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Fixed point of almost generalized $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$-contractive type mappings in weak partial metric spaces

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. This study introduces almost generalized $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$-contractive type mappings and investigates some fixed point theorems for such mappings in weak partial metric spaces.Our results extend the implications of Altun and Durmaz [15] and other prior results in this area. Additionally, We present some examples that support our findings.

AMS Subject Classifications: 47H10, 54H25.

Keywords: Triangular $\hat{\alpha}$-admissible mapping, almost generalized $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$-contractive type mapping, weak partial metric spaces.

Contents

1. Introduction and Background

Banach fixed point theorem has been expanded in numerous ways and it has undergone numerous generalisations in various metric spaces. Partial metric space (PMS), which Matthews [1] introduced in 1992, is a very intriguing generalisation of the metric space in which the self distance not required to be zero.By establishing a new class of contractive type mappings known as $\hat{\alpha} - \hat{\psi}$ contractive type mappings, Samet et al. [3] further expanded and generalised the Banach contraction principle. The $\hat{\alpha} - \hat{\psi}$ contractive type mappings were generalised by Karapinar and Samet[4].On the other hand, Berinde [7, 8] introduced the concept of almost contractions in metric spaces. The concept of weak partial metric spaces, a generalisation of partial metric spaces, was first introduced by Heckmann [14] in 1999. Some results for mappings in weak partial metric spaces have recently been obtained in [17], [18],[19] and [20].

Definition 1.1. *[12] Let Ψ be the set of functions $\hat{\psi} : [0, \infty) \to [0, \infty)$ such that*

(a) $\hat{\psi}$ *is non decreasing and continuous;*

(b) $\hat{\psi}(u) = 0 \Leftrightarrow u = 0.$

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Definition 1.2. *[3] Let $\Gamma : W_p \to W_p$ and $\hat{\alpha} : W_p \times W_p \to [0, \infty)$. Γ is said to $\hat{\alpha}$-admissible if*

$$\hat{\alpha}(\eta_p, \zeta_p) \geq 1 \Rightarrow \hat{\alpha}(\Gamma\eta_p, \Gamma\zeta_p) \geq 1$$

for all $\eta_p, \zeta_p \in W_p$.

Definition 1.3. *[5] Let $\Gamma : W_p \to W_p$ and $\hat{\alpha} : W_p \times W_p \to [0, \infty)$ be two functions. Then Γ is said to triangular $\hat{\alpha}$-admissible if Γ is $\hat{\alpha}$-admissible and for $\eta_p, \zeta_p, \delta_p \in W_p$, $\hat{\alpha}(\eta_p, \delta_p) \geq 1$ and $\hat{\alpha}(\delta_p, \zeta_p) \geq 1 \Rightarrow \hat{\alpha}(\eta_p, \zeta_p) \geq 1$.*

Lemma 1.4. *[5] Let $\Gamma : W_p \to W_p$ be a triangular $\hat{\alpha}$-admissible mapping. Suppose that there exists $\eta_{p_0} \in W_p$ such that $\hat{\alpha}(\eta_{p_0}, \Gamma\eta_{p_0}) \geq 1$.If we define a sequence $\{\eta_{p_i}\}$ by $\eta_{p_{i+1}} = \Gamma\eta_{p_i}$ for every $i \in \mathbb{N}_0$. Then we have $\hat{\alpha}(\eta_{p_j}, \eta_{p_i}) \geq 1$ for all $j, i \in \mathbb{N}$ with $j > i$.*

In 1992, Matthews [1] presented generalization of metric space as follows:

Definition 1.5. *([1]) Let W_p be a set which is non-empty. A mapping $\mathfrak{d}_\varrho : W_p \times W_p \to [0, \infty)$ is known as partial metric on W_p if the following conditions are satisfied:*

(PMS1) $\eta_p = \zeta_p \Leftrightarrow \mathfrak{d}_\varrho(\eta_p, \eta_p) = \mathfrak{d}_\varrho(\zeta_p, \zeta_p) = \mathfrak{d}_\varrho(\eta_p, \zeta_p)$;

(PMS2) $\mathfrak{d}_\varrho(\eta_p, \eta_p) \leq \mathfrak{d}_\varrho(\eta_p, \zeta_p)$;

(PMS3) $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \mathfrak{d}_\varrho(\zeta_p, \eta_p)$:

(PMS4) $\mathfrak{d}_\varrho(\eta_p, \zeta_p) \leq \mathfrak{d}_\varrho(\eta_p, \delta_p) + \mathfrak{d}_\varrho(\delta_p, \zeta_p) - \mathfrak{d}_\varrho(\delta_p, \delta_p)$. *for all $\eta_p, \zeta_p, \delta_p \in W_p$.*

