MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(04)**(2022), 354–363.
http://doi.org/10.26637/mjm1004/006

Cone S-metric spaces and some new fixed point results for contractive mapping satisfying ϕ-maps with implicit relation

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*R*eceived 14 March 2022; *A*ccepted 16 September 2022

Abstract. In this article, we use implicit relations to establish some new fixed point results in the setting of cone S-Metric spaces for phi-map type contractive conditions. An example is provided to support our results. Our results extend, unify, and generalize several results from the current literature. In particular, the results presented in this paper improve and generalize the corresponding results of Sedghi and Dung[8], which used the ideas of Saluja, G. S. [19].

AMS Subject Classifications: 47H10, 54H25.

Keywords: Cone metric spce, Cone S- mtric space, Fixed point, Unique fixed point, Implicit relation.

Contents

1. Introduction and Background

The metric fixed point theory is very important and useful in mathematics. There are a number of generalizations of metric spaces and the Banach contraction principle(1922). From 1922, when Stefen Banach [1] formulated the notion of contraction and proved the famous theorem, scientists around.The world is publishing new results that are connected either to establishing a generalization of metric spaces or to getting an improvement in contractive conditions.

Huang and Zhang [2] recently introduced the concept of cone metric space as a generalization of metric spaces by replacing the set of real numbers with a general Banach space E that is partially ordered with respect to a cone $\mathcal{P} \subset E$ and establishing some fixed point theorems for contractive conditions in normal cone metric spaces. Subsequently, other mathematicians have generalized the results of Huang and Zhang [2].

Very recently, Sedghi, et al. [3] introduced the concept of S-metric space, which is different from other spaces, and proved fixed point theorems in S-metric space. They also give some examples. As a result, numerous authors

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have investigated the fixed point for mapping satisfying contractive conditions in complete S-metric spaces (see, for example, [4,5,6,7,8,9,10]).

Dhamodharan and Krishna Kumar [11] recently introduced the concept of S-metric space and proved some fixed point theorems in the space using various contractive conditions. On the other hand, Berinde and Vetro [12] surveyed an implicit contraction type condition instead of the usual explicit condition.

For this path of research, we prepared a consistent literature on fixed point and common fixed point theorems in various ambient spaces [see, 13,14,15,16,1,18].. Here we prove a paramount result of cone S-metric space and obtain some classical fixed point theorems with corollaries.

In the setting of complete cone S-metric spaces, our results extend and generalize several results from the existing literature, particularly the result of Sedghi Dung [8] and the idea of Saluja, G. S. [19].

2. Basic concept and mathematical preliminaries

Now, we begin with some besic concept, notations, Defifnitions and Lemmas, which needs in the sequel.

Definition 2.1. *Let E be a real Banach space. A sub set $\mathcal{P}$ of E is called a cone whenever the following conditions hold:*

(1). *$\mathcal{P}$ is closed, non empty and $\mathcal{P} \neq \{0\}$;*

(2). *$a, b \in \mathcal{R}$, $a, b \geq 0$ and $x, y \in \mathcal{P} \Rightarrow ax + by \in \mathcal{P}$;*

(3). $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$.

Given a cone $\mathcal{P} \subset E$, we define a partial ordering $\leq$ in E with respect to $\mathcal{P}$ by $x \leq y$ if and only if $y - x \in \mathcal{P}$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathcal{P}^0$, where $\mathcal{P}^0$ stands for the interior of $\mathcal{P}$. If $\mathcal{P}^0 \neq \phi$, then $\mathcal{P}$ is called a solid cone (see[20]). There exists two kinds of cones normal (with the normal constant $\mathcal{K}$) and non normal ones([21]).

Let E be a real Banach space,$\mathcal{P} \subset E$ a cone and $\leq$ partial ordering defined by$\mathcal{P}$, then $\mathcal{P}$ is called normal if there is number $\mathcal{K} > 0$ such that for all $x, y \in \mathcal{P}$,

$$0 \leq x \leq y \Rightarrow ||x|| \leq \mathcal{K}||y||$$

(1.1) or equivalently, if for all n, $x_n \leq y_n \leq z_n$ and $\lim\limits_{x \to n} = \lim\limits_{z \to n} x \Rightarrow \lim\limits_{y \to n} = x$.The least positive number $\mathcal{K}$ satisfying (1.1) is called normal constant of $\mathcal{P}$.

