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Integrity and vertex neighbor integrity of some graphs

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Abstract. The integrity $I(G)$ of a noncomplete connected graph G is a measure of network vulnerability and is defined by $I(G) = \min_{S\subset V(G)}\{|S| + m(G - S)\}$, where S and $m(G - S)$ denote the subset of V and the order of the largest component of $G - S$, respectively. The vertex neigbor integrity denoted as $VNI(G)$ is the concept of the integrity of a connected graph G and is defined by $VNI(G) = \min_{S\subset V(G)}\{|S| + m(G - S)\}$, where S is any vertex subversion strategy of G and $m(G - S)$ is the number of vertices in the largest component of $G - S$. If a network is modelled as a graph, then the integrity number shows not only the difficulty to break down the network but also the damage that has been caused. This article includes several results on the integrity of the $k - ary\ tree\ H_n^k$, the diamond-necklace N_k, the diamond-chain L_k and the thorn graph of the cycle graph and the vertex neighbor integrity of the H_n^2, H_n^3.

AMS Subject Classifications: 03D20, 05C07, 05C69, 68M10.

Keywords: Integrity, vertex neighbor integrity, vulnerability.

Contents

1. Introduction

Let $G = (V(G), E(G))$ be a graph. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. The number of the neighbor vertices of the vertex v is called degree of v and denoted by $deg_G(v)$. The minimum and maximum degrees of a vertex of G are denoted by $\delta(G)$ and $\Delta(G)$. A vertex v is said to be pendant vertex if $deg_G(v) = 1$. A vertex u is called support if u is adjacent to a pendant vertex [6]. The complement $\overline{G}$ of a graph G is a graph whose vertex set is $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are nonadjacent in G [6].
Let G be a graph and $S \subseteq V(G)$. We denote by $< S >$ the subgraph of G induced by S. A set S is said to be an *independent set* of G, if no pair of vertices of S are adjacent in G. The *independence number* of G, denoted by $\beta(G)$, is the cardinality of a maximum independent set of G. We denote by $\Omega(G)$ the set of all maximum independent sets of G. A vertex and an edge are said to *cover* each other if they are incident. A set of vertices

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which cover all the edges of a graph G is called a *vertex cover* for G, while a set of edges which covers all the vertices is an *edge cover*. The smallest number of vertices in any vertex cover for G is called its *vertex covering number* and is denoted by $\alpha(G)$ [6]. For any graph G of order n, $\alpha(G) + \beta(G) = n$. For a graph G, we denote the minimum number of colors necessary to color G by $\chi(G)$, the chromatic number of the graph G. The connectivity $\kappa = \kappa(G)$ of a graph G is the minimum number of points whose removal results in a disconnected or trivial graph.

The vulnerability of a communication network gives us an idea of the resilience and robustness of the network after some centers in the network have failed. Vulnerability can be measured by certain parameters. In the analysis of the vulnerability of a communication network to disruption, attention is paid to the number of non-working elements and the size of the largest remaining group in it. Especially in a hostile relationship where mutual communication still takes place, it is desirable that the adversary's network be such that the two quantities can be made small at the same time. [1]

One of the parameters used to measure the vulnerability of the graph is the integrity value. Formally, the vertex integrity (frequintly called just the integrity) is

$$I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}$$

where $m(G - S)$ denotes the order of a largest component of $G - S$. This concept was introduced by Barefoot, Entringer and Swart [2], who discovered many of the early results on the subject. If G is a graph of order n, then $1 \leq I(G) \leq n$ and if H is any subgraph of G, then $I(H) \leq I(G)$. The integrity of the binomial trees was given in [8].

The concept of the vertex neighbor-integrity was introduced as a measure of graph vulnerability by M.B.Cozzens and Shu-Shih Y.Wu [4]. Let u be a vertex of a graph $G = (V, E)$. Then $N(u) = \{v \in V(G), v \; and \; u \; are \; adjacent\}$ is the open neighborhood of u, and $N[u] = \{u\} \cup N(u)$ denotes the closed neighborhood of u. A vertex u of a graph G is said to be subverted if the closed neighborhood $N[u]$ is deleted from G. A set of vertices $S = \{u_1, u_2, ..., u_m\}$ is called a vertex subversion strategy of G if each of the vertices in S has been subverted from G. Let $G - S$ be the survival subgraph when S has been a vertex subversion strategy of G. The vertex neighbor integrity of a graph G, $VNI(G)$, is defined to be

$$VNI(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}$$

where S is any vertex subversion strategy of G and $m(G - S)$ is the number of vertices in the largest component of $G - S$. The set S is called the $VNI - set$ of a graph G, which gives its neighbor integrity. The neighbor integrity for total graphs was given in [9]

In this article, we give a recursive formula on the integrity of H_n^k and we calculate the integrity of the diamond-necklace N_k, the diamond-chain L_k and the thorn graph G^* of a cycle C_n. Then, we present a result on the vertex neighbor-integrity of H_n^2 and H_n^3 .

