (4.7) and applying (4.3), $(FNS3)$, we find that

$$\begin{aligned} F\left(\frac{1}{\mathcal{T}^n}\,\mathcal{N}_n(\mathcal{T}^{m+1}v) - \mathcal{N}_n(\mathcal{T}^m v), \frac{c}{\mathcal{T}^n}\right) &\geq F'\left(\mathcal{M}\left(\mathcal{T}^m v\right), c\right) \\ &\geq F'\left(O^m\mathcal{M}(v), c\right) \\ &= F'\left(\mathcal{M}(v), \frac{c}{O^m}\right); \forall\ v \in \mathcal{R}_1;\ \ c > 0. \end{aligned} \tag{4.8}$$

With the help of $(FNS3)$ it follows from (4.8), that

$$F\left(\frac{1}{\mathcal{T}^{n+mn}}\,\mathcal{N}_n(\mathcal{T}^{m+1}v) - \frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^m v), \frac{c}{\mathcal{T}^n\,.\,\mathcal{T}^{mn}}\right) \geq F'\left(\mathcal{M}(v), \frac{c}{O^m}\right); \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.9}$$

Changing c by $O^m c$ in (4.9), we achieve that

$$F\left(\frac{1}{\mathcal{T}^{n+mn}}\,\mathcal{N}_n(\mathcal{T}^{m+1}v)-\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v),\frac{c}{\mathcal{T}^n}\cdot\left[\frac{O}{\mathcal{T}^n}\right]^m\right)\geq F'\left(\mathcal{M}\left(v\right),c\right);\forall\;\; v\in\mathcal{R}_1;\;\; c>0. \quad (4.10)$$

It is easy to see that

$$\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v)-\mathcal{N}_n(v)=\sum_{r=0}^{m-1}\left[\frac{1}{\mathcal{T}^{n+rn}}\,\mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\,\mathcal{N}_n(\mathcal{T}^{r}v)\right];\quad \forall\;\; v\in\mathcal{R}_1. \quad (4.11)$$

for all $v\in\mathcal{R}_1$. From equations (4.10) and (4.11), we obtain that

$$\begin{aligned}
&F\left(\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v)-\mathcal{N}_n(v),\frac{c}{\mathcal{T}^n}\cdot\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^n}\right]^r\right)\\
&\quad\geq \min F\left(\sum_{r=0}^{m-1}\left[\frac{1}{\mathcal{T}^{n+rn}}\,\mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\,\mathcal{N}_n(\mathcal{T}^{r}v)\right],\frac{c}{\mathcal{T}^n}\cdot\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^n}\right]^r\right)\\
&\quad\geq \min\bigcup_{r=0}^{m-1}\left\{F\left(\left[\frac{1}{\mathcal{T}^{n+rn}}\,\mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\,\mathcal{N}_n(\mathcal{T}^{r}v)\right],\frac{c}{\mathcal{T}^n}\cdot\left[\frac{O}{\mathcal{T}^n}\right]^r\right)\right\}\\
&\quad\geq \min\bigcup_{r=0}^{m-1}\left\{F'\Big(\mathcal{M}\left(v\right),c\Big)\right\}=F'\left(\mathcal{M}\left(v\right),c\right);\forall\;\; v\in\mathcal{R}_1;\;\; c>0. \qquad (4.12)
\end{aligned}$$

Replacing v by $\mathcal{T}^{\kappa}v$ in (4.12) and applying (4.3), $(FNS3)$, we find that

$$\begin{aligned}
&F\left(\frac{1}{\mathcal{T}^{mn+\kappa n}}\,\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)-\frac{1}{\mathcal{T}^{\kappa n}}\mathcal{N}_n(\mathcal{T}^{\kappa}v),\frac{c}{\mathcal{T}^n\cdot\mathcal{T}^{\kappa n}}\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^n}\right]^r\right)\\
&\geq F'\left(\mathcal{M}\left(\mathcal{T}^{\kappa}v\right),c\right)\geq F'\left(O^{\kappa}\mathcal{M}\left(v\right),c\right)=F'\left(\mathcal{M}\left(v\right),\frac{c}{O^{\kappa}}\right);\forall\;\; v\in\mathcal{R}_1;\;\; c>0. \qquad (4.13)
\end{aligned}$$

Changing c by $O^{\kappa}c$ in (4.13), we achieve that

$$F\left(\frac{1}{\mathcal{T}^{mn+\kappa n}}\,\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)-\frac{1}{\mathcal{T}^{\kappa n}}\mathcal{N}_n(\mathcal{T}^{\kappa}v),\frac{c}{\mathcal{T}^n}\cdot\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^{n+\kappa}}\right]^r\right)\geq F'\left(\mathcal{M}\left(v\right),c\right);\forall\;\; v\in\mathcal{R}_1;\;\; c>0. \quad (4.14)$$

for all $\kappa>m\geq 0$. It follows from (4.14), we see that

$$F\left(\frac{1}{\mathcal{T}^{mn+\kappa n}}\,\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)-\frac{1}{\mathcal{T}^{\kappa n}}\mathcal{N}_n(\mathcal{T}^{\kappa}v),c\right)\geq F'\left(\mathcal{M}\left(v\right),\frac{c}{\frac{1}{\mathcal{T}^n}\sum_{r=0}^{m-1}\left[\frac{O}{\mathcal{T}^{n+\kappa}}\right]^r}\right);\forall\;\; v\in\mathcal{R}_1;\;\; c>0. \quad (4.15)$$