Lemma 1.6. *([1]) Let $(W_p, \mathfrak{d}_\varrho)$ be a partial metric space.*

(a) A sequence $\{\eta_{p_i}\}$ in the space $(W_p, \mathfrak{d}_\varrho)$ converges to a point $\eta_p \in W_p \Leftrightarrow$

$$\mathfrak{d}_\varrho(\eta_p, \eta_p) = \lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_p),$$

(b) If $\lim_{j,i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j})$ exists and finite then the sequence $\{\eta_{p_i}\}$ is a Cauchy sequence in space $(W_p, \mathfrak{d}_\varrho)$,

(c) If every Cauchy sequence $\{\eta_{p_i}\}$ in W_p converges to a point $\eta_p \in W_p$, such that

$$\mathfrak{d}_\varrho(\eta_p, \eta_p) = \lim_{j,i\to\infty} \mathfrak{d}_\varrho(\eta_{p_j}, \eta_{p_i}) = \lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_p) = \mathfrak{d}_\varrho(\eta_p, \eta_p)$$

Then $(W_p, \mathfrak{d}_\varrho)$ is complete.

Lemma 1.7. *([11],[1],[2]) Let $\mathfrak{d}_\varrho$ be a partial metric on W_p, then the mapping $\mathfrak{d}_\varrho^m : W_p \times W_p \to \mathbb{R}^+$ such that*

$$\begin{aligned} \mathfrak{d}_\varrho^m(\eta_p, \zeta_p) =& \max\{\mathfrak{d}_\varrho(\eta_p, \zeta_p) - \mathfrak{d}_\varrho(\eta_p, \eta_p), \mathfrak{d}_\varrho(\eta_p, \zeta_p) - \mathfrak{d}_\varrho(\zeta_p, \zeta_p)\} \\ =& \mathfrak{d}_\varrho(\eta_p, \zeta_p) - \min\{\mathfrak{d}_\varrho(\eta_p, \eta_p), \mathfrak{d}_\varrho(\zeta_p, \zeta_p)\} \end{aligned} \tag{1.1}$$

is metric on W_p. Furthermore, $(W_p, \mathfrak{d}_\varrho^m)$ is metric space.
Let $(W_p, \mathfrak{d}_\varrho^m)$ be a partial metric space. Then

1. *A sequence $\{\eta_{p_i}\}$ in $(W_p, \mathfrak{d}_\varrho^m)$ is a Cauchy sequence $\Leftrightarrow$ $\{\eta_{p_i}\}$ is a Cauchy sequence in the metric space $(W_p, \mathfrak{d}_\varrho^m)$,*

2. *$(W_p, \mathfrak{d}_\varrho^m)$ is complete $\Leftrightarrow$ $(W_p, \mathfrak{d}_\varrho)$ is complete. Moreover*
$\lim_{i\to\infty} \mathfrak{d}_\varrho^m(\eta_{p_i}, \eta_p) = 0 \Leftrightarrow \mathfrak{d}_\varrho(\eta_p, \eta_p) = \lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_p) = \lim_{i,j\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j})$.

Lemma 1.8. *([18]) Suppose that $\{\eta_{p_i}\}$ be a sequence $\eta_{p_i} \to \delta_p$ as $i \to \infty$ in a partial metric space $(W_p, \mathfrak{d}_\varrho)$ such that $\mathfrak{d}_\varrho(\delta_p, \delta_p) = 0$. Then $\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \zeta_p) = \mathfrak{d}_\varrho(\delta_p, \zeta_p)$ for every $\zeta_p \in W_p$.*

Lemma 1.9. *[18] If $\{\eta_{p_i}\}$be a sequence with $\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) = 0$ such that $\{\eta_{p_i}\}$ is not a Cauchy sequence in $(W_p, \mathfrak{d}_\varrho)$, and there exist two sequences $\{i(u)\}$ and $\{j(u)\}$ of positive integers such that $i(u) > j(u) > u$, then following sequences*

$$\mathfrak{d}_\varrho(\eta_{p_{j(u)}}, \eta_{p_{i(u)+1}}), \mathfrak{d}_\varrho(\eta_{p_{j(u)}}, \eta_{p_{i(u)}}),$$
$$\mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)+1}}), \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)}})$$

tend to $\mu_p > 0$ when $u \to \infty$

Lemma 1.10. *([12], [16])Let W_p be a set which is non-empty. Suppose that $(W_p, \mathfrak{d}_\varrho)$ be a partial metric space.*

1. *If $\eta_p \neq \zeta_p$ then $\mathfrak{d}_\varrho(\eta_p, \zeta_p) > 0$,*
2. *if $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = 0$ then $\eta_p = \zeta_p$.*

By omitting the small self-distance axiom in partial metric spaces, Heckmann [14] introduced the concept of weak partial metric space as follows:

Definition 1.11. *[14] Let W_p be a set which is non-empty . A mapping $\mathfrak{d}_\varrho : W_p \times W_p \to [0, \infty)$ is known as weak partial metric on W_p if the following conditions are satisfied:*