2. Basic Results on Integrity and Vertex Neighbor Integrity

Theorem 2.1. *[1] Define the comet $C_{t,r}$ to be the graph obtained by identifying one end of the path P_t with the center of the star $K_{1,r}$. Then, $I(C_{t,r}) \leq I(C_{t+1,r-1}) \leq I(C_{t,r}) + 1$.*

Theorem 2.2. *[1] The integrity of*
(a) *the complete graph K_p is p;*
(b) *the null graph $\overline{K_p}$ is 1;*
(c) *the star $K_{1,n}$ is 2;*
(d) *the path P_n is $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$.;*

(e) *the cycle C_n is $I(C_n) = \lceil 2\sqrt{n}\rceil - 1$.;*
(f) *the comet $C_{p-r,r}$ is $I(P_p)$, if $r \leq \sqrt{p+1} - \frac{5}{4}$; $\lceil 2\sqrt{p-r}\rceil - 1$, otherwise;*
(g) *the complete bipartite graph $K_{m,n}$ is $1 + minm, n$;*
(h) *any complete multipartite graph of order p and largest partite set of order r is $p - r + 1$.*

Theorem 2.3. *[1] Let G be a graph of order p.*
(a) *$I(G) = 1$ if and only if G is null.*
(b) *$I(G) = 2$ if and only if all nontrivial components of G are edges or the only nontrivial component is a star.*
(c) *$I(G) = p - 1$ if and only if G is not complete and $\overline{G}$ has girth at least 5.*
(d) *$I(G) = p$ if and only if G is complete.*

Theorem 2.4. *[1] If in graph G, v is a vertex for which $deg(v) \geq I(G - v)$, then $I(G) = 1 + I(G - v)$.*

Parameters that will be discussed here include the following:

- δ, the minimum vertex degree;
- κ, the connectivity;
- α, the covering number;
- β, the independence number;
- χ, the chromatic number.

Theorem 2.5. *For any graph G,*
(a) $I(G) \leq \alpha + 1$.
(b) $I(G) \geq \delta + 1$.
(c) $I(G) \geq \chi$.
(d) $I(G) \geq (p - \kappa(G))/\beta(G) + \kappa(G)$.
(e) *$I(G) = \kappa(G) + 1$ if and only if $\kappa(G) = \alpha(G)$;*
(f) *$I(G) = \alpha(G) + 1$ if and only if G does not contain $2K_2$ as an induced subgraph;*
(g) *$I(G) = \delta(G) + 1$ if and only if $G \cong rK_n$ or $G \cong rK_n + F$ for some graph F satisfying $\delta(F) \geq |G| - (2r - 1)n - 1$.*

Theorem 2.6. *[4] Let P_n be the path on n vertices. Then we have*

$$VNI(P_n) = \begin{cases} \lceil 2\sqrt{n+3}\rceil - 4, & if\ n \geq 2 \\ 1, & if\ n = 1 \end{cases}$$

Theorem 2.7. *[4] Let C_n be the $n - cycle$, where $n \geq 3$. Then*

$$VNI(C_n) = \begin{cases} \lceil 2\sqrt{n}\rceil - 3, & if\ n > 4, \\ 2, & if\ n = 4, \\ 1, & if\ n = 3. \end{cases}$$

3. Integrity of H_n^k, N_k, L_k and the Thorn Graph

Integrity of G is defined to be $I(G) = \min\limits_{S \subset V(G)}\{|S| + m(G - S)\}$, where $m(G - S)$ denotes the order of a largest component of $G - S$. In this section we calculated a recursive formula about the integrity of H_n^3 and we give a common result for the integrity of H_n^k.

Definition 3.1. *[3] The complete $k-ary\ tree$ H_n^k of depth n is the rooted tree in which all vertices at level $n-1$ or less have exactly k children, and all vertices at level n are leaves. A $2-ary\ tree$, H_4^2 is illustrated in Figure 1.*

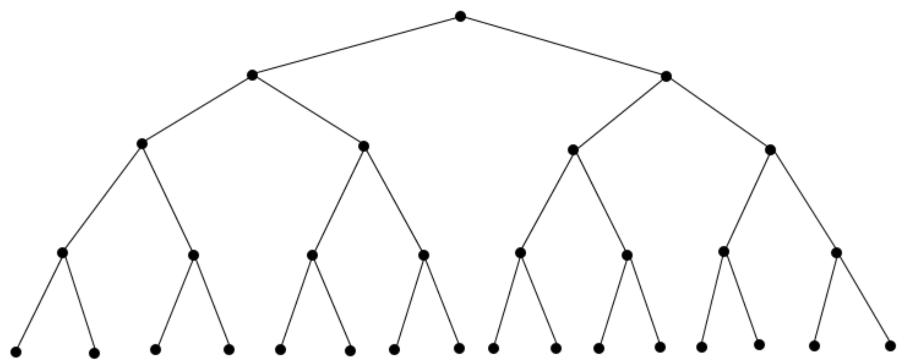

Figure 1: 2-ary tree H_4^2

Theorem 3.2. *The integrity of a complete $3-ary\ tree$ H_n^3 is given by*

$$I(H_n^3)=\begin{cases}3^{\frac{n}{2}}+I(H_{n-2}^3), & if\ n\equiv 0(mod\ 2)\\ 3^{\lfloor\frac{n}{2}\rfloor}+I(H_{n-1}^3), & if\ n\equiv 1(mod\ 2)\end{cases}$$

Proof. **Case 1.** $n\equiv 0(\text{mod } 2)$
In this case S consists of the all vertices at the level $n/2$. Hence, $m(H_n^3-S)=\sum_{i=0}^{(n/2)-1}3^i$, $|S|=3^{n/2}$ and $I(H_n^3)=\sum_{i=0}^{n/2}3^i=\frac{3^{(n/2)+1}-1}{2}$. To express this function as a recursive function we use induction. Certainly $I(H_4^3)=3^2+I(H_2^3)=9+4=13$. Assume $I(H_{n-2}^3)=\frac{3^{\frac{n-2}{2}+1}-1}{2}$, then we need to show that $I(H_n^3)=\frac{3^{(n/2)+1}-1}{2}$.

$$I(H_n^3)=3^{n/2}+I(H_{n-2}^3) \tag{3.1}$$
$$=3^{n/2}+\frac{3^{\frac{n-2}{2}+1}-1}{2} \tag{3.2}$$
$$=3^{n/2}+\frac{3^{\frac{n}{2}}-1}{2} \tag{3.3}$$
$$=\frac{3^{\frac{n}{2}+1}-1}{2} \tag{3.4}$$

Case 2. $n\equiv 1(\text{mod } 2)$
In this case S consists of the all vertices at the level $\lfloor n/2\rfloor$. Hence, $m(H_n^3-S)=\sum_{i=0}^{\lfloor(n/2)\rfloor}3^i$, $|S|=3^{\lfloor n/2\rfloor}$ and $I(H_n^3)=\frac{3^{\lfloor(n/2)\rfloor+1}-1}{2}+3^{\lfloor n/2\rfloor}$. To express this function as a recursive function we use induction. Certainly $I(H_3^3)=3+I(H_2^3)=3+4=7$. Assume $I(H_{n-1}^3)=\frac{3^{\lfloor\frac{n}{2}\rfloor-1+1}-1}{2}+3^{\lfloor\frac{n}{2}\rfloor}$, then we need to show that $I(H_n^3)=\frac{3^{\lfloor(n/2)\rfloor+1}-1}{2}+3^{\lfloor\frac{n}{2}\rfloor}$.

$$I(H_n^3) = 3^{\lfloor n/2 \rfloor} + I(H_{n-1}^3) \tag{3.5}$$

$$= 3^{\lfloor n/2 \rfloor} + \frac{3^{\lfloor \frac{n}{2} \rfloor} - 1}{2} + 3^{\lfloor n/2 \rfloor} \tag{3.6}$$

$$= \frac{3.3^{\lfloor n/2 \rfloor} - 1}{2} + 3^{\lfloor n/2 \rfloor} \tag{3.7}$$

$$= \frac{3^{\lfloor (n/2) \rfloor + 1} - 1}{2} + 3^{\lfloor \frac{n}{2} \rfloor} \tag{3.8}$$

The proof is completed. ■

Corollary 3.3. *The integrity of a complete $k-ary\ tree$ H_n^k, $k \geq 3$ is given by*

$$I(H_n^k) = \begin{cases} k^{\frac{n}{2}} + I(H_{n-2}^k), & if\ n \equiv 0(mod\ 2) \\ k^{\lfloor \frac{n}{2} \rfloor} + I(H_{n-1}^k), & if\ n \equiv 1(mod\ 2) \end{cases}$$

Definition 3.4. *[7] For $k \geq 2$ an integer, let $D_1, D_2, ..., D_k$ be k disjoint copies of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and where $a_i b_i$ is the missing edge in D_i. N_k is obtained from the disjoint union of these k diamonds by adding the edges $\{a_i b_{i+1} | i = 1, 2, ..., k-1\}$ and adding the edge $a_k b_1$. We call N_k a diamond-necklace with k diamonds.*

Definition 3.5. *[7] For $k \geq 1$, we define a diamond-chain L_k with k diamonds as follows. Let L_k be obtained from a diamond-necklace N_{k+1} with $k+1$ diamonds $D_1, D_2, ..., D_{k+1}$ by removing the diamond D_{k+1} and adding two disjoint triangles T_1 and T_2 and adding an edge joining b_1 to a vertex of T_1 and adding an edge joining a_k to a vertex of T_2. A diamond-necklace, N_6, with six diamonds and a diamond-chain, L_2, with two diamonds is illustrated in Figure 2.*

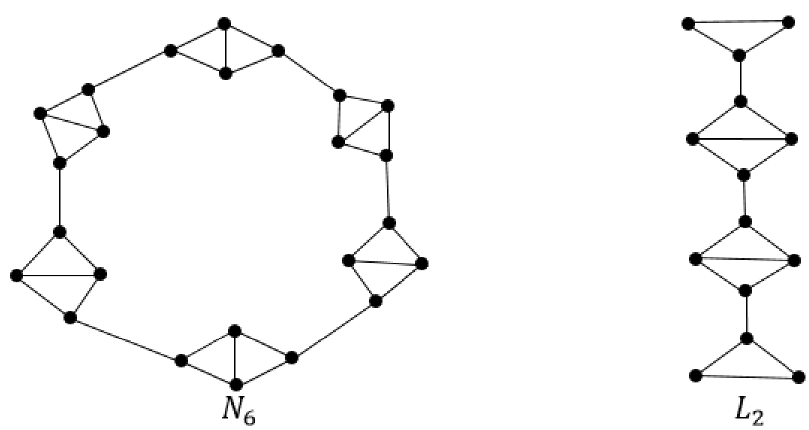

Figure 2: A diamond-necklace N_6 and a diamond-chain L_2

Theorem 3.6. *Let N_k be a diamond-necklace with k vertices.The integrity of N_k is $I(N_k) = \lceil 4\sqrt{k} - 1 \rceil$.*

Proof. For the set S, we choose the vertices among a_i, b_i , where $i \in \{\overline{1,k}\}$ to minimalized the value $|S| + m(N_k - S)$. So, if we remove r vertices from N_k, then we have r components. Number of vertices of a largest component $m(N_k - S) \geq \frac{4k-r}{r}$.

$$I(N_k) = \min_{S \subset V(N_k)} \{|S| + m(N_k - S)\} \tag{3.9}$$
$$\geq \min_{r} \{r + \frac{4k-r}{r}\} \tag{3.10}$$

The function $f(r) = r + \frac{4k-r}{r}$ takes its minimum value at $r = 2\sqrt{k}$. We substitute the minimum value in the function $f(r)$. As the integrity is integer valued, we round this up to get a lower bound and obtain $I(N_k) = \lceil 4\sqrt{k} - 1 \rceil$. ■

Theorem 3.7. *Let L_k be a diamond-chain with $4k + 6$ vertices.The integrity of L_k is $I(L_k) = \lceil \frac{(-1+\sqrt{4k+7})^2+4k+6}{\sqrt{4k+7}} \rceil$.*

Proof. For the set S, we choose the vertices among a_i, b_i and the vertex of order 3 in T_2, where $i \in \{\overline{1,k}\}$ to minimalized the value $|S| + m(L_k - S)$. So, if we remove r vertices from L_k, then we have $r + 1$ components. Number of vertices of a largest component $m(L_k - S) \geq \frac{4k+6-r}{r+1}$.

$$I(L_k) = \min_{S \subset V(L_k)} \{|S| + m(L_k - S)\} \tag{3.11}$$
$$\geq \min_{r} \{r + \frac{4k+6-r}{r+1}\} \tag{3.12}$$

The function $f(r) = \frac{r^2+4k+6}{r+1}$ takes its minimum value at $r = -1 + \sqrt{4k+7}$. We substitute the minimum value in the function $f(r)$. As the integrity is integer valued, we round this up to get a lower bound and obtain $I(L_k) = \lceil \frac{(-1+\sqrt{4k+7})^2+4k+6}{\sqrt{4k+7}} \rceil$. ■

Definition 3.8. *[5] Let $p_1, p_2, ..., p_n$ be non-negative integers and G be such a graph, $V(G) = n$. The thorn graph of the graph G, with parameters $p_1, p_2, ..., p_n$, is obtained by attaching p_i new vertices of degree 1 to the vertex u_i of the graph G, $i = 1, 2, ..., n$. The thorn graph of the graph G will be denoted by G^* or by $G^*(p_1, p_2, ..., p_n)$, if the respective parameters need to be specified. The thorn graph G^* of the cycle C_6 is illustrated in Figure 3.*

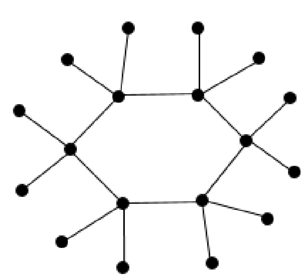

Figure 3: The thorn graph G^*

Theorem 3.9. *If G^* is a thorn graph of C_n with $p_1 = p_2 = ... = p_n = p$, then $I(G^*) = 2\lceil \sqrt{n}\sqrt{p+1} \rceil - p - 1$.*

Proof. For the set S, we choose the vertices of order $p+2$ in C_n to minimalized the value $|S|+m(G^*-S)$. So, if we remove r vertices from C_n, then we have r components. Number of vertices of a largest component $m(C_n-S) \geq \frac{(n-r)p+n-r}{r}$.

$$I(N_k) = \min_{S \subset V(G^*)} \{|S| + m(G^* - S)\} \tag{3.13}$$

$$\geq \min_r \{r + \frac{(n-r)p+n-r}{r}\} \tag{3.14}$$

The function $f(r) = r + \frac{(n-r)p+n-r}{r}$ takes its minimum value at $r = \sqrt{np+n}$. Hence if we substitute the minimum value in the function $f(r)$, we have $I(C_n) = 2\sqrt{n}\sqrt{p+1} - p - 1$. Since the integrity is integer valued, we round this up to get a lower bound and obtain $I(C_n) = 2\lceil \sqrt{n}\sqrt{p+1} \rceil - p - 1$. ■

4. Vertex Neighbor Integrity of H_n^2 and H_n^3

The vertex neighbor integrity of G is defined to be $VNI(G) = \min_{S \subset V(G)} \{|S| + m(G-S)\}$, where $m(G-S)$ is the largest connected component in the graph $G-S$. Here, we calculated the $VNI(H_n^2)$ and $VNI(H_n^3)$.

Theorem 4.1. *The vertex neighbor integrity of a complete* $2-ary\ tree$ H_n^2 *is given by*

$$VNI(H_n^2) = \lfloor \frac{\sqrt{2^{n+3}-3}-1}{2} \rfloor$$

Proof. We have $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ vertices in H_n^2. If we remove $|S| = r$ vertices from H_n^2, then we have $4r+1$ components in $H_n^2 - S$. Hence, $m(H_n^2 - S) \geq \frac{2^{n+1}-1-r}{4r+1}$.

$$VNI(H_n^2) = \min_{S \subset V(H_n^2)} \{|S| + m(H_n^2 - S)\} \tag{4.1}$$

$$\geq \min_r \{r + \frac{2^{n+1}-1-r}{4r+1}\} \tag{4.2}$$

The function $f(r) = r + \frac{2^{n+1}-1-r}{4r+1} = \frac{2^{n+1}+4r^2-1}{4r+1}$ takes its minimum value at $r = \frac{\sqrt{2^{n+3}-3}-1}{4}$. Hence if we substitute the minumum value in the function $f(r)$, we have $VNI(H_n^2) = \lfloor \frac{\sqrt{2^{n+3}-3}-1}{2} \rfloor$.

The proof is completed. ■

Theorem 4.2. *The vertex neighbor integrity of* H_n^3*, is given by*

$$VNI(H_n^3) = \begin{cases} 3^{\frac{n}{2}} + VNI(H_{n-2}^3), & if\ n \equiv 1 (mod\ 2) \\ 3^{\lfloor \frac{n}{2} \rfloor} + VNI(H_{n-3}^3), & if\ n \equiv 0 (mod\ 2) \end{cases}$$

Proof. The proof is similar to the proof of Theorem 3.2. ■

The following table gives the vertex neighbor integrity values of H_n^2 and H_n^3 for $n = 2, ..., 10$.

Table 1: The vertex neighbor integrity of H_n^2 and H_n^3

n	2	3	4	5	6	7	8	9	10
H_n^2	2	3	5	7	11	15	23	31	47
H_n^3	3	4	7	13	31	40	94	121	283

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