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Some properties of b-linear functional in linear n-normed space

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Abstract. Some results in linear n-normed space have been discussed. Several nice properties of bounded b-linear functional in linear n-normed space are presented. We see that the collection of all bounded b-linear functionals after introducing suitable operations, is a normed space.

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Contents

1. Introduction and Background

If one is interested in the study of a certain class of mathematical objects, it is natural and fruitful to investigate the set of operators from one such class to another which preserve some or all of the structures defined on the objects. The linear operators from one normed space to another normed space over the same scalar field preserve the algebraic structure. The fact that a normed space gives rise to a metric, induced by the norm, provides naturally to the extremely important classification of linear operators into continuous and discontinuous ones. In normes spaces, this distinction is facilitated by a very simple criterion for continuity; namely, any linear operator between normed space is continuous if and only if it is bounded. The collection of all these bounded linear operators is a normed space.

The idea of linear 2-normed space was first introduced by S. Gahler [5] and thereafter the geometric structure of linear 2-normed spaces was developed by A. White, Y. J. Cho, R. W. Freese [1, 9]. The concept of 2-Banach space is briefly discussed in [9]. In recent times, some important results in classical normed spaces have been proved into 2-norm setting by many researchers. P. Ghosh et al. [4] studied some fundamental theorem in classical normed space into 2-normed space. H. Gunawan and Mashadi [6] developed the generalization of a linear 2-normed space for $n \geq 2$. Some results of classical normed space with respect to b-linear functional in linear n-normed space were established by P. Ghosh and T. K. Samanta [2]. They also studied the reflexivity of linear n-normed space with respect to b-linear functional [3].

In this paper, some results in linear n-normed space are being described. We shall verify that the collection of all bounded b-linear functionals after introducing suitable operations, is a normed space. Some properties of bounded b-linear functional are going to be established.

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Definition 1.1. *[7] Let X be a linear space and M be subspace of X. Then M is said to be a convex set if $x, y \in M$, $tx + (1-t)y \in M$ for $0 \leq t \leq 1$.*

Definition 1.2. *[7] A set M in a linear space X is called a hyperplane if X can be expressed as a direct sum of M and one-dimensional subspace of X, i. e., $X = M + \langle x \rangle$, for some $x \in X$.*

Definition 1.3. *[6] Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field with dim $X \geq n$, where n is a positive integer. A real valued function $\|\cdot, \cdots, \cdot\| : X^n \to \mathbb{R}$ is called an n-norm on X if*

(i) $\|x_1, x_2, \cdots, x_n\| = 0$ *if and only if $x_1, \cdots, x_n$ are linearly dependent,*

(ii) $\|x_1, x_2, \cdots, x_n\|$ *is invariant under permutations of $x_1, x_2, \cdots, x_n$,*

(ii) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \, \|x_1, x_2, \cdots, x_n\| \quad \forall \ \alpha \in \mathbb{K}$,

(iv) $\|x + y, x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$

hold for all $x, y, x_1, x_2, \cdots, x_n \in X$. The pair $(X, \|\cdot, \cdots, \cdot\|)$ is then called a linear n-normed space.

Throughout this paper, X will denote linear n-normed space over the field $\mathbb{K}$ of complex or real numbers, associated with the n-norm $\|\cdot, \cdots, \cdot\|$.

Definition 1.4. *[6] A sequence $\{x_k\} \subseteq X$ is said to converge to $x \in X$ if*

$$\lim_{k \to \infty} \|x_k - x, e_2, \cdots, e_n\| = 0$$

for every $e_2, \cdots, e_n \in X$ and it is called a Cauchy sequence if

$$\lim_{l, k \to \infty} \|x_l - x_k, e_2, \cdots, e_n\| = 0$$

for every $e_2, \cdots, e_n \in X$. The space X is said to be complete or n-Banach space if every Cauchy sequence in this space is convergent in X.

Definition 1.5. *[8] For $a \in X$, define the following open and closed ball in X:*

$$B_{\{e_2, \cdots, e_n\}}(a, \delta) = \{x \in X : \|x - a, e_2, \cdots, e_n\| < \delta\} \text{ and}$$

$$B_{\{e_2, \cdots, e_n\}}[a, \delta] = \{x \in X : \|x - a, e_2, \cdots, e_n\| \leq \delta\},$$

for every $e_2, \cdots, e_n \in X$ and δ be a positive number.

Definition 1.6. *[8] A subset G of X is said to be open in X if for all $a \in G$, there exist $e_2, \cdots, e_n \in X$ and $\delta > 0$ such that $B_{\{e_2, \cdots, e_n\}}(a, \delta) \subseteq G$.*

Definition 1.7. *[8] Let $A \subseteq X$. Then the closure of A is defined as*

$$\overline{A} = \left\{ x \in X \mid \exists \{x_k\} \in A \text{ with } \lim_{k \to \infty} x_k = x \right\}.$$

The set A is said to be closed if $A = \overline{A}$.