Since $0<t<\mathcal{T}^n$ and $\sum_{r=0}^{m}\left(\frac{c}{\mathcal{T}^n}\right)^r<\infty$, the Cauchy criterion for convergence and $(FNS5)$ implies that $\left\{\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v)\right\}$ is a Cauchy sequence in $(\mathcal{R}_2,N')$ and it is complete, this sequence converges to some point $\Gamma_n\in\mathcal{R}_2$. So one can define the mapping $\Gamma_n:\mathcal{R}_1\to\;\mathcal{R}_2$ by

$$\lim_{m\to\infty}F\left(\Gamma_n(v)-\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^{m}v),c\right)=1 \quad (4.16)$$

for all $v \in \mathcal{R}_1$ and all $s > 0$. Letting $\kappa = 0$ and $m \to \infty$ in (4.15), we achieve that

$$F\ \left(\Gamma_n(v) - \mathcal{N}_n(v), c\right) \geq F'\left(\mathcal{M}\left(v\right), c \cdot \left(\mathcal{T}^n - O\right)\right); \forall\ \ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.17}$$

To prove Γ_n satisfies the (1.7), replacing v by $\mathcal{T}^m v$ in (4.3), we arrive that

$$\begin{aligned} &F\ \left(\frac{1}{\mathcal{T}^{nm}} \cdot \mathcal{N}_{n;n+1}(\mathcal{T}\,\mathcal{T}^m v) - \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\right.\\ &\left.-\frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big), c\ \right) \geq F'\ \left(\mathcal{M}\left(\mathcal{T}^m v\right), \mathcal{T}^{nm} c\ \right); \forall\ \ v \in \mathcal{R}_1;\ \ c > 0. \end{aligned} \tag{4.18}$$

Now,

$$\begin{aligned} &F\ \left(\Gamma_n(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big), c\ \right)\\ &\geq min\left\{F\ \left(\Gamma_n(\mathcal{T}v) - \frac{1}{\mathcal{T}^{nm}} \cdot \mathcal{N}_{n;n+1}(\mathcal{T}\,\mathcal{T}^m v), \frac{c}{4}\right),\right.\\ &\quad F\ \left(-\frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) + \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big), \frac{c}{4}\right),\\ &\quad F\ \left(-\frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big) + \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big), \frac{c}{4}\right),\\ &\quad F\ \left(\frac{1}{\mathcal{T}^{nm}} \cdot \mathcal{N}_{n;n+1}(\mathcal{T}\,\mathcal{T}^m v) - \frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) - \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big)\right.\\ &\qquad\quad \left.\left.-\frac{1}{\mathcal{T}^{nm}} \cdot \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(\mathcal{T}^m v) + \mathcal{N}_{n;n+1}(-\mathcal{T}^m v)\Big), c\ \right)\right\} \end{aligned} \tag{4.19}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Applying (4.16), (4.18), $(FNS5)$ in (4.19), we observe that

$$\begin{aligned} &F\ \left(\Gamma_n(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big), c\ \right)\\ &\qquad \geq \min\left\{1, 1, 1, F'\ \left(\mathcal{M}\left(\mathcal{T}^m v\right), \mathcal{T}^{nm} c\ \right)\right\}; \forall\ \ v \in \mathcal{R}_1;\ \ c > 0. \end{aligned} \tag{4.20}$$

Approaching m tends to infinity in (4.20) and applying (4.3), we achieve that

$$F\ \left(\Gamma_n(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big), c\ \right) = 1 \tag{4.21}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Applying $(FNS2)$ in (4.21) we identify that

$$\Gamma_n(\mathcal{T}v) = \frac{\mathcal{T}^n}{2}\Big(\Gamma_n(v) - \Gamma_n(-v)\Big) + \frac{\mathcal{T}^{n+1}}{2}\Big(\Gamma_n(v) + \Gamma_n(-v)\Big); \forall\ \ v \in \mathcal{R}_1;\ \ c > 0.$$

for all $v \in \mathcal{R}_1$. Hence Γ_n satisfies the functional equation (1.7). To prove $\Gamma_n(v)$ is unique, let $\Gamma'_n(v)$ be another additive functional equation satisfying (1.7) and (4.5). So,

$$\begin{aligned} N(\Gamma_n(v) - \Gamma'_n(v), s) &= F\ \left(\frac{\Gamma_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \frac{\Gamma'_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}, c\right)\\ &\geq \min\left\{F\ \left(\frac{\Gamma_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \frac{\Gamma'_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}, \frac{c}{2}\right), F\ \left(\frac{\Gamma'_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}} - \frac{\Gamma_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}, \frac{c}{2}\right)\right\}\\ &\geq F'\ \left(\mathcal{M}\left(\mathcal{T}^m v\right), \frac{c\left(\mathcal{T}^n - O\right)\mathcal{T}^{nm}}{2}\right)\\ &= F'\ \left(\mathcal{M}\left(v\right), \frac{c\left(\mathcal{T}^n - O\right)\mathcal{T}^{nm}}{2\,O^m}\right) \end{aligned}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Since $\lim_{m\to\infty} \frac{c\,(\mathcal{T}^n - O)\mathcal{T}^{nm}}{2\,O^m} = \infty$, it follows that $\lim_{m\to\infty} F'\left(\mathcal{M}(v), \frac{c\,(\mathcal{T}^n-0)\,\mathcal{T}^{nm}}{2\,O^m}\right) = 1$ for all $v \in \mathcal{R}_1$ and all $c > 0$. Thus

$$N(\Gamma_n(v) - \Gamma_n'(v), s) = 1$$

for all $v \in \mathcal{R}_1$ and all $c > 0$, hence $\Gamma_n(v) = \Gamma_n'(v)$. Therefore $\Gamma_n(v)$ is unique. Hence for $t = 1$ the theorem holds.