The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is , if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y_n$ for some $y \in E$, then there is $x \in E$ such that,

$$||x_n - x|| \to 0$$

as $n \to \infty$. Equivalntly, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. it is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with int $P \neq \phi and \leq$ is partial ordering in E with respect to P.

Lemma 1([22]):

Every regular cone is normal.

Definition 2.2 ([2]): Let X be a nonempty set. Let the mapping $d : X \times X \longrightarrow E$ satisfy:

(i) $0 \leq d(x, y)$, for all $x, y \in \mathcal{X}$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2 ([2]): Let $E = R^2$, $P = \{(x, y) \in R^2 : x \geq 0, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y), \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space with normal cone P, where $K = 1$.

Definition 2.3 ([3,6]): Let X be a non empty set and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, t \in X$;

1. $S(x, y, z) \geq 0$;
2. $S(x, y, z) = 0$if and only if $x = y = z$;
3. $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.

Then the function S is called S-metric on X and the pair (X, S) is called an S-metric space or simplySMS.
Example 3([23]): Let X be a nonempty set and d be the ordering metric on X. Then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.

Definition 2.4 ([11]): Suppose that, E is a real Banach space. P is a cone E with int $P \neq \phi$ and $\leq$ is partial ordering with respect to P. Let X be a nonempty set and let the function $S : X^3 \to E$ satisfy the following condtions:

1. $S(x, y, z) \geq 0$;
2. $S(x, y, z) = 0 if and only if x = y = z$;
3. $S(x, y, z) \leq S(x, xa) + S(y, y, a) + (z, z, a)$, for all $x, y, z, a \in X$.

Thenthefunction S is called a cone S-metric on X and the pair (X, S) is called a cone S-metric space or simply$CSMS$.

Example 4 ([11]): Let $E = R^2$, $P = \{(x, y) \in R^2 : x \geq 0, y \geq 0\}$, $X = R$ and d be the ordering metric on X. Then the dunction $S : X^3 \to E$ defined by $S(x, y, z) = (d(x, y) + d(y, z)), \alpha(d(x, z) + d(y, z))$, where $\alpha > 0$ is a cone S-metric space on X.
Lemma 2([11]), Let $S(X, S)$ be a cone S-metric spaces. Then we have $S(x, x, y) = S(y, y, x)$.
Definition 2.5([11]), Let (X, S) be a cone S-metric space Then a

1. sequence $\{x_n\}$ in X converges to $x \in X$if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, there exists $n_0 \in N$ such that for all $n \geq n_0, S(x_n, x_n, x) \ll c$ for each $c \in E, 0 \ll c$. We denote this
$$\lim_{n\to\infty} x_n = x, or \lim_{n\to\infty} S(x_n, x_n, x) = 0$$
.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if $(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.That is, there exists $n_0 \in N$ such that for all $n, m \geq n_0, S(x_n, x_n, x_m) \ll$, for each $c \in E, 0 \ll c$.
3. The cone S-metric space (X, S) is called complete if every Cauchy sequence is convergent.

Lemma 3 ([8]): Let $T : X \times X \to Y$ be a map from an S-metric space X to an S-metric space Y. Then T is continuous at $x \in X$ iff $T(x_n) \to T(x)$, whenever $x_n \to x$.
Definition 2.6 ([24]), ϕ Maps
: In 1977, Matkowski[24] introduced the ϕ maps as the following:
Let Φ be the set of all functions ϕ such that $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function function satisfying
$$\lim_{n\to\infty} \phi^n(t) = 0, forall, t \in (0, \infty)$$
. if $\phi \in \Phi$, then ϕ is called Φ-maps. Furthermore, if ϕ is a Φ map, then

1. $\phi(t) < t$ for all $t \in (0, \infty)$;

2. $\phi(0) = 0$.
   From now on, unless otherwise stated ϕ is meant the Φ-map.

Now, we'll look at some fixed point theorems on cone S-metric spaces using an implicit relation. **Definition 2.6**: Let Ψ be the family of all continuous functions of the five variables $\phi : R_+^5 \to R_+$, for some $K \in [0, 1)$, we consider the following conditions:

1. For all $x, y, z \in R_+$, if $Y \leq \phi(x, x, 0, z, y)$ with $z \leq 2x + y$, then $Y \in kx$.

2. For all $y \in R_+$, if $y \leq \phi(y, 0, y, y, 0)$. Then $y = 0$.

3. If $x_i \leq y_i + z_i$, for all $x_i, y_i, z_i \in R_+$, $i \leq 5$, then
   $\phi(x_1, ...x_5) \leq \phi(y_1....y_5) + \phi(z_1...z_5)$. Moreover, for all $y \in X, \phi(0, 0, 0, y, 2y) \leq 1 < y$.

Remark 1: Note the coefficient $\mathcal{K}$ in conditions (1) and (3) may be different for k_1 and k_3 respectively, But we may assume that they are equal by taking $k = max\{k_1, k_3\}$.

3. Main Results

In this section, we will prove some new unique fixed point theorems in the framework of cone S-metric space for the Phi type of contractive conditions.

Theorem 3.1. *Let (X, S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose the mapping $f : X \times X \to X$ satifies the following conditions*

$$S(fx, fx, fy) \leq \phi(S(x, x, y), S(x, fx, fx), S(y, fx, fx), S(x, fy, fy), S(y, fy, fy)), \tag{3.1}$$

for all $x, y \in X$ and some $\phi \in \psi$.
Then we have

1. *If ϕ satisfies the condition (1). Then f has a fixed point. Moreover, for any $w_0 \in X$ and the fixed point w, we have*
$$S(fw_n, fw_n, w) \leq \frac{2(\lambda)^n}{1 - \lambda}$$

2. *If ϕ satisfies the condition (2) and f has a fixed point, then the fixed point is unique.*

3. *If ϕ satisfies the condition (3) and f has a fixed point w, f is continuous at w.*

Proof. (i) For each $w_0 \in X$ and $n \in N$, put $w_{n+1} = fw_n$. It follows from (2) and Lemma 2, that

$$\begin{aligned} S(w_{n+1}, w_{n+1}, w_{n+2} &= S(fw_n, fw_n, fw_{n+1}) \\ &\leq \phi(S(w_n, w_n, w_{n+1}), S(w_n, fw_n, fw_n), S(w_{n+1}, fw_n, fw_n), \\ &\quad S(w_n, fw_{n+1}, fw_{n+1}), S(w_{n+1}, fw_{n+1}, fw_{n+1}) \\ &= \phi(S(w_n, w_n, w_{n+1}), S(w_n, w_{n+1}, w_{n+1}), S(w_{n+1}, w_{n+1}, w_{n+1}), \\ &\quad S(w_{n+1}, w_{n+2}, w_{n+2}), S(w_{n+1}, w_{n+2}, w_{n+2}) \\ &= \phi(S(w_n, w_n, w_{n+1}), S(w_n, w_n, w_{n+1}), 0, S(w_n, w_n, w_{n+2}), \\ &\quad S(w_{n+1}, w_{n+1}, w_{n+2}). \end{aligned}$$

By the definition 2.4(3) and Lemma 2, we have

$S(w_n, w_n, w_{n+2}) \leq 2S(w_n, w_n, w_{n+1}) + S(w_{n+2}, w_{n+2}, w_{n+1}). \quad = 2S(w_n, w_n, w_{n+1}) + S(w_{n+1}, w_{n+1}, w_{n+2})$

Since ϕ satisfies the condition (1), there exists $\lambda \in [0, 1)$ such that

$$\begin{aligned} S(w_{n+1}, w_{n+1}, w_{n+2}) &\leq S(w_n, w_n, w_{n+1}) \\ &\leq \lambda^{n+1} S(w_0, w_0, w_1). \end{aligned} \tag{3.2}$$