Definition 1.8. *[2] Let W be a subspace of X and $b_2, b_3, \cdots, b_n$ be fixed elements in X and $\langle b_i \rangle$ denote the subspaces of X generated by b_i, for $i = 2, 3, \cdots, n$. Then a map $T : W \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to \mathbb{K}$ is called a b-linear functional on $W \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$, if for every $x, y \in W$ and $k \in \mathbb{K}$, the following conditions hold:*

(i) $T(x+y, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) + T(y, b_2, \cdots, b_n)$

(ii) $T(kx, b_2, \cdots, b_n) = k\,T(x, b_2, \cdots, b_n)$.

A b-linear functional is said to be bounded if $\exists$ *a real number* $M > 0$ *such that*

$$|T(x, b_2, \cdots, b_n)| \leq M \, \|x, b_2, \cdots, b_n\| \quad \forall\, x \in W.$$

The norm of the bounded b-linear functional T *is defined by*

$$\|T\| = \inf\{M > 0 : |T(x, b_2, \cdots, b_n)| \leq M \, \|x, b_2, \cdots, b_n\| \quad \forall\, x \in W\}.$$

2. Main Results

Theorem 2.1. *Let* X *be a linear n-normed space. Then*

$$|\,\|x, e_2, \cdots, e_n\| - \|y, e_2, \cdots, e_n\|\,| \leq \|x - y, e_2, \cdots, e_n\|$$

hold for all $x, y, e_2, e_3, \cdots, e_n \in X$.

Proof. Take $x, y, e_2, \cdots, e_n \in X$. Then

$$\begin{aligned} &\|x, e_2, \cdots, e_n\| = \|x - y + y, e_2, \cdots, e_n\| \\ &\leq \|x - y, e_2, \cdots, e_n\| + \|y, e_2, \cdots, e_n\| \\ &\Rightarrow \|x, e_2, \cdots, e_n\| - \|y, e_2, \cdots, e_n\| \leq \|x - y, e_2, \cdots, e_n\|. \end{aligned}$$

Also interchanging x and y we get

$$-(\|x, e_2, \cdots, e_n\| - \|y, e_2, \cdots, e_n\|) \leq \|x - y, e_2, \cdots, e_n\|.$$

Combining the above two inequality the result follows. ■

Theorem 2.2. *Let* M *be a subspace of a linear n-normed space* X*. Then the closure* $\overline{M}$ *of* M *is also subspace.*

Proof. Let $x, y \in \overline{M}$. Then corresponding to $\epsilon > 0$, $\exists\, u, v \in M$ such that

$$\|x - u, e_2, \cdots, e_n\| < \epsilon \text{ and } \|y - v, e_2, \cdots, e_n\| < \epsilon$$

for every $e_2, e_3, \cdots, e_n \in X$. Now, for non zero scalars α, β,

$$\begin{aligned} &\|(\alpha x + \beta y) - (\alpha u + \beta v), e_2, \cdots, e_n\| \\ &= \|\alpha(x - u) + \beta(y - v), e_2, \cdots, e_n\| \\ &\leq |\alpha|\, \|x - u, e_2, \cdots, e_n\| + |\beta|\, \|y - v, e_2, \cdots, e_n\| \\ &< \epsilon(|\alpha| + |\beta|) = \epsilon', \text{ say} \end{aligned}$$

This shows that $\alpha u + \beta v \in B_{\{e_2, \cdots, e_n\}}(\alpha x + \beta y, \epsilon')$. As $\alpha u + \beta v \in M$ and $\epsilon' > 0$ is arbitrary, it follows that $\alpha x + \beta y \in \overline{M}$. Hence the proof. ■

Theorem 2.3. *The sets* $B_{\{e_2, \cdots, e_n\}}[a, \delta]$ *and* $B_{\{e_2, \cdots, e_n\}}(a, \delta)$ *in a linear* n*-normed space* X *are convex.*

Proof. Let $x, y \in B_{\{e_2, \cdots, e_n\}}[a, \delta]$. Then

$$\|x - a, e_2, \cdots, e_n\| \leq \delta \text{ and } \|y - a, e_2, \cdots, e_n\| \leq \delta.$$

Take $z = tx + (1-t)y \in M$ for $0 \leq t \leq 1$. Then we have

$$\begin{aligned}
&\|z - a, e_2, \cdots, e_n\| = \|tx + (1-t)y - a, e_2, \cdots, e_n\| \\
&= \|tx + (1-t)y - ta - (1-t)a, e_2, \cdots, e_n\| \\
&\leq \|tx - ta, e_2, \cdots, e_n\| + \|(1-t)y - (1-t)a, e_2, \cdots, e_n\| \\
&= t\|x - a, e_2, \cdots, e_n\| + (1-t)\|y - a, e_2, \cdots, e_n\| \leq t\delta + (1-t)\delta = \delta.
\end{aligned}$$