Replacing v by $\dfrac{v}{\mathcal{T}}$ in (4.6), we notice that

$$F\left(\mathcal{N}_n(v) - \mathcal{T}^n\,\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right), c\right) \geq F'\left(\mathcal{M}\left(\frac{v}{\mathcal{T}}\right), c\right); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{4.22}$$

The rest of the proof is similar lines to that of case $t = 1$ Hence the theorem holds for the case $t = -1$. This completes the proof of the theorem. ■

The following corollary is the immediate consequence of Theorem 4.6 concerning the stabilities of (1.7).

Corollary 4.7. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\begin{aligned} F\Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big),\ c\Bigg) \geq \begin{cases} F'\ (s, c\,); \\ F'\ (s\|v\|^{\mu}, c\,); \end{cases} \end{aligned} \tag{4.23}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$F\ (\Gamma_n(v) - \mathcal{N}_n(v), c) \geq \begin{cases} F'\,(s, c \cdot |\mathcal{T}^n - 1|)\,, \\ F'\,(s\,\|v\|^{\mu}, c \cdot |\mathcal{T}^n - \mathcal{T}^{\mu}|)\,,\ \mu \neq n; \end{cases} \tag{4.24}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

4.3. Stability Results: Even Case

The proof of the following theorem and corollary is similar clues that of Theorem 4.6 and Corollary 4.7 with the help of (1.10). Hence the details of the proof are omitted.

Theorem 4.8. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even mapping fullfilling the inequality*

$$\begin{aligned} F\Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ -\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big), c\Bigg) \geq F'\ (\mathcal{M}\,(v)\,, c\,) \end{aligned} \tag{4.25}$$

where $\mathcal{M} : \mathcal{R}_1 \to \mathcal{R}_3$ with the conditions

$$\lim_{m\to\infty} F'\ \left(\mathcal{M}\left(\mathcal{T}^{2nmt}v\right), \mathcal{T}^{2nmt}c\,\right) = 1; \tag{4.26}$$

$$F'\ \left(\mathcal{M}\left(\mathcal{T}^t v\right), c\,\right) \geq F'\ \left(O^t \mathcal{M}\,(v)\,, c\,\right); \tag{4.27}$$

with $0 < \left(\dfrac{c}{\mathcal{T}^{2n}}\right)^t < 1$. Then there exists a unique $(n+1)^{th}$ order mapping $\Gamma_{n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ which satisfies (1.7) and

$$F\ (\mathcal{N}_n(v) - \Gamma_{n+1}(v), c) \geq F'\ \left(\mathcal{M}\,(v)\,,\ c\,|\mathcal{T}^{2n} - O|\right). \tag{4.28}$$

The mapping $\Gamma_{n+1}(v)$ is defined by

$$\lim_{m\to\infty} F\left(\Gamma_{n+1}(v) - \frac{\mathcal{N}_n(\mathcal{T}^{2nmt}v)}{\mathcal{T}^{2nmt}}, c\right) = 1 \tag{4.29}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$ with $t \pm 1$.

Corollary 4.9. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function fulfilling the inequality*

$$\begin{aligned} F\ \Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) &- \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &-\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big),\ c\Bigg) \geq \begin{cases} F'\ (s, c\ ); \\ F'\ (s||v||^{\mu}, c\ ); \end{cases} \end{aligned} \tag{4.30}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$F\ (\Gamma_{n+1}(v) - \mathcal{N}_n(v), c) \geq \begin{cases} F'\left(s, c\cdot|\mathcal{T}^{2n} - 1|\right), \\ F'\left(\ s\ ||v||^{\mu}, c\cdot|\mathcal{T}^{2n} - \mathcal{T}^{\mu}|\right),\ \mu \neq 2n; \end{cases} \tag{4.31}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

4.4. Stability Results: Odd-Even Case

Theorem 4.10. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function satisfies the inequality*

$$\begin{aligned} F\ \Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) &- \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &-\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big), c\Bigg) \geq F'\ (\mathcal{M}\ (v)\ , c\ ) \end{aligned} \tag{4.32}$$

where $\mathcal{M} : \mathcal{R}_1 \to [0,\infty)$ satisfying the conditions (4.2), (4.3), (4.26) and (4.27) with $0 < \left(\frac{c}{\mathcal{T}^{2n}}\right)^t < \left(\frac{c}{\mathcal{T}^{n}}\right)^t < 1$. Then there exists one and only n^{th} order mapping $\Gamma_n : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ which satisfies (1.7) and

$$F\ (\mathcal{N}_n(v) - \Gamma_n(v) - \Gamma_{n+1}(v), 2c) \geq \min\left\{F'\ \left(\Big(\mathcal{M}\ (v) + \mathcal{M}\ (-v)\Big),\ 2c\left(|\mathcal{T}^n - O| + |\mathcal{T}^{2n} - O|\right)\right)\right\} \tag{4.33}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$ with $t \pm 1$. The mappings $\Gamma_n(v)$ and $\Gamma_{n+1}(v)$ are respectively defined in (4.5) and (4.29).