Thus for all $n < m$, by using $Definition 2.4(3)$, Lemma 2 and equation (5), we have

$$\begin{aligned} S(w_n, w_n, w_m) &\leq 2S(w_n, w_n, w_{n+1}) + S(w_m, w_m, w_{n+1}) \\ &= 2S(w_n, w_n, w_{n+1}) + S(w_{n+1}, w_{n+1}, w_m) \\ &\leq ... \leq \\ &\leq 2[\lambda^n + ... + \lambda^{m-1}] S(w_0, w_0, w_1) \\ &\leq \frac{2(\lambda)^n}{1-\lambda} \end{aligned} \tag{3.3}$$

This implies that $||S(w_n, w_n, w_m)|| \leq \frac{2(\lambda)^n}{1-\lambda}.K||S(w_0, w_0, w_1)||$.
Taking the limit as $n, m \to \infty$. we get

$$||(w_n, w_n, w_m)|| \to 0$$

. Since $0 < \lambda < 1$. Thus, we have $S(w_n, w_n, w_m) \to 0$ as $n, m \to \infty$.
This proves that w_n is a Cauchy sequence in the complete S-metric space (X, S). By the completeness of the space, we have

$$\lim_{n \to \infty} w_n = w \in X$$

. Moreover, taking limit as $n \to \infty$, we get

$$S(w_n, w_n, w) \leq \frac{2\lambda^{n+1}}{1-\lambda} S(w_0, w_0, w_1)$$

. It implies that

$$S(fw_n, fw_n, w) \leq \frac{2\lambda^n}{1-\lambda} S(w_0, w_0, fw_0)$$

. Now we prove that w is a fixed point of f. By using inequality (2) again we obtain

$$\begin{aligned} S(w_{n+1}, w_{n+1}, fw &= S(fw_n, fw_n, fw) \\ &\leq \phi(S(w_n, w_n, w), S(w, fw_n, fw_n), S(w_n, fw_n, fw_n), \\ &\quad S(w_n, fw, fw), S(w, fw, fw)) \\ &= \phi(S(w_n, w_n, w), S(w, w_{n+1}, w_{n+1}), \\ &\quad S(w_n, w_{n+1}, w_{n+1}), \end{aligned}$$

Note that $\phi \in \psi$, then using Lemma (3) and taking the limit as $n \to \infty$, we get

$$S(w, w, fw) \leq \phi(0, 0, 0, S(w, fw, fw), S(w, fw, fw))$$

Since ϕ satisfies the definition (3.6) of condition (1), then $S(w, w, fw) \leq \lambda 0 = 0$. This proves that $fw = w$. Thus w is a fixed point of f.

(ii) Let w_1, w_2 be fixed points of f. We shall prove that $w_1 = w_2$. It follows from (2) and Lemma 2 that

$$\begin{aligned} S(w_1, w_1, w_2) &= S(fw_1, fw_1, fw_2) \\ &\leq \phi(S(w_1, w_1, w_2), S(w_1, fw_1, fw_1), S(w_2, fw_1, fw_1), \\ &\quad S(w_1, fw_2, fw_2), S(w_2, fw_2, fw_2)) \\ &= (S(w_1, w_1, w_2), S(w_1, w_1, w_1), S(w_2, w_1, w_1), S(w_1, w_2, w_2), S(w_2, w_2, w_2)) \\ &= (S(w_1, w_1, w_2), 0, S(w_2, w_1, w_1), S(w_1, w_2, w_2), 0) \\ &= (S(w_1, w_1, w_2), 0, S(w_1, w_1, w_2), S(w_1, w_1, w_2), 0) \end{aligned}$$

Sinc ϕ satisfies the condition (2), then $S(w_1, w_1, w_2) = 0$. This prove that $w_1 = w_2$.
Thus the fixed point of f is unique.
(iii) Let w be the fixed point of f and $x_n \to w \in X$. By Lemma 3, we need to prove that $fw_n \to fw$. It follows from (2) and Lemma 2 that

$$\begin{aligned} S(w, w, fx_n) &= S(fw, fw, fx_n) \\ &\leq \phi(S(w, w, x_n), S(w, fw, fw), S(x_n, fw, fw), S(w, fx_n, fx_n), \\ &\quad S(x_n, fx_n, fx_n)) \\ &\leq \phi(S(w, w, x_n), S(w, w, w), S(x_n, w, w), S(w, fx_n, fx_n), S(x_n, fx_n, fx_n) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2)} \end{aligned}$$

Since ϕ satisfies the condition (3), by Lemma 2 and (CSM_3), we have
$S(fx_n, fx_n, x_n) \leq 2S(w, fx_n, fx_n) + S(w, x_n, x_n)$, then we have

$$\begin{aligned} S(w, w, fx_n) &\leq \phi(S(w, w, x_n), 0, S(x_n, w, w), 0S(x_n, w, w) \\ &\quad + \phi(0, 0, 0, S(w, fx_n, fx_n), 2S(w, fx_n, fx_n) \\ &\leq \phi(S(w, w, x_n), 0, S(x_n, w, w), 0, S(x_n, w, w) + \lambda S(w, fx_n, fx_n) \end{aligned}$$

Therefore,
$S(w, w, fx_n) \leq \frac{1}{1-\lambda}.\phi(S(x_n, w, w), 0, S(x_n, w, w,), 0, S(x_n, w, w))$.
Note that $\phi \in \psi$, hence taking limit as $n \to \infty$, we get

$$(fx_n, w, w) \to \infty.$$

This proves that $fx_n \to w = fw$. This completes the proof.

■

Next, we give some anaogues of fixed point theorems in metric spaces for cone S-metric spaces by combining Theorem 1 with $\phi \in \psi$ and ϕ satisfies the definition (2.6) of conditions (1), (2), and (3).
The following Corollary is analogues of Hardy and Rogers result in [25].
Corollary3.2: Let (X, S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose the mapping $f : X \times X \to X$ satisfies the followingcondition:
$S(fx, fx, fy) \leq a_1 S(x, x, y) + a_2 S(x, fx, fx) + a_3 S(y, fx, fx) + a_4 S(x, fy, fy) + a_5 S(y, fy, fy)$ for some $a_1 ... a_5 \geq 0$ Such that

$$max\{a_1 + a_2 + 3a_4 + a_5, a_1 + a_3 + a_4, a_4 + 2a_5\} < 1$$

and for all $x, y \in X$. Then f has a unique fixed point in X. Moreover, f is continuous at the fixed point.

Proof. The assertion follows using Theorem 1 with $\phi(x, y, z, t) = a_1x + a_2y + a_3z + a_4s + a_5t \geq 0$ for some $a_1...a_5 \geq 0$ such that

$$max\{a_1 + a_2 + 3a_4 + a_5, a_1 + a_3 + a_4 + 2a_5\} < 1$$

and all $x, y, z, t \in R_+$. Indeed, ϕ is continuous.
First we have

$$\phi(x, x, 0, z,) = a_1x + a_2x + a_4z + a_5y.$$

So, if $y \leq \phi(x, x, 0, z, y)$ with $z \leq 2x + y$, then

$$\begin{aligned} y &\leq a_1x + a_2x + a_4z + a_5y \\ &\leq a_1x + a_2x + a_4(2x + y) + a_5y.Then \\ y &\leq \frac{a_1 + a_2 + 2a_4}{(1 - a_4 - a_5)x} with \frac{a_1 + a_2 + 2a_4}{1 - a_4 - a_5} < 1. \end{aligned} \tag{3.4}$$

Therefore, f satisfies by definition $(3, 6)$ of the condition (1). Next

$$\begin{aligned} y &\leq \phi(y, 0, y, y, 0) \\ &= a_1y + a_3y + a_4y \\ &= (a_1 + a_3 + a_4)y, \end{aligned} \tag{3.5}$$

then $y = 0$. Since $a_1 + a_3 + a_4 < 1$. Therefore, f is satisfies by Definition (3.6) of the condition (2).
Finally, if $x_i \leq y \leq +z_i$, for $i \leq 5$, then

$$\begin{aligned} \phi(x_1, ...x_5) &= a_1x_1 + ... + a_5x_5 \\ &\leq a_1(y_1 + z_1) + ... + a_5(y_5 + z_5) \\ &= (a_1y_1 + a_1z_1 + ... + a_5y_5) \\ &= (a_1y_1 + ... + a_5y_5) + (a_1z_1 + ... + a_5z_5) \\ &= \phi(y_1, ..., y_5) + \phi(z_1, ..., z_5) \end{aligned} \tag{3.6}$$