(WPMS1) $\eta_p = \zeta_p \Leftrightarrow \mathfrak{d}_\varrho(\eta_p, \eta_p) = \mathfrak{d}_\varrho(\zeta_p, \zeta_p) = \mathfrak{d}_\varrho(x, \zeta_p)$ *;*

(WPMS2) $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \mathfrak{d}_\varrho(\zeta_p, \eta_p)$;

(WPMS3) $\mathfrak{d}_\varrho(\eta_p, \zeta_p) \leq \mathfrak{d}_\varrho(\eta_p, \delta_p) + \mathfrak{d}_\varrho(\delta_p, \zeta_p) - \mathfrak{d}_\varrho(\delta_p, \delta_p)$. *for all $\eta_p, \zeta_p, \delta_p \in W_p$.*

and the pair $(W_p, \mathfrak{d}_\varrho)$ is called weak partial metric space (in short WPMS).
Additionally, Heckmann [14] demonstrates that the weak small self-distance feature follows if $\mathfrak{d}_\varrho$ is a weak partial metric on W_p i.e.

$$\mathfrak{d}_\varrho(\eta_p, \zeta_p) \geq \frac{\mathfrak{d}_\varrho(\eta_p, \eta_p) + \mathfrak{d}_\varrho(\zeta_p, \zeta_p)}{2}$$

for all $\eta_p, \zeta_p \in W_p$.
Every partial metric space is obviously a weak partial metric space, but the converse may not be true. For example, for $\eta_p, \zeta_p \in \mathbb{R}$ the function $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \frac{e^{\eta_p} + e^{\zeta_p}}{2}$ is a weak partial metric space but not a partial metric on $\mathbb{R}$.

Lemma 1.12. *[15] Let $(W_p, \mathfrak{d}_\varrho)$ be a weak partial metric space(WPMS).*

(i) *$\{\eta_{p_i}\}$ is a Cauchy sequence in $(W_p, \mathfrak{d}_\varrho) \Leftrightarrow$ it is a Cauchy sequence in $(W_p, \mathfrak{d}_\varrho^m)$;*

(ii) *$(W_p, \mathfrak{d}_\varrho)$ is complete $\Leftrightarrow (W_p, \mathfrak{d}_\varrho^m)$ is complete.*

Lemma 1.13. *[17] Let $(W_p, \mathfrak{d}_\varrho)$ be a weak partial metric space and $\{\eta_{p_i}\}$ is a sequence in $(W_p, \mathfrak{d}_\varrho)$. If $\lim_{i\to\infty} \eta_{p_i} = \eta_p$ and $\mathfrak{d}_\varrho(\eta_p, \eta_p) = 0$, then $\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \zeta_p) = \mathfrak{d}_\varrho(\eta_p, \zeta_p)$, for all $\zeta_p \in W_p$.*

Definition 1.14. *[13] Let Φ be the set of all functions $\hat{\varphi} : [0, \infty) \to [0, \infty)$ satisfying the following conditions:*

(i) *$\hat{\varphi}(u) < \hat{\psi}(u)$ for all $u > 0$*

(ii) $\hat{\varphi}(0) = 0$

Definition 1.15. *[21] Let Θ be the set of functions $\hat{\vartheta} : [0, \infty) \to [0, \infty)$ such that*

(i) *$\hat{\vartheta}$ is continuous;*

(ii) $\hat{\vartheta}(u) = 0 \Leftrightarrow u = 0$.

Remark 1.16. *The convergence of sequences, Cauchy sequences, and completeness in a weak partial metric space are defined as being in a partial metric space.*

2. Main Results

Definition 2.1. *Let $(W_p, \mathfrak{d}_\varrho)$ be a weak partial metric space and $\Gamma : W_p \to W_p$ be a given self map. We say that Γ is almost generalized $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$-contractive mapping if there exists $\hat{\alpha} : W_p \times W_p \to [0, \infty)$ and $\hat{\psi} \in \Psi$, $\hat{\varphi} \in \Phi$, $\hat{\vartheta} \in \Theta$ and $L \geq 0$ such that for all $\eta_p, \zeta_p \in W_p$ we have*

$$\hat{\alpha}(\eta_p, \zeta_p)\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p)) \leq \hat{\varphi}(\tilde{\mathcal{M}}(\eta_p, \zeta_p)) + L\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_p, \zeta_p)) \tag{2.1}$$

Where

$$\tilde{\mathcal{M}}(\eta_p, \zeta_p) = \max\left\{\mathfrak{d}_\varrho(\eta_p, \zeta_p), \mathfrak{d}_\varrho(\eta_p, \Gamma\eta_p), \mathfrak{d}_\varrho(\zeta_p, \Gamma\zeta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p, \Gamma\zeