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On vertex-edge corona of graphs and its spectral polynomial

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Abstract. Given a graph G_1, the vertex-corona (corona) and the edge-corona focus only on vertices and edges respectively, in forming the corona product with other graphs. In the present work, we define a new corona by considering both vertices and edges simultaneously in forming the corona aproduct with other graphs, called vertex-edge corona. Further, we study the spectral polynomial for the vertex-edge corona of three arbitrary graphs, followed by some corollaries related to regular graphs for their spectrum, energy and equienergetic graphs.

AMS Subject Classifications: 05C50.

Keywords: vertex-corona, edge-corona, vertex-edge corona, spectral polynomial.

Contents

1. Introduction

In 1969, R. Frucht et al. defined a new operation on two graphs G_1 and G_2, called their corona [9] while studying the isomorphism between the group associated with the new graph and the wreath product of the groups G_1 and G_2. The corona of two graphs so defined, focus only on the vertices in forming the corona product with the other graph, hance can be called as vertex-corona. Graph is associated with many concepts like: edges, neighbours, subdivision of edges and more. With the advent of reseacrh various corona products are defined, namely:

1. edge corona (2010) [12],
2. neighbourhood corona (2011) [13],
3. subdivision-vertex and subdivision-edge corona (2013) [18],
4. subdivision-vertex and subdivision-edge neighbourhood coronae (2013) [16],
5. R-vertex, R-edge, R-vertex neighbourhood and R-edge neighbourhood corona (2015) [14],
6. N-vertex, N-edge, C-vertex and C-edge corona (2015) [1],

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7. Extended and extended neighbourhood corona (2016) [2].

The work related to their spectrum and various polynomials can be seen in [4–7, 15, 17, 20].

It is observed that, for a given graph G_1, the vertex-corona (corona) and the edge-corona focus only on vertices and edges, respectively, in forming the corona product with other graphs. We thought to focus both on vertices and edges simultaneously in forming the corona product with other graphs, hence define new corona, called vertex-edge corona, which involves two more graphs G_2 and G_3 one corresponding to vertices and other to edges of G_1. Further, we study the spectral polynomial for the vertex-edge corona of three arbitrary graphs, followed by some corollaries related to regular graphs for their spectrum, energy and equienergetic graphs.

Remarkable observation is that, vertex-edge corona can be considered as the generalization of: vertex-corona, edge-corona, R-vertex corona and C-edge corona, which is possible with the suitable selection of the graphs G_2 and G_3.

2. Preliminaries

All graphs considered here are simple, finite and undirected. If G is a graph on n vertices $v_1, v_2, \ldots, v_n$ and m edges $e_1, e_2, \ldots, e_m$ then its adjacency matrix, $A(G) = [a_{ij}]_{n\times n}$ in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent, and 0 otherwise, and the vertex-edge incidence (incidence) matrix $R(G) = [b_{ij}]_{n\times m}$ in which $b_{ij} = 1$ if the vertex v_i and edge e_j are incident, and 0 otherwise. The polynomial $\phi\Big(A(G); x\Big) = det\Big(xI_n - A(G)\Big)$ associated with $A(G)$ is called the spectral polynomial and the roots of the equation, $\phi\Big(A(G); x\Big) = 0$ are the eigenvalues of G, which constitute the spectrum of G. If G has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $n_1, n_2, \ldots, n_k$ respectively, then we can write: $\begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ n_1 & n_2 & \ldots & n_k \end{pmatrix}$ for the spectrum of G, where $\sum_{i=1}^{k} n_i = n$. The aggregate of the absolute values of these graph eigenvalues, called energy[10] of G is defined as: $\mathcal{E}(G) = \sum\limits_{i=1}^{k} n_i \mid \lambda_i \mid$. The degree of a vertex v_i in G denoted by d_i is the number of edges incident to it, if $d_i = r$ (a constant) for all the vertices v_i then G is called an r-regular graph. If G is r-regular graph, then $R(G)R(G)^T = A(G) + rI_n$. The Kronecker product $C \otimes D$ of two matrices $C = [c_{ij}]_{m\times n}$ and $D = [d_{ij}]_{p\times q}$ is the $mp \times nq$ matrix obtained from C by replacing each entry c_{ij} by $c_{ij}D$ [11]. For matrices C, D, E and F such that products CE and DF exist, $(C \otimes D)(E \otimes F) = CE \otimes DF$, $(C \otimes D)^{-1} = C^{-1} \otimes D^{-1}$ and $(C \otimes D)^T = C^T \otimes D^T$. 1_n denotes the column vector of dimension n. $K_n, K_{p,q}$ denotes complete graph and complete bipartite graphs respectively. Zero order graph is a graph with no vertices. For undefined graph theoretical terminologies and notations, we follow the book [8].

Proposition 2.1. *(Schur Complement [3]) Suppose that the order of all four matrices D_{11}, D_{12}, D_{21} and D_{22} satisfy the rules of operations on matrices. Then we have,*

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = \begin{cases} |D_{22}|\ |D_{11} - D_{12}D_{22}^{-1}D_{21}|, & \text{if } D_{22} \text{ is a non-singular matrix,} \\ |D_{11}|\ |D_{22} - D_{21}D_{11}^{-1}D_{12}|, & \text{if } D_{11} \text{ is a non-singular matrix.} \end{cases}$$

Definition 2.2. *[9] Given a graph G_1 on n_1 vertices, the vertex-corona (corona) $G_1 \circ G_2$ of G_1 with the graph G_2 is the graph obtained by taking one copy of G_1 and n_1 copies of G_2, then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.*

Definition 2.3. *[12] Given a graph G_1 with m_1 edges, the edge-corona $G_1 \diamond G_2$ of G_1 with the graph G_2 is the graph obtained by taking one copy of G_1 and m_1 copies of G_2, then joining two end vertices of the i^{th} edge of G_1 to every vertex in the i^{th} copy of G_2.*

Definition 2.4. *[19] Given a graph G on n vertiecs with the graph matrix M, where M is viewed as a matrix over the field of rational functions $\mathbb{C}(x)$ with $det(xI_n - M)$ non zero. The M-coronal $\Gamma_M(x) \in \mathbb{C}(x)$ of G is, $\Gamma_M(x) = 1_n^T \left(xI_n - M\right)^{-1} 1_n$. If M has a constant row sum r, $\Gamma_M(x) = \dfrac{n}{x-r}$.*

3. Vertex-edge corona of graphs

Given a graph G_1, the vertex-corona (corona) and the edge-corona focus only on vertices and edges, respectively, in forming the corona product with other graphs. Our prime purpose here is to focus both on vertices and edges simultaneously in forming the corona product with other graphs, hence define a new corona, called vertex-edge corona, which involves two more graphs G_2 and G_3 one corresponding to vertices and other to edges of G_1.

Definition 3.1. *Let G_1, G_2, G_3 be any three graphs on n_1, n_2, n_3 vertices and m_1, m_2, m_3 edges, respectively. The vertex-edg corona $G_1 \circ G_2 \diamond G_3$ of G_1, G_2 and G_3 is the graph obtained by taking one copy of G_1, n_1 copies of G_2 and m_1 copies of G_3, then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 and two end vertices of the i^{th} edge of G_1 to every vertex in the i^{th} copy of G_3.*

It is noted that $G_1 \circ G_2 \diamond G_3$ has $n_1 + n_1n_2 + m_1n_3$ vertices and $m_1 + n_1m_2 + m_1m_3 + n_1n_2 + 2m_1n_3$ edges.

Example 3.2. *Let P_n denotes the path on n vertices. Figure 1 depicts $P_4 \circ P_3 \diamond P_2$.*

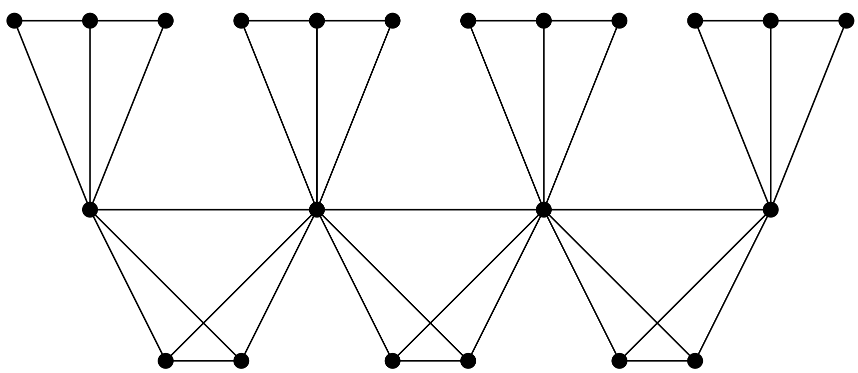

Figure 1:

In the following section, we study the spectral polynomial for the vertex-edge corona of three arbitrary graphs, followed by some corollaries related to regular graphs for their spectrum and energy. We also construct infinitely many pairs of cospectral graphs by applying these results.

4. Spectral polynomial of vertex-edge corona of three graphs

Theorem 4.1. *Let G_1, G_2, G_3 be any theree graphs on n_1, n_2, n_3 vertices respectively. If G_1 has m_1 edges then the spectral polynomial of the vertex-edge corona $G_1 \circ G_2 \diamond G_3$ of three graphs is*

$$\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big)$$
$$= \phi\Big(A\left(G_2\right);\, x\Big)^{n_1} \phi\Big(A\left(G_3\right);\, x\Big)^{m_1} det\Big(xI_{n_1} - A\left(G_1\right) - \Gamma_{A(G_3)}\left(x\right) R\left(G_1\right) R\left(G_1\right)^T - \Gamma_{A(G_2)}\left(x\right) I_{n_1}\Big).$$

Proof. The general adjacency matrix of the vertex-edge corona $G_1 \circ G_2 \diamond G_3$ of G_1, G_2, G_3 on n_1, n_2, n_3 vertices respectively with m_1 edges in G_1 is,

$$A\left(G_1 \circ G_2 \diamond G_3\right) = \begin{pmatrix} A(G_1) & I_{n_1} \otimes 1_{n_2}^T & R(G_1) \otimes 1_{n_3}^T \\ I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes A(G_2) & O_{n_1n_2 \times n_3m_1} \\ R(G_1)^T \otimes 1_{n_3} & O_{n_3m_1 \times n_1n_2} & I_{m_1} \otimes A(G_3) \end{pmatrix}.$$

The spectral polynomial is,

$$\begin{aligned}
&\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big)\\
&= det\Big(x\, I_{n_1+n_1n_2+m_1n_3} - A\left(G_1 \circ G_2 \diamond G_3\right)\Big)\\
&= det\left(\begin{array}{cc|c} xI_{n_1} - A(G_1) & -I_{n_1}\otimes 1_{n_2}^T & -R(G_1)\otimes 1_{n_3}^T \\ -I_{n_1}\otimes 1_{n_2} & I_{n_1}\otimes\Big(xI_{n_2} - A(G_2)\Big) & O_{n_1n_2\times n_3m_1} \\ \hline -R(G_1)^T\otimes 1_{n_3} & O_{n_3m_1\times n_1n_2} & I_{m_1}\otimes\Big(xI_{n_3} - A(G_3)\Big) \end{array}\right).
\end{aligned}$$

Applying Proposition 2.1, we have

$$\begin{aligned}
\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big) &= \phi\Big(A\left(G_3\right);\, x\Big)^{m_1}\\
&\quad det\left(\begin{pmatrix} xI_{n_1} - A(G_1) & -I_{n_1}\otimes 1_{n_2}^T \\ -I_{n_1}\otimes 1_{n_2} & I_{n_1}\otimes\Big(xI_{n_2} - A(G_2)\Big) \end{pmatrix} - S\right)
\end{aligned}$$

where,

$$\begin{aligned}
S &= \begin{pmatrix} -R(G_1)\otimes 1_{n_3}^T \\ O_{n_1n_2\times n_3m_1} \end{pmatrix}\Big(I_{m_1}\otimes\Big(xI_{n_3} - A(G_3)\Big)\Big)^{-1}\left(-R(G_1)^T\otimes 1_{n_3}\ O_{n_3m_1\times n_1n_2}\right)\\
&= \begin{pmatrix} R(G_1)R(G_1)^T\otimes 1_{n_3}^T\Big(xI_{n_3} - A(G_3)\Big)^{-1}1_{n_3} & O \\ O & O \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big) &= \phi\Big(A\left(G_3\right);\, x\Big)^{m_1}\\
&\quad det\left(\begin{array}{c|c} xI_{n_1} - A(G_1) - S_{11} & -I_{n_1}\otimes 1_{n_2}^T \\ \hline -I_{n_1}\otimes 1_{n_2} & I_{n_1}\otimes\Big(xI_{n_2} - A(G_2)\Big) \end{array}\right),
\end{aligned}$$

where $S_{11} = R(G_1)R(G_1)^T\otimes 1_{n_3}^T\Big(xI_{n_3} - A(G_3)\Big)^{-1}1_{n_3}$.
Again applying Proposition 2.1,

$$\begin{aligned}
\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big) &= \phi\Big(A\left(G_3\right);\, x\Big)^{m_1}\phi\Big(A\left(G_2\right);\, x\Big)^{n_1}\\
&\quad det\Big(xI_{n_1} - A(G_1) - S_{11} - I_{n_1}\otimes 1_{n_2}^T\Big(xI_{n_2} - A(G_2)\Big)^{-1}1_{n_2}\Big)\\
&= \phi\Big(A\left(G_3\right);\, x\Big)^{m_1}\phi\Big(A\left(G_2\right);\, x\Big)^{n_1}\\
&\quad det\Big(xI_{n_1} - A(G_1) - R(G_1)R(G_1)^T\Gamma_{A(G_3)}(x) - I_{n_1}\Gamma_{A(G_2)}(x)\Big),
\end{aligned}$$

on re-arrangement result follows. ∎

Corollary 4.2. *If G_1, G_2, G_3 are all r_1, r_2, r_3 regular graphs, respectively. If $r_1 = \lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ are the eigenvalues of G_1, then*

$$\begin{aligned}
\phi\Big(A\left(G_1 \circ G_2 \diamond G_3\right);\, x\Big) &= \frac{\phi\Big(A\left(G_3\right);\, x\Big)^{m_1}\phi\Big(A\left(G_2\right);\, x\Big)^{n_1}}{(x - r_3)^{n_1}(x - r_2)^{n_1}}\\
&\quad \prod_{i=1}^{n_1}\Big[x^3 - (\lambda_i + r_2 + r_3)x^2 + (\lambda_i r_2 + \lambda_i r_3 - \lambda_i n_3 + r_2 r_3 - r_1 n_3 - n_2)x\\
&\quad +(\lambda_i r_2 n_3 - \lambda_i r_2 r_3 + r_1 r_2 n_3 + n_2 r_3)\Big].
\end{aligned}$$

Proof. Substituting $R(G_1)R(G_1)^T = A(G_1) + r_1 I_{n_1}$, $\Gamma_{A(G_3)}(x) = \dfrac{n_3}{x-r_3}$ and $\Gamma_{A(G_2)}(x) = \dfrac{n_2}{x-r_2}$ in Theorem 4.1, and expanding the determinant interms of λ_i, result follows. ■

Corollary 4.3. *If G_1, G_2, G_3 are all r_1, r_2, r_3 regular graphs, respectively. If $r_1 = \lambda_1, \lambda_2, \ldots, \lambda_{n_1}$, $r_2 = \mu_1, \mu_2, \ldots, \mu_{n_2}$, and $r_3 = \delta_1, \delta_2, \ldots, \delta_{n_3}$ are the eigenvalues of G_1,G_2 and G_3 respectively. Then*

$$\mathcal{E}\Big(G_1 \circ G_2 \diamond G_3\Big) = m_1\mathcal{E}(G_3) + n_1\mathcal{E}(G_2) - n_1(r_2 + r_3) + \sum_{i=1}^{n_1}\Big[\mid \gamma_{1i} \mid + \mid \gamma_{2i} \mid + \mid \gamma_{3i} \mid\Big]$$

where $\gamma_{1i}, \gamma_{2i}, \gamma_{3i}$ are roots of the polynomial,

$$\Big[x^3 - (\lambda_i + r_2 + r_3)x^2 + (\lambda_i r_2 + \lambda_i r_3 - \lambda_i n_3 + r_2 r_3 - r_1 n_3 - n_2)x + (\lambda_i r_2 n_3 - \lambda_i r_2 r_3 + r_1 r_2 n_3 + n_2 r_3)\Big].$$

Proof. Equating the polynomial in Corollary 4.2 for eigenvalues and applying the definition of energy, result follows. ■

Corollary 4.4. *If G_1 is an r_1-regular graph, $G_2 = G_3$ is an r_2-regular graph, then*

$$\phi\Big(A\left(G_1 \circ G_2 \diamond G_2\right);\, x\Big) = \frac{\phi\Big(A\left(G_2\right);\, x\Big)^{m_1+n_1}}{(x-r_2)^{n_1}} \prod_{i=1}^{n_1}\Big[x^2 - (\lambda_i + r_2)x + (\lambda_i r_2 - r_1 n_2 - \lambda_i n_2 - n_2)\Big].$$

where $r_1 = \lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ are the eigenvalues of G_1.

Proof. Substituting $G_2 = G_3$, $R(G_1)R(G_1)^T = A(G_1) + r_1 I_{n_1}$, $\Gamma_{A(G_2)}(x) = \dfrac{n_2}{x-r_2}$ in Theorem 4.1 and expanding the determinant interms of λ_i, result follows. ■

Corollary 4.5. *If G_1 is an r_1-regular graph, $G_2 = G_3$ is an r_2-regular graph, then spectrum of $G_1 \circ G_2 \diamond G_2$ is:*

$$\begin{pmatrix} r_2 & \mu_2 & \mu_3 & \ldots & \mu_{n_2} & \dfrac{r_2 + \lambda_i \pm \sqrt{(r_2-\lambda_i)^2 + 4n_2(r_1 + \lambda_i + 1)}}{2} \\ m_1 & m_1 + n_1 & m_1 + n_1 & \ldots & m_1 + n_1 & 1 \end{pmatrix}$$

for $i = 1, 2, \ldots, n_1$. Hence, energy

$$\mathcal{E}\Big(G_1 \circ G_2 \diamond G_2\Big) = (n_1 + m_1)\mathcal{E}(G_2) - n_1 r_2 + \sum_{i=1}^{n}\left|\frac{r_2+\lambda_i \pm \sqrt{(r_2-\lambda_i)^2 + 4n_2(r_1+\lambda_i+1)}}{2}\right|.$$

Proof. Equating the polynomial in Corollary 4.4 to zero for the eigenvalues and applying the definition of energy, result follows. ■

Corollary 4.6. *If G_1 is an r_1-regular graph, and $G_2 = G_3 = K_{p,q}$ with $p \neq q$ then*

$$\phi\Big(A\left(G_1 \circ K_{p,q} \diamond K_{p,q}\right);\, x\Big) = x^{(p+q-2)(n_1+m_1)}(x^2 - pq)^{m_1} \prod_{i=1}^{n_1}\Big[x^3 - \lambda_i x^2 - (pq + pr + qr + p\lambda_i + q\lambda_i + p + q)x - pq(\lambda_i + 2r + 2)\Big].$$

Remark

1. If H_1, H_2 be a pair of cospectral graphs with same order and of same regularity, then for two regular graphs G_2, G_3, the graphs $H_1 \circ G_2 \diamond G_3$ and $H_2 \circ G_2 \diamond G_3$ are also cospectral.
2. In $G_1 \circ G_2 \diamond G_3$:
    - if G_3 is a zero order garph, then resulting corona is vertex-corona (corona).
    - if G_2 is a zero order garph, then resulting corona is edge-corona.
    - if G_3 is K_1, then the resulting corona is R-vertex corona.
    - if G_2 is K_1, then the resulting corona is C-edge corona.

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