So, $z \in B_{\{e_2, \cdots, e_n\}}[a, \delta]$. This shows that $B_{\{e_2, \cdots, e_n\}}[a, \delta]$ is a convex set. Similarly, it can be shown that $B_{\{e_2, \cdots, e_n\}}(a, \delta)$ is also a convex set. This completes the proof. ■

Theorem 2.4. *Let X be a linear n-normed space and M be a convex subset of X. Then the closure of M, $\overline{M}$ is convex.*

Proof. Let $x, y \in \overline{M}$. Then corresponding to $\epsilon > 0$, $\exists\ u, v \in M$ such that

$$\|x - u, e_2, \cdots, e_n\| < \epsilon \text{ and } \|y - v, e_2, \cdots, e_n\| < \epsilon$$

for every $e_2, e_3, \cdots, e_n \in X$. Let $0 \leq t \leq 1$, then

$$\|\{tx + (1-t)y\} - \{tu + (1-t)v\}, e_2, \cdots, e_n\|$$

$$\leq t\|x - u, e_2, \cdots, e_n\| + (1-t)\|y - v, e_2, \cdots, e_n\| < t\epsilon + (1-t)\epsilon = \epsilon.$$

Since M is convex, $tu + (1-t)v \in M$ and because $\epsilon > 0$ is arbitrary, $tx + (1-t)y \in \overline{M}$. Hence, $\overline{M}$ is convex. ■

Theorem 2.5. *Let X be a linear n-normed space and x_0 be a fixed element in X and $\alpha \neq 0$ be a fixed scalar. Then the mappings $x \to x_0 + x$ and $x \to \alpha x$ are sequentially continuous.*

Proof. Let $\{x_k\}$ be a sequence in X such that $x_k \to x$ as $k \to \infty$. Then

$$\lim_{k\to\infty} \|x_k - x, e_2, \cdots, e_n\| = 0\ \forall\ e_2, \cdots, e_n \in X. \tag{2.1}$$

Firstly, we consider the mapping, $f(x) = x_0 + x$. Then,

$$\begin{aligned}
\|f(x_k) - f(x), e_2, \cdots, e_n\| &= \|(x_0 + x_k) - (x_0 + x), e_2, \cdots, e_n\| \\
&= \|x_k - x, e_2, \cdots, e_n\|
\end{aligned}$$

Taking limit both sides as $k \to \infty$, we get

$$\begin{aligned}
\lim_{k\to\infty} \|f(x_k) - f(x), e_2, \cdots, e_n\| &= \lim_{k\to\infty} \|x_k - x, e_2, \cdots, e_n\| \\
&= 0\ [\text{using } (2.1)].
\end{aligned}$$

This shows that $f(x_k) \to f(x)$ as $k \to \infty$.

Now, take $g(x) = \alpha x$. Then, for each $e_2, \cdots, e_n \in X$, we have

$$\begin{aligned}
\|g(x_k) - g(x), e_2, \cdots, e_n\| &= \|\alpha x_k - \alpha x, e_2, \cdots, e_n\| \\
&= \|\alpha(x_k - x), e_2, \cdots, e_n\| \\
&= |\alpha|\,\|x_k - x, e_2, \cdots, e_n\|
\end{aligned}$$

So by (2.1), $g(x_k) \to g(x)$ as $k \to \infty$. Therefore, the mappings $x \to x_0 + x$ and $x \to \alpha x$ are sequentially continuous. ■

3. Main results

In this section, some properties of bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ are discussed.

Theorem 3.1. *Let T be a bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Then*

(i) $|T(x, b_2, \cdots, b_n)| \leq \|T\| \|x, b_2, \cdots, b_n\| \;\; \forall x \in X$.

(ii) $\|T\| = \sup\{|T(x, b_2, \cdots, b_n)| : \|x, b_2, \cdots, b_n\| \leq 1\}$.

(iii) $\|T\| = \sup\{|T(x, b_2, \cdots, b_n)| : \|x, b_2, \cdots, b_n\| = 1\}$.

(iv) $\|T\| = \sup\left\{\dfrac{|T(x, b_2, \cdots, b_n)|}{\|x, b_2, \cdots, b_n\|} : \|x, b_2, \cdots, b_n\| \neq 0\right\}$.

Proof. (i) For arbitrary $\epsilon > 0$, it follows by the definition of norm of T that

$$|T(x, b_2, \cdots, b_n)| \leq (\|T\| + \epsilon)\|x, b_2, \cdots, b_n\| \;\; \forall x \in X. \tag{3.1}$$

If possible, suppose that there exists $x_1 \in X$ such that

$$|T(x_1, b_2, \cdots, b_n)| > \|T\| \|x_1, b_2, \cdots, b_n\|.$$

Then for some $\epsilon > 0$,

$$|T(x_1, b_2, \cdots, b_n)| > \|T\| \|x_1, b_2, \cdots, b_n\| + \epsilon \|x_1, b_2, \cdots, b_n\|$$
$$= (\|T\| + \epsilon)\|x_1, b_2, \cdots, b_n\|$$

which contradicts (3.1). Hence

$$|T(x, b_2, \cdots, b_n)| \leq \|T\| \|x, b_2, \cdots, b_n\| \;\; \forall x \in X.$$

(ii) If $\|x, b_2, \cdots, b_n\| \leq 1$, then

$$|T(x, b_2, \cdots, b_n)| \leq \|T\| \|x, b_2, \cdots, b_n\| \leq \|T\|$$
$$\Rightarrow \sup\{|T(x, b_2, \cdots, b_n)| : \|x, b_2, \cdots, b_n\| \leq 1\} \leq \|T\|. \tag{3.2}$$

On the other hand, by definition, for every $\epsilon > 0$ there exists $x' \neq \theta$ such that

$$|T(x', b_2, \cdots, b_n)| > (\|T\| - \epsilon)\|x', b_2, \cdots, b_n\|.$$

Take $x_1 = \dfrac{x'}{\|x', b_2, \cdots, b_n\|}$, then we get

$$|T(x_1, b_2, \cdots, b_n)| = \frac{1}{\|x', b_2, \cdots, b_n\|}|T(x', b_2, \cdots, b_n)|$$
$$> \frac{1}{\|x', b_2, \cdots, b_n\|}(\|T\| - \epsilon)\|x', b_2, \cdots, b_n\| = \|T\| - \epsilon.$$

Since $\|x_1, b_2, \cdots, b_n\| = 1$, we get

$$\sup\{|T(x, b_2, \cdots, b_n)| : \|x, b_2, \cdots, b_n\| \leq 1\} \geq |T(x_1, b_2, \cdots, b_n)|$$

$$> \| T \| - \epsilon \geq \| T \| \ [\text{ since } \epsilon > 0 \text{ is arbitrary }].$$

Combining this with (3.2), the proof of (II) is complete.

(iii) The proof is follows from (II), replacing $\| x, b_2, \cdots, b_n \| \leq 1$ by $\| x, b_2, \cdots, b_n \| = 1$.

(iv) Let $\alpha = \sup \left\{ \dfrac{| T(x, b_2, \cdots, b_n) |}{\| x, b_2, \cdots, b_n \|} : \| x, b_2, \cdots, b_n \| \neq 0 \right\}$. Now, for arbitrary $\epsilon > 0$, there exists an elements $x_1 \neq \theta$ with $x_1, b_2, \cdots, b_n$ are linearly independent such that

$$| T(x_1, b_2, \cdots, b_n) | > (\alpha - \epsilon) \| x_1, b_2, \cdots, b_n \|.$$

It follows from the definition of norm that $\| T \| > \alpha - \epsilon$ and since $\epsilon > 0$ is arbitrary, we obtain $\| T \| \geq \alpha$. If possible, suppose that $\| T \| > \alpha$.
Let $\epsilon = | T(x_1, b_2, \cdots, b_n) | - \alpha$, then $\alpha < \| T \| - \dfrac{\epsilon}{2}$. So, for arbitrary x,

$$\frac{| T(x, b_2, \cdots, b_n) |}{\| x, b_2, \cdots, b_n \|} \leq \alpha < \| T \| - \frac{\epsilon}{2}$$

$$\Rightarrow | T(x, b_2, \cdots, b_n) | < \left(\| T \| - \frac{\epsilon}{2} \right) \| x, b_2, \cdots, b_n \| \ \ \forall x \in X.$$

This contradicts the fact that $\| T \|$ is the lower bound of all those M for which

$$| T(x, b_2, \cdots, b_n) | \leq M \, \| x, b_2, \cdots, b_n \| \ \ \forall x \in X.$$

So $\| T \|$ cannot be greater that α, i. e., $\| T \| = \alpha$. This proves the theorem. ■

Theorem 3.2. *The set X_F^* of all bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ is a linear space.*

Proof. Let $S, T \in X_F^*$. Then there exists $L, M > 0$ such that

$$| S(x, b_2, \cdots, b_n) | \leq L \, \| x, b_2, \cdots, b_n \|, \text{ and}$$

$$| T(x, b_2, \cdots, b_n) | \leq M \, \| x, b_2, \cdots, b_n \| \ \ \forall x \in X.$$

$$\Rightarrow | (S + T)(x, b_2, \cdots, b_n) | \leq | S(x, b_2, \cdots, b_n) | + | T(x, b_2, \cdots, b_n) |$$

$$\leq (L + M) \| x, b_2, \cdots, b_n \| \ \ \forall x \in X, \text{ and}$$

$$\Rightarrow | (\lambda T)(x, b_2, \cdots, b_n) | \leq | \lambda | M \, \| x, b_2, \cdots, b_n \| \ \ \forall x \in X \text{ and } \lambda \in \mathbb{K}.$$

This shows that $S + T \in X_F^*$ and $\lambda T \in X_F^*$. Hence, X_F^* is a linear space. ■

Theorem 3.3. *Let X_F^* be the linear space of all bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Define $\| \cdot \| : X_F^* \to \mathbb{R}$ by*

$$\| T \| = \sup \{ | T(x, b_2, \cdots, b_n) | : x \in X, \| x, b_2, \cdots, b_n \| \leq 1 \}.$$

Then $(X_F^*, \| \cdot \|)$ is a Banach space.