Proof. The proof is similar lines to that of Theorem 2.5. ■

Corollary 4.11. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function fulfilling the inequality*

$$\begin{aligned} F\ \Bigg(\mathcal{N}_{n;n+1}(\mathcal{T}v) &- \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) \\ &-\frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big),\ c\Bigg) \geq \begin{cases} F'\ (s, c\ ); \\ F'\ (s||v||^{\mu}, c\ ); \end{cases} \end{aligned} \tag{4.34}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only n^{th} order mapping $\Gamma_n : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ which satisfies (1.7) and

$$F\ \left(\Gamma_{n+1}(v) - \mathcal{N}_n(v), 2c\right) \geq \begin{cases} F'\left(s, c\Big(|\mathcal{T}^n - 1| + |\mathcal{T}^{2n} - 1|\Big)\right), \\ F'\left(s, c\Big(|\mathcal{T}^n - \mathcal{T}^{\mu}| + |\mathcal{T}^{2n} - \mathcal{T}^{\mu}|\Big)\right), \ \mu \neq n, 2n; \end{cases} \tag{4.35}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

5. Stability In Random Banach Space of (1.7)

In this section, we investigate the generalized Ulam - Hyers stability of the functional equations (1.7) in Fuzzy Banach space. To prove stability results, let us take $\mathcal{R}_1$ and $(\mathcal{R}_2, \eta, c)$ are linear space and Random Banach space.

5.1. Definitions on Random Banach Spaces

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces as in [46, 47].

From now on, D^+ is the space of distribution functions, that is, the space of all mappings

$$F : R \cup \{-\infty, \infty\} \to [0, 1],$$

such that F is left continuous and nondecreasing on $R, F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of D^+ consisting of all functions $F \in D^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is,

$$l^- f(x) = \lim_{t \to x^-} f(t).$$

The space D^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for D^+ in this order is the distribution function ϵ_0 given by

$$\epsilon_0(t) = \begin{cases} 0, \ if\ t \leq 0, \\ 1, \ if\ t > 0. \end{cases} \tag{5.1}$$

Definition 5.1. *A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm (briefly, a continuous $t-$norm) if T satisfies the following conditions:*

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a, 1) = a$ for all $a \in [0, 1]$;

(d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous $t-$norms are $T_P(a, b) = ab, T_M(a, b) = min(a, b)$ and $T_L(a, b) = max(a + b - 1, 0)$ (the Lukasiewicz $t-$norm). Recall (see [15, 16]) that if T is a $t-$norm and x_n is a given sequence of numbers in $[0, 1]$, then $T_{i=1}^n x_{n+i}$ is defined recurrently by

$$T_{i=1}^1 x_i = x_1 \qquad and \qquad T_{i=1}^n x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right) \qquad for \qquad n \geq 2.$$

$T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$. It is known [16] that, for the Lukasiewicz $t-$norm, the following implication holds:

$$\lim_{n \to \infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty \tag{5.2}$$

Definition 5.2. *A random normed space (briefly, RN-space) is a triple (X, η, T), where X is a vector space, T is a continuous $t-$norm and η is a mapping from X into D^+ satisfying the following conditions:*

(RBS1) $\eta_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$*;*

(RBS2) $\eta_{\alpha\, x}(t) = \eta_x(t/|\alpha|)$ *for all* $x \in X$*, and* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 0$*;*

(RBS3) $\eta_{x+y}(t+s) \geq T\left(\eta_x(t), \eta_y(s)\right)$ *for all* $x, y \in X$ *and* $t, s \geq 0$*.*

Example 5.3. *Every normed spaces $(X, ||\cdot||)$ defines a random normed space (X, η, T_M), where*

$$\eta_x(t) = \frac{t}{t + ||x||}$$

and T_M is the minimum $t-$norm. This space is called the induced random normed space.

Definition 5.4. *Let (X, η, T) be a RN-space.*

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\eta_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\eta_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq m \geq N$.

(3) A RN-space (X, η, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X.

Theorem 5.5. *If (X, η, T) is a RN-space and $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, then $\lim\limits_{n\to\infty} \eta_{x_n}(t) = \eta_x(t)$ almost everywhere.*

To prove stability results, let us take

$$\mathcal{N}^{\mathcal{T}}_{n;n+1}(v) = \mathcal{N}_{n;n+1}(\mathcal{T}v) - \frac{\mathcal{T}^n}{2}\Big(\mathcal{N}_{n;n+1}(v) - \mathcal{N}_{n;n+1}(-v)\Big) - \frac{\mathcal{T}^{n+1}}{2}\Big(\mathcal{N}_{n;n+1}(v) + \mathcal{N}_{n;n+1}(-v)\Big)$$

5.2. Stability Results: Odd Case

Theorem 5.6. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\,c\,) \geq \eta'_v(\,c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.3}$$

for which there exist a function $\eta' : \mathcal{R}_1 \rightarrow D^+$ with the condition

$$\lim_{m\to\infty} T^{\infty}_{r=0}\eta'_{\mathcal{T}^{(m+r)t}v}\left(\mathcal{T}^{(m+r+1)t}c\right) = 1 = \lim_{m\to\infty} \eta'_{\mathcal{T}^{mt}v}\left(\mathcal{T}^{mt}c\,\right); \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.4}$$

Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\Gamma_n(v) - \mathcal{N}_n(v)}(\,c\,) \geq T^{\infty}_{r=0}\eta'_{\mathcal{T}^{rt}v}(\mathcal{T}^{(rn+n)t}\,c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.5}$$

with $t = \pm 1$. The mapping $\Gamma_n(v)$ is defined by

$$\eta_{\Gamma_n(v)}(\,c\,) = \lim_{m\to\infty} \eta_{\frac{\mathcal{N}_n(\mathcal{T}^{mt}v)}{\mathcal{T}^{nmt}}}(\,c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.6}$$