Moreover,

$$\begin{aligned} \phi(0, 0, 0, y, 2y) &= a_4y + 2a_5y \\ &= (a_4 + 2a_5)y, \text{ where } a_4 + 2a_5 < 1. \end{aligned}$$

Therefore, f is satisfies by definition(3.6) of the condition(3).

■

The following Corollary is an analogue of Reich, S. result in [26].

Corollary 3.3: Let (X, S) be a complete cone S-metric space and$P$ be a normal cone with normal constant K. Suppose the mapping $f : X \times X \to X$ satisfies the followingcondition:
$S(fx, fx, fy) \leq aS(x, x, y) + bS(x, fx, fx) + cS(y, fy, fy)$ for all $x, y \in X$ where $a, b, c, \geq 0$ are constants with $a + b + c < 1$. Then f has a unique fixed point in X. Moreover, if $c < \frac{1}{2}$, then f is continuous at the fixed point.
The following Corollary is an analogur of Kannans resultin [27]:

Corollary 3.4: Let (X, S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose the mapping $f : X \times X \to X$ satisfies the followingcondition:
$S(fx, fx, fy) \leq \alpha[S(x, fx, fx) + S(y, fy, fy)]$ for some $\alpha \in [0, \frac{1}{2}$ for all $x, y \in X$. Then f has a unique fixed point in X. Moreover, f is continuous at the fixed point.

Next, the following Corollary is an analogue of Banach contraction principle:
Corollary 3.5: Let (X, S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose the mapping $f : X \times X \to X$ satisfies the following condition:

$$S(fx, fx, fy) \leq \lambda S(x, x, y)$$

for some $\lambda \in [0, 1)$ for all $x, y \in X$. Then f has a unique fixed point in X. Moreover, f is continuous at the fixed point.
Example 5 Let $E = R^2$, the ecludian plane, $P = (x, y) \in R^2 : x \geq 0, y \geq 0$ a normal cone in E and $X = R$. Then the function $S : X^3 \to E$ defined by $(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Then (X, S) is a cone S-metric space. Now we consider the mapping $f : X \times X \to X$ by $fx = \frac{x}{5}$. Then

$$\begin{aligned} S(fx, fx, fy) &= |fx - fy| + |fx - fy| \\ &= 2|fx - fy| \\ &= 2\left|\left(\frac{x}{5}\right) - \left(\frac{x}{5}\right)\right| \\ &= \frac{2}{5}|x - y| \\ &\leq \frac{1}{4}\,(2|x - y|) \\ &= hS(x, x, y), \text{ where } \frac{1}{4} < 1. \end{aligned}$$

Thus f satisfies all the conditions of the theore 3.1 Corollary with 3.2 and clearly $0 \in X$ is the unique fixed point of f.

Example 3.6 Let $E = R^2$, the ecludian plane, $P = (x, y) \in R^2 : x \geq 0, y \geq 0$ a normal cone in E and $X = R$. Then the function $S : X^3 \to E$ defined by $(x, y, z) = |x - 2| + |y - 2|$ for all $x, y, z \in X$. Then (X, S) is a cone S-metric space. Now we consider the mapping $f : X \times X \to X$ by $fx = \frac{x}{2}$ and $\{x_n\} = \{\frac{1}{2^n}\}$ for all $n \in N$ is a sequence converging to zero.

4. Conclusion

In the present work, we obtained some unique fixed pont results by using the ϕ- maps type contractive condition in cone S -metric space with an implicit function.Our theorem 1 with corollaries 1,2,3,and 4 of extend and improve some recent results of Sedghi and Dung [8] by idea of Saluja,G.S.[19]. Also gave the example to support our result. Further scope for higher teaching and higher learning mathematics, in particular nonlinear space transformation.

5. Acknowledgement

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