Proof. Since every $T \in X_F^*$ is bounded b-linear functional, the norm $\| \cdot \|$ on X_F^* is well defined.

(i) $\| T \| \geq 0 \ \forall T \in X_F^*$.

(ii) $\|T\| = 0 \Rightarrow |T(x, b_2, \cdots, b_n)| = 0, \forall x \in X, \|x, b_2, \cdots, b_n\| \leq 1$

$$\Rightarrow T(x, b_2, \cdots, b_n) = 0, \forall x \in X, \|x, b_2, \cdots, b_n\| \leq 1.$$

Let $x \in X$ with $\|x, b_2, \cdots, b_n\| = \alpha > 1$ and suppose $y = \dfrac{x}{\beta}$, where $\beta > \alpha$. Then

$$\|y, b_2, \cdots, b_n\| = \frac{\|x, b_2, \cdots, b_n\|}{\beta} = \frac{\alpha}{\beta} < 1$$

and so $T(y, b_2, \cdots, b_n) = 0$. But

$$0 = T(y, b_2, \cdots, b_n) = T\left(\frac{x}{\beta}, b_2, \cdots, b_n\right) = \frac{1}{\beta} T(x, b_2, \cdots, b_n).$$

Therefore $T(x, b_2, \cdots, b_n) = 0$. So, $T(x, b_2, \cdots, b_n) = 0, \forall x \in X$ and therefore $T = 0$. Conversely, if $T = 0$, then clearly $\|T\| = \|0\| = 0$.

(iii) For any $\lambda \in \mathbb{K}$, we have

$$\|\lambda T\| = \sup\{|(\lambda T)(x, b_2, \cdots, b_n)| : x \in X, \|x, b_2, \cdots, b_n\| \leq 1\}$$
$$= \sup\{|\lambda| |T(x, b_2, \cdots, b_n)| : x \in X, \|x, b_2, \cdots, b_n\| \leq 1\} = |\lambda| \|T\|.$$

(iv) For $T, S \in X_F^*$, we have

$$\|T + S\| = \sup\{|(T + S)(x, b_2, \cdots, b_n)| : x \in X, \|x, b_2, \cdots, b_n\| \leq 1\}$$
$$\leq \sup\{|T(x, b_2, \cdots, b_n)| : x \in X, \|x, b_2, \cdots, b_n\| \leq 1\} + \sup\{|S(x, b_2, \cdots, b_n)| : x \in X, \|x, b_2, \cdots, b_n\| \leq 1\}$$
$$= \|T\| + \|S\|.$$

Therefore $\|\cdot\|$ defines a norm on X_F^*.

To prove the completeness of X_F^* under the norm $\|\cdot\|$, let $\{T_k\}$ be a Cauchy sequence in X_F^*. Now, for every $x \in X$, we have

$$|T_l(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n)| = |(T_l - T_k)(x, b_2, \cdots, b_n)|$$
$$\leq \|T_l - T_k\| \|x, b_2, \cdots, b_n\|.$$

This calculation shows that $\{T_k(x, b_2, \cdots, b_n)\}$ is a Cauchy sequence in $\mathbb{K}$ for each $x \in X$. Since $\mathbb{K}$ is complete, $\{T_k(x, b_2, \cdots, b_n)\}$ converges in $\mathbb{K}$. Let $T_k(x, b_2, \cdots, b_n) \to T(x, b_2, \cdots, b_n)$. We shall now show that $T \in X_F^*$.

(i) T is b-linear: For $x, y \in X$ and $\lambda \in \mathbb{K}$, we have

$$T(x + y, b_2, \cdots, b_n) = \lim_{k \to \infty} T_k(x + y, b_2, \cdots, b_n)$$
$$= \lim_{k \to \infty} T_k(x, b_2, \cdots, b_n) + \lim_{k \to \infty} T_k(y, b_2, \cdots, b_n) \text{ [since } T_k \text{ is } b\text{-linear]}$$
$$= T(x, b_2, \cdots, b_n) + T(y, b_2, \cdots, b_n),$$

and $T(\lambda x, b_2, \cdots, b_n) = \lim_{k \to \infty} T_k(\lambda x, b_2, \cdots, b_n)$

$$= \lambda \lim_{k \to \infty} T_k(x, b_2, \cdots, b_n) = \lambda T(x, b_2, \cdots, b_n).$$

This verifies that T is a b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$.

(ii) Since for each k, T_k is bounded b-linear functional, it follows that

$$|T_k(x, b_2, \cdots, b_n)| \leq \|T_k\| \, \|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X.$$

Also, the sequence $\{T_k\}$ being Cauchy sequence in X_F^*, it is bounded and hence there is a constant $K > 0$ such that $\|T_k\| \leq K \;\; \forall\, k \in \mathbb{N}$. Consequently,

$$|T_k(x, b_2, \cdots, b_n)| \leq K \, \|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X \text{ and } k \in \mathbb{N}.$$

Thus, for each $x \in X$ and $k \in \mathbb{N}$, we have $|T(x, b_2, \cdots, b_n)|$

$$\leq |T(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n)| + |T_k(x, b_2, \cdots, b_n)|$$

$$\leq |T(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n)| + K \, \|x, b_2, \cdots, b_n\|.$$

Since k is arbitrary, letting $k \to \infty$ and using $T_k(x, b_2, \cdots, b_n) \to T(x, b_2, \cdots, b_n)$, we obtain

$$|T(x, b_2, \cdots, b_n)| \leq K \, \|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X.$$

Hence, T is bounded. Therefore (i) and (ii) verify that $T \in X_F^*$.

Finally, we show that $T_k \to T$ in $(X_F^*, \|\cdot\|)$. Since $\{T_k\}$ is a Cauchy sequence in X_F^*, for each $\epsilon > 0$, there exists an integer $N > 0$ such that

$$\|T_l - T_k\| < \epsilon \;\; \forall\, k, l \leq N. \text{ Therefore}$$

$$|T_l(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n)| < \epsilon \, \|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X.$$

Taking $l \to \infty$, for all $k \geq N$, we get

$$|T(x, b_2, \cdots, b_n) - T_k(x, b_2, \cdots, b_n)| < \epsilon \, \|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X.$$

This gives, $\|T_k - T\|$

$$= \sup\{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| : x \in X, \|x, b_2, \cdots, b_n\| \leq 1\}$$

$$\leq \epsilon \;\; \forall\, k \geq N.$$

Hence, $T_k \to T$, as $k \to \infty$, in $(X_F^*, \|\cdot\|)$. This completes the proof. ∎

Theorem 3.4. *Let X be a linear n-normed space and T be a b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Then T is bounded if and only if T maps bounded sets in X into bounded sets in $\mathbb{K}$.*

Proof. Suppose T is bounded and S is any bounded subset of X. Then, there exists $M_1 > 0$ such that

$$|T(x, b_2, \cdots, b_n)| \leq M_1 \, \|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X$$

and in particular, $\forall\, x \in S$. The set S being bounded, for some real number $M > 0$, we have

$$|T(x, b_2, \cdots, b_n)| \leq M \;\; \forall\, x \in S \Rightarrow \text{ the set } \{T(x, b_2, \cdots, b_n) : x \in S\}$$

is bounded in $\mathbb{K}$ and hence T maps bounded sets in X into bounded sets in $\mathbb{K}$.

Conversely, for the closed unit ball

$$B_{\{e_2, \cdots, e_n\}}[0, 1] = \{x \in X : \|x, e_2, \cdots, e_n\| \leq 1\},$$

the set $\{T(x, b_2, \cdots, b_n) : x \in B_{\{e_2, \cdots, e_n\}}[0, 1]\}$ is bounded set in $\mathbb{K}$. Therefore, there exists $K > 0$ such that

$$|T(x, b_2, \cdots, b_n)| \leq K \quad \forall x \in B_{\{e_2, \cdots, e_n\}}[0, 1].$$

If $x = 0$, then $T(x, b_2, \cdots, b_n) = 0$ and the assertion

$|T(x, b_2, \cdots, b_n)| \leq K \|x, b_2, \cdots, b_n\|$ is obvioully true. If $x \neq 0$, then

$\dfrac{x}{\|x, e_2, \cdots, e_n\|} \in B_{\{e_2, \cdots, e_n\}}[0, 1]$, and for particular $e_2 = b_2, \cdots, e_n = b_n$

$$\left| T\left(\frac{x}{\|x, b_2, \cdots, b_n\|}, b_2, \cdots, b_n \right) \right| \leq K$$

$$\Rightarrow |T(x, b_2, \cdots, b_n)| \leq K \|x, b_2, \cdots, b_n\| \quad \forall x \in X.$$

Hence, T is a bounded b-linear functional. ■

Theorem 3.5. *Let X be a linear n-normed space and T be a b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Then T is bounded if the set Ker$(T) = \{x \in X : T(x, b_2, \cdots, b_n) = 0\}$ is closed.*

Proof. Let $T : X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to \mathbb{K}$ be a b-linear functional. Suppose T is not the zero b-linear functional. Clearly, $X - \text{Ker}(T) \neq \phi$. Then, there exists $y \in X - \text{Ker}(T)$ such that $T(y, b_2, \cdots, b_n) \neq 0$. Letting $z = \dfrac{y}{T(y, b_2, \cdots, b_n)}$, note that $T(z, b_2, \cdots, b_n) = 1$ and $z \in X - \text{Ker}(T)$. Since $X - \text{Ker}(T)$ is open, there exist $e_2, \cdots, e_n \in X$ and $r > 0$ such that

$$B_{\{e_2, \cdots, e_n\}}(z, r) \subset X - \text{Ker}(T).$$

Now, we shall first prove that

$$|T(x, b_2, \cdots, b_n)| < 1 \quad \forall x \in B_{\{e_2, \cdots, e_n\}}(0, r).$$

If possible suppose there exists some $x_1 \in B_{\{e_2, \cdots, e_n\}}(0, r)$ such that

$$|T(x_1, b_2, \cdots, b_n)| \geq 1.$$

Then, for $t = \dfrac{-x_1}{T(x_1, b_2, \cdots, b_n)}$, we have $T(t, b_2, \cdots, b_n) = -1$ and

$$\|t, e_2, \cdots, e_n\| = \left\| \frac{-x_1}{T(x_1, b_2, \cdots, b_n)}, e_2, \cdots, e_n \right\|$$

$$= \frac{1}{|T(x_1, b_2, \cdots, b_n)|} \|x_1, e_2, \cdots, e_n\| \leq \|x_1, e_2, \cdots, e_n\|.$$

Thus, $t \in B_{\{e_2, \cdots, e_n\}}(0, r)$ and this implies $z + t \in B_{\{e_2, \cdots, e_n\}}(z, r)$. Also,

$$T(z + t, b_2, \cdots, b_n) = T(z, b_2, \cdots, b_n) + T(t, b_2, \cdots, b_n) = 0$$

$$\Rightarrow z + t \in \text{Ker}(T)$$

Therefore,

$$\text{Ker}(T) \cap B_{\{e_2, \cdots, e_n\}}(z, r) \neq \phi$$

This contradicts the fact that $B_{\{e_2, \cdots, e_n\}}(z, r) \subset X - \text{Ker}(T)$. Thus

$$|T(x, b_2, \cdots, b_n)| < 1 \;\forall x \in B_{\{e_2, \cdots, e_n\}}(0, r). \tag{3.3}$$

Now, take any $x \neq 0$. Then

$$\left\| \frac{r\,x}{2\,\|\,x,\,e_2,\,\cdots,\,e_n\,\|},\,e_2,\,\cdots,\,e_n \right\| = \frac{r}{2\,\|\,x,\,e_2,\,\cdots,\,e_n\,\|}\,\|\,x,\,e_2,\,\cdots,\,e_n\,\| = \frac{r}{2} < r$$

$$\Rightarrow \frac{r\,x}{2\,\|\,x,\,e_2,\,\cdots,\,e_n\,\|} \in B_{\{e_2,\cdots,e_n\}}(0,\,r).$$

Then by (3.3), for particular $e_2 = b_2, \cdots, e_n = b_n$

$$\left| T\left(\frac{r\,x}{2\,\|\,x,\,b_2,\,\cdots,\,b_n\,\|},\,b_2,\,\cdots,\,b_n \right) \right| < 1$$

$$\Rightarrow \frac{r}{2\,\|\,x,\,b_2,\,\cdots,\,b_n\,\|}\,|\,T\,(\,x,\,b_2,\,\cdots,\,b_n\,)\,| < 1$$

$$\Rightarrow |\,T\,(\,x,\,b_2,\,\cdots,\,b_n\,)\,| < \frac{2}{r}\,\|\,x,\,b_2,\,\cdots,\,b_n\,\|.$$

Also, if $x = 0$, we have

$$|\,T\,(\,x,\,b_2,\,\cdots,\,b_n\,)\,| = 0 = \|\,x,\,b_2,\,\cdots,\,b_n\,\|$$

$$\Rightarrow |\,T\,(\,x,\,b_2,\,\cdots,\,b_n\,)\,| \leq \frac{2}{r}\,\|\,x,\,b_2,\,\cdots,\,b_n\,\|.$$

Thus, we have shown that

$$|\,T\,(\,x,\,b_2,\,\cdots,\,b_n\,)\,| \leq \frac{2}{r}\,\|\,x,\,b_2,\,\cdots,\,b_n\,\| \quad \forall\, x \in X.$$

Hence, T is a bounded b-linear functional defined on $X \times \langle\, b_2\,\rangle \times \cdots \times \langle\, b_n\,\rangle$. ■