Proof. Applying oddness of $\mathcal{N}_{n;n+1}$ in (5.3) and by (1.9), we observe that

$$\eta_{\mathcal{N}_n(\mathcal{T}v)-\mathcal{T}^n\mathcal{N}_n(v)}(\, c\,) \geq \eta'_v(\, c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.7}$$

Applying $(RBS2)$ in (5.7), we obtain that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}v)}{\mathcal{T}^n}-\mathcal{N}_n(v)}\left(\frac{c}{\mathcal{T}^n}\right) \geq \eta'_v(\, c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.8}$$

Changing c by $\mathcal{T}^n\ c$ in (5.8), we notice that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}v)}{\mathcal{T}^n}-\mathcal{N}_n(v)}(c) \geq \eta'_v(\mathcal{T}^n\ c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.9}$$

Replacing v by $\mathcal{T}^m v$ in (5.9), we see that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}\ \mathcal{T}^m v)}{\mathcal{T}^n}-\mathcal{N}_n(\mathcal{T}^m v)}(c) \geq \eta'_{\mathcal{T}^m v}(\mathcal{T}^n\ c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.10}$$

Applying $(RBS2)$ in (5.10), we achieve that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}^{m+1}v)}{\mathcal{T}^{nm+n}}-\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}}\left(\frac{c}{\mathcal{T}^{nm}}\right) \geq \eta'_{\mathcal{T}^m v}(\mathcal{T}^n\ c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.11}$$

Changing c by $\mathcal{T}^m\ c$ in (5.11), we obtain that

$$\eta_{\frac{\mathcal{N}_n(\mathcal{T}^{m+1}v)}{\mathcal{T}^{nm+n}}-\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}}(\, c\,) \geq \eta'_{\mathcal{T}^m v}(\mathcal{T}^{nm+n}\ c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.12}$$

It is easy to see that

$$\frac{1}{\mathcal{T}^{mn}}\,\mathcal{N}_n(\mathcal{T}^m v) - \mathcal{N}_n(v) = \sum_{r=0}^{m-1}\left[\frac{1}{\mathcal{T}^{n+rn}}\,\mathcal{N}_n(\mathcal{T}^{r+1}v) - \frac{1}{\mathcal{T}^{rn}}\,\mathcal{N}_n(\mathcal{T}^r v)\right]; \quad \forall\ v \in \mathcal{R}_1. \tag{5.13}$$

From equations (5.12) and (5.13) and $(RBS3)$, we observe that

$$\begin{aligned}
\eta_{\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}-\mathcal{N}_n(v)}(\, c\,) &= \eta_{\sum\limits_{r=0}^{m-1}\left[\frac{1}{\mathcal{T}^{n+rn}}\ \mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\ \mathcal{N}_n(\mathcal{T}^r v)\right]}(\, c\,) \\
&\geq T_{r=0}^{m-1}\eta_{\left[\frac{1}{\mathcal{T}^{n+rn}}\ \mathcal{N}_n(\mathcal{T}^{r+1}v)-\frac{1}{\mathcal{T}^{rn}}\ \mathcal{N}_n(\mathcal{T}^r v)\right]}(\mathcal{T}^{rn+n}\ c\,) \\
&\geq T_{r=0}^{m-1}\eta'_{\mathcal{T}^r v}(\mathcal{T}^{rn+n}\ c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0.
\end{aligned} \tag{5.14}$$

In order to prove the convergence of the sequence $\left\{\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^m}\right\}$, replacing v by $\mathcal{T}^\kappa v$ in (5.14) and applying $(RBS2)$, (5.4), we arrive

$$\begin{aligned}
\eta_{\frac{\mathcal{N}_n(\mathcal{T}^{m+\kappa}v)}{\mathcal{T}^{nm+n\kappa}}-\frac{\mathcal{N}_n(\mathcal{T}^\kappa v)}{\mathcal{T}^{n\kappa}}}(\, c\,) &\geq T_{r=0}^{m-1}\eta'_{\mathcal{T}^{r+\kappa}v}(\mathcal{T}^{nm+nr+n}\ c\,) \\
&\to 1 \quad as \quad m \to \infty; \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0.
\end{aligned}$$

Thus $\left\{\frac{1}{\mathcal{T}^{nm}}\,\mathcal{N}_n(\mathcal{T}^m v)\right\}$ is a Cauchy sequence $\mathcal{R}_2$ and it is complete, this sequence converges to some point $\Gamma_n \in \mathcal{R}_2$. So one can define the mapping $\Gamma_n : \mathcal{R}_1 \to\ \mathcal{R}_2$ by

$$\eta_{\Gamma_n(v)}(\, c\,) = \lim_{m\to\infty}\eta_{\frac{\mathcal{N}_n(\mathcal{T}^m v)}{\mathcal{T}^{nm}}}(\, c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0.$$

Letting $m \to \infty$ in (5.14), we identify that (5.5) holds for $t = 1$, for all $v \in \mathcal{R}_1$ and all $c > 0$. To prove that Γ_n satisfies (1.7), replacing v by $\mathcal{T}^m v$ in (5.3), we find that

$$\eta_{\frac{1}{\mathcal{T}^{nm}}\ \mathcal{N}^{\mathcal{T}}_{n;n+1}(\mathcal{T}^m v)}(\, c\,) \geq \eta'_{\mathcal{T}^m v}\,(\mathcal{T}^{nm}\ c)\,; \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0.$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Letting $n \to \infty$ in the overhead inequality and applying the definition of $\Gamma_n(v)$, we identify that Γ_n satisfies (1.7) for all $v \in \mathcal{R}_1$. To prove $\Gamma_n(v)$ is unique, let $\Gamma'_n(v)$ be another mapping satisfying (1.7) and (5.6). So,

$$\begin{aligned}\eta_{\Gamma_n(v)-\Gamma'_n(v)}(2\,c\,) &= \eta_{\Gamma_n(\mathcal{T}^m v)-\mathcal{N}_n(\mathcal{T}^m v)+\mathcal{N}_n(\mathcal{T}^m v)-\Gamma'_n(\mathcal{T}^m v)}(\mathcal{T}^{nm}\cdot 2\,c\,)\\ &\geq T\left(\eta_{\Gamma_n(\mathcal{T}^m v)-\mathcal{N}_n(\mathcal{T}^m v)}(\mathcal{T}^{nm}\,c\,), \eta_{\mathcal{N}_n(\mathcal{T}^m v)-\Gamma'_n(\mathcal{T}^m v)}(\mathcal{T}^{nm}\,c\,)\right)\\ &\geq T\left(T_{r=0}^{\infty}\eta'_{\mathcal{T}^r v}(\mathcal{T}^{rn+nm+n})\,c\,), T_{r=0}^{\infty}\eta'_{\mathcal{T}^r v}(\mathcal{T}^{rn+nm+n}\,c\,)\right)\\ &\to\ 1 \quad as \quad m\to\infty;\quad \forall\ v\in\mathcal{R}_1;\ \ c>0.\end{aligned}$$

Hence, Γ_n is unique. Hence for $t = 1$ the theorem holds.

Replacing v by $\dfrac{v}{\mathcal{T}}$ in (5.7), we notice that

$$\eta_{\mathcal{N}_n(v)-\mathcal{T}^n\mathcal{N}_n\left(\frac{v}{\mathcal{T}}\right)}(\,c\,) \geq \eta'_v(\,c\,);\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.15}$$

The rest of the proof is similar lines to that of case $t = 1$ Hence the theorem holds for the case $t = -1$. This completes the proof of the theorem. ■

The following corollary is the immediate consequence of Theorem 5.6 concerning the stability of (1.7).

Corollary 5.7. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an odd function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\,c\,) \geq \begin{cases}\eta'_s(\,c\,);\\ \eta'_{s||v||^\mu}(\,c\,); & \mu\neq n,\end{cases} \tag{5.16}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\Gamma_n(v)-\mathcal{N}_n(v)}(\,c\,) \geq \begin{cases}\eta'_{|s|}\,(|\mathcal{T}^n-1|\ c\,)\\ \eta'_{s||v||^\mu}(|\mathcal{T}^n-\mathcal{T}^\mu|\ c\,)\end{cases} \tag{5.17}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

5.3. Stability Results: Even Case

The proof of the following theorem and corollary is similar clues that of Theorem 5.6 and Corollary 5.7 with the help of (1.10). Hence the details of the proof are omitted.

Theorem 5.8. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\,c\,) \geq \eta'_v(\,c\,);\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.18}$$

for which there exist a function $\eta' : \mathcal{R}_1 \to D^+$ with the condition

$$\lim_{m\to\infty} T_{r=0}^{\infty}\eta'_{\mathcal{T}^{(m+r)t}v}\left(\mathcal{T}^{2(m+r+1)t}c\right) = 1 = \lim_{m\to\infty}\eta'_{\mathcal{T}^{mt}v}\left(\mathcal{T}^{2mt}c\right);\forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.19}$$

Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\Gamma_{n+1}(v)-\mathcal{N}_n(v)}(\,c\,) \geq T_{r=0}^{\infty}\eta'_{\mathcal{T}^{rt}v}(\mathcal{T}^{2(rn+n)t}\,c\,);\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.20}$$

with $t = \pm 1$. The mapping $\Gamma_{n+1}(v)$ is defined by

$$\eta_{\Gamma_{n+1}(v)}(\,c\,) = \lim_{m\to\infty}\eta_{\frac{\mathcal{N}_n(\mathcal{T}^{mt}v)}{\mathcal{T}^{2nmt}}}(\,c\,);\quad \forall\ v\in\mathcal{R}_1;\ \ c>0. \tag{5.21}$$

Corollary 5.9. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be an even function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\,c\,) \geq \begin{cases} \eta'_s(\,c\,); & \\ \eta'_{s||v||^{\mu}}(\,c\,); & \mu \neq 2n, \end{cases} \tag{5.22}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\Gamma_{n+1}(v)-\mathcal{N}_n(v)}(\,c\,) \geq \begin{cases} \eta'_{|s|}(|\mathcal{T}^{2n}-1|\,c\,) \\ \eta'_{s||v||^{\mu}}(|\mathcal{T}^{2n}-\mathcal{T}^{\mu}|c\,) \end{cases} \tag{5.23}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

5.4. Stability Results: Odd - Even Case

Theorem 5.10. *Assume $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\,c\,) \geq \eta'_v(\,c\,); \quad \forall\ v \in \mathcal{R}_1;\ \ c > 0. \tag{5.24}$$

for which there exist a function $\eta' : \mathcal{R}_1 \to D^+$ with the conditions (5.4) and (5.19) . Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ and one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\begin{aligned} &\eta_{\mathcal{N}_n(v)-\Gamma_n(v)-\Gamma_{n+1}(v)}(\,c\,) \\ &\quad \geq T^3\Big(T^{\infty}_{r=0}\eta'_{\mathcal{T}^{rt}v}(\mathcal{T}^{(rn+n)t}\,c\,), T^{\infty}_{r=0}\eta'_{\mathcal{T}^{rt}-v}(\mathcal{T}^{(rn+n)t}\,c\,), \\ &\qquad T^{\infty}_{r=0}\eta'_{\mathcal{T}^{2rt}v}(\mathcal{T}^{(rn+n)t}\,c\,), T^{\infty}_{r=0}\eta'_{\mathcal{T}^{2rt}-v}(\mathcal{T}^{(rn+n)t}\,c\,)\Big); \forall\ v \in \mathcal{R}_1;\ \ c > 0. \end{aligned} \tag{5.25}$$

with $t = \pm 1$. The mappings $\Gamma_n(v)$ and $\Gamma_{n+1}(v)$ are respectively defined in (5.6) and (5.21) .

Corollary 5.11. *Assume s and μ be positive numbers. Let $\mathcal{N}_{n;n+1} : \mathcal{R}_1 \to \mathcal{R}_2$ be a function fulfilling the inequality*

$$\eta_{\mathcal{N}^{\mathcal{T}}_{n;n+1}(v)}(\,c\,) \geq \begin{cases} \eta'_s(\,c\,); & \\ \eta'_{s||v||^{\mu}}(\,c\,); & \mu \neq n, 2n, \end{cases} \tag{5.26}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$. Then there exists one and only n^{th} order mapping $\Gamma_n(v) : \mathcal{R}_1 \to \mathcal{R}_2$ one and only $(n+1)^{th}$ order mapping $\Gamma_{n+1}(v) : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying the functional equation (1.7) and

$$\eta_{\mathcal{N}_n(v)-\Gamma_n(v)-\Gamma_{n+1}(v)}(\,c\,) \geq \begin{cases} \eta'_{|s|}\Big((|\mathcal{T}^n-1|+|\mathcal{T}^{2n}-1|)\cdot c\Big) \\ \eta'_{s||v||^{\mu}}\Big((|\mathcal{T}^n-\mathcal{T}^{\mu}|+|\mathcal{T}^{2n}-\mathcal{T}^{\mu}|)\cdot c\Big) \end{cases} \tag{5.27}$$

for all $v \in \mathcal{R}_1$ and all $c > 0$.

References

[1] J. ACZEL AND J. DHOMBRES, Functional Equations in Several Variables, Cambridge Univ, Press, 1989.

[2] I. AMEMIYA, On the representation of complemented modular lattices, *J. Math. Soc. Jpn.,* **9**, 263-279 (1957).

[3] T. AOKI, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, **2** (1950), 64–66.

[4] M. ARUNKUMAR, C. DEVI SHYAMALA MARY, G. SHOBANA, Simple AQ And Simple CQ Functional Equations, *Journal Of Concrete And Applicable Mathematics (JCAAM)*, **13**, Issue 1/2 , Jan - Apr 2015, 120 - 151.

[5] M.ARUNKUMAR, E.SATHYA, C. DEVI SHYAMALA MARY, S. HEMA LATHA, AQ and CQ Functional Equations, *Malaya Journal of Matematik*, **6**, Issue 1, 2018, 182-205.

[6] M. ARUNKUMAR, E.SATHYA, C.POOJA, Single Variable Generalized Additive - Quadratic And Generalized Cubic- Quartic Functional Equations In Various Banach Spaces, *International Journal of Mathematics And its Applications*, **8** (3) (2020), 1-42.

[7] T. BAG, S.K. SAMANTA, Finite dimensional fuzzy normed linear spaces, *J. Fuzzy Math.*, **11** (3) (2003) 687-705.

[8] T. BAG AND S.K. SAMANTA, Fuzzy bounded linear operators, *Fuzzy Sets and Systems*, **151** (2005) 513-547.

[9] E. CASTILLO, A. IGLESIAS AND R. RUIZ-COHO, Functional Equations in Applied Sciences, Elsevier, B.V.Amslerdam, 2005.

[10] S.C. CHENG AND J.N. MORDESON, Fuzzy linear operator and fuzzy normed linear spaces, *Bull. Calcutta Math. Soc.*, **86** (1994) 429-436.

[11] S. CZERWIK, Functional Equations and Inequalities in Several Variables, World Scientific, River Edge, NJ, 2002.

[12] H. FAST, Sur la convergence statistique, *Colloq Math.*, **2** (1951), 241-244.

[13] JA. FRIDY, On statistical convergence, *Analysis.*, 5 (1985), 301-313.

[14] P. GĂVRUTĂ, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), no. 3, 431–436.

[15] O. HADZIC AND E. PAP, Fixed Point Theory in Probabilistic Metric Spaces, vol. 536 of Mathematics and Its Applications, Kluwer Academic, Dordrecht, The Netherlands, 2001.

[16] O. HADZIC, E. PAP AND M. BUDINCEVIC, Countable extension of triangular norms and their applications to the fixed point theory in probabilistic metric spaces, *Kybernetika*, **38** 3, pp. (2002), 363-82.

[17] D. H. HYERS, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.*, U. S. A. **27** (1941), 222–224.

[18] J.L.W.V. JENSEN, Sur les fonctions convexes et les inegalities entre les valeurs myennes, Acta Math., **30**, (1965), 179-193.

[19] S.M. JUNG, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.

[20] PL. KANNAPPAN, Functional Equations and Inequalities with Applications, Springer Monographs in Mathematics, 2009.