Theorem 3.6. *Let T be a non-zero b-linear functional defined on $X \times \langle\, b_2\,\rangle \times \cdots \times \langle\, b_n\,\rangle$, where X be a linear n-normed space and let $x_0 \in X - Ker\,(\,T\,)$. Then any $x \in X$ can be expressed uniquely in the form $x = y + \alpha\, x_0$, where $y \in Ker\,(\,T\,)$ and α is some scalar.*

Proof. Since $x_0 \in X - \text{Ker}\,(\,T\,)$, so $T\,(\,x_0\,,\,b_2\,,\,\cdots\,,\,b_n\,) \neq 0$. Take

$$\alpha = \frac{T\,(\,x\,,\,b_2\,,\,\cdots\,,\,b_n\,)}{T\,(\,x_0\,,\,b_2\,,\,\cdots\,,\,b_n\,)} \quad \text{and define} \quad y = x - \frac{T\,(\,x\,,\,b_2\,,\,\cdots\,,\,b_n\,)}{T\,(\,x_0\,,\,b_2\,,\,\cdots\,,\,b_n\,)}\,x_0.$$

Then $x = y + \alpha\, x_0$ and $T\,(\,y,\,b_2,\,\cdots,\,b_n\,)$

$$= T\,(\,x,\,b_2,\,\cdots,\,b_n\,) - \frac{T\,(\,x,\,b_2,\,\cdots,\,b_n\,)}{T\,(\,x_0,\,b_2,\,\cdots,\,b_n\,)}\,T\,(\,x_0,\,b_2,\,\cdots,\,b_n\,) = 0.$$

Thus $y \in \text{Ker}\,(\,T\,)$. For the uniqueness, we assume that $x = y + \alpha\, x_0$ and $x = y_1 + \alpha_1\, x_0$. If $\alpha = \alpha_1$, then $y = y_1$. If $\alpha \neq \alpha_1$, then $x_0 = \dfrac{y - y_1}{\alpha - \alpha_1}$ and

$$T\,(\,x_0,\,b_2,\,\cdots,\,b_n\,) = \frac{1}{\alpha - \alpha_1}\,\{T\,(\,y,\,b_2,\,\cdots,\,b_n\,) - T\,(\,y_1,\,b_2,\,\cdots,\,b_n\,)\} = 0.$$

Therefore $x_0 \in \text{Ker}\,(\,T\,)$, which contradicts the assumption that $x_0 \in X - \text{Ker}\,(\,T\,)$. This completes the proof. ■

Theorem 3.7. *Let X be a linear n-normed space and $0 \neq T \in X_F^*$ and*

$$M_T = \{\,x \in X\,:\,T\,(\,x\,,\,b_2\,,\,\cdots\,,\,b_n\,) = 1\,\}.$$

Then M_T is a hyperplane and $\inf\limits_{x \in M_T} \|\,x\,,\,b_2\,,\,\cdots\,,\,b_n\,\| = \dfrac{1}{\|\,T\,\|}$.

Proof. Since T be a non-zero b-linear functional, there exists $x_1 \in X - \text{Ker}(T)$ such that $T(x_1, b_2, \cdots, b_n) \neq 0$. Take $x_0 = \dfrac{x_1}{T(x_1, b_2, \cdots, b_n)}$, then $T(x_0, b_2, \cdots, b_n) = 1$. Now,

$$M_T = \{x \in X : T(x, b_2, \cdots, b_n) = 1 = T(x_0, b_2, \cdots, b_n)\}$$
$$= \{x \in X : T(x - x_0, b_2, \cdots, b_n) = 0\} = x_0 + \text{Ker}(T)$$

and therefore M_T is a hyperplane. Since T is a bounded b-linear functional,

$$|T(x, b_2, \cdots, b_n)| \leq \|T\| \, \|x, b_2, \cdots, b_n\| \;\; \forall\, x \in X.$$

In particular, for all $x \in M_T$,

$$\|T\| \, \|x, b_2, \cdots, b_n\| \geq 1 \Rightarrow \|x, b_2, \cdots, b_n\| \geq \frac{1}{\|T\|}$$
$$\Rightarrow \inf_{x \in M_T} \|x, b_2, \cdots, b_n\| \geq \frac{1}{\|T\|}.$$

Further,

$$\|T\| = \sup\left\{ \frac{|T(x, b_2, \cdots, b_n)|}{\|x, b_2, \cdots, b_n\|} : x \in X, \|x, b_2, \cdots, b_n\| \neq 0 \right\}$$

it follows that, there exists $0 \neq y \in X$ such that

$$\frac{1}{\|T\|} > \left\| \frac{y}{|T(y, b_2, \cdots, b_n)|}, b_2, \cdots, b_n \right\|$$
$$\geq \inf_{x \in M_T} \|x, b_2, \cdots, b_n\| \left[\text{since } \frac{y}{|T(y, b_2, \cdots, b_n)|} \in M_T \right]$$

and hence the result follows. ■

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