[21] S. KARAKUS, Statistical convergence on probabilistic normed spaces, *Math Commun.*, **12** (2007) 11-23.

[22] H. M. Kim, H. Y. Shin, Refined stability of additive and quadratic functional equations in modular spaces, *Journal of Inequalities and Applications*, (2017), 2017:146. DOI 10.1186/s13660-017-1422-z.

[23] M.A. Khamsi , Quasicontraction mappings in modular spaces without Δ_2-condition, *Fixed Point Theory Appl.*, Article ID 916187 (2008).

[24] E. Kolk, The statistical convergence in Banach spaces, *Tartu Ul Toime.*, **928**, (1991), 41-52.

[25] S. Koshi, T. Shimogaki, On F-norms of quasi-modular spaces, *J. Fac. Sci., Hokkaido Univ. Ser.*, **1** 15, (1961), 202-218.

[26] M. Krbec, Modular interpolation spaces, *I. Z. Anal. Anwend.*, **1**, (1982), 25-40 .

[27] W.A. Luxemburg, Banach function spaces, PhD thesis, Delft University of Technology, Delft, The Netherlands (1959).

[28] L. Maligranda, Orlicz Spaces and Interpolation, *Seminários de Mathemática*, **5**. Universidade Estadual de Campinas, Departamento de Matemática, Campinas (1989).

[29] B. Mazur, Modular curves and the Eisenstein ideal, *Publ. Math. IHES*, **47**, (1978), 33-186.

[30] A. K. Mirmostafaee, M. S. Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, *Fuzzy Sets and Systems*, **159**, no. 6, (2008), 720-729.

[31] A. K. Mirmostafaee, M. Mirzavaziri, M. S. Moslehian, Fuzzy stability of the Jensen functional equation, *Fuzzy Sets and Systems*, **159**, no. 6, (2008), 730-738.

[32] A. K. Mirmostafaee, M. S. Moslehian, Fuzzy approximately cubic mappings, *Information Sciences*, **178** no. 19, (2008), 3791-3798.

[33] A. K. Mirmostafaee, M. S. Moslehian, Fuzzy almost quadratic functions, *Results in Mathematics*, **52**, no. 1-2, (2008), 161-177.

[34] J. Musielak Orlicz Spaces and Modular Spaces, Lecture Notes in Mathematics, **1034**, Springer, Berlin (1983).

[35] M. Mursaleen, λ-statistical convergence, *Math. Slovaca*, **50** (2000), 111–115 .

[36] M. Mursaleen, S. A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, *J. Comput. Anal. Math.*, **233** ,(2009), 142–149. doi:10.1016/j.cam.2009.07.005

[37] H. Nakano, Modulared Semi-Ordered Linear Spaces, *Maruzen*, Tokyo (1950).

[38] W. Orlicz, Collected Papers, vols. I, II. *PWN, Warszawa*, (1988).

[39] J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.*, **46** (1982), no. 1, 126–130.

[40] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), no. 2, 297–300.

[41] Th.M. Rassias, Functional Equations, Inequalities and Applications, Kluwer Acedamic Publishers, Dordrecht, Bostan London, 2003.

[42] K. Ravi, M. Arunkumar and J.M. Rassias, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *International Journal of Mathematical Sciences*, Autumn 2008 **3**, No. 08, 36-47.

[43] P. K. SAHOO, PL. KANNAPPAN, Introduction to Functional Equations, Chapman and Hall/CRC Taylor and Francis Group, 2011.

[44] T. SALAT, On the statistically convergent sequences of real numbers, *Math Slovaca.*, **30**, (1980), 139-150.

[45] G. SADEGHI, A fixed point approach to stability of functional equations in modular spaces, *Bull. Malays. Math. Sci. Soc.*, **37** (2), (2014), 333-344.

[46] B. SCHWEIZER AND A. SKLAR, Probabilistic Metric Spaces, Elsevier, North Holand, New York, 1983.

[47] A.N. ŠERSTNEV, On the notion of a random normed space, *Dokl Akad Nauk SSSR*, **149**, (1963), 280283.

[48] H. STEINHAUS, Sur la convergence ordinaire et la convergence asymptotique, *Colloq Math.*, **2**, (1951), 73-34.

[49] P. TURPIN, Fubini inequalities and bounded multiplier property in generalized modular spaces, *Comment. Math.*, **1**, (1978), 331-353.

[50] S. M. ULAM, Problems in modern mathematics, Science Editions John Wiley & Sons, Inc., New York, 1964.

[51] A. WILANSKY, Modern Methods in Topological Vector Space, McGraw-Hill International Book Co., New York (1978).

[52] K. WONGKUM, P. CHAIPUNYA, P. KUMAM, On the generalized Ulam-Hyers-Rassias stability of quadratic mappings in modular spaces without $\Delta_2$conditions, *J. Funct. Spaces*, Article ID 461719 (2015).

[53] K. WONGKUM, P. KUMAM, Y.J. CHO, P. CHAIPUNYA, On the generalized Ulam-Hyers-Rassias stability for quartic functional equation in modular spaces , *J. Nonlinear Sci. Appl.* **10**, (2017), 1399-1406 .

[54] J. Z. XIAO, X. H. ZHU, Fuzzy normed spaces of operators and its completeness, *Fuzzy Sets and Systems*, **133**, (2003) 389-399.

[55] S. YAMAMURO, On conjugate spaces of Nakano spaces, *Trans. Am. Math. Soc.* , **90**, (1959), 291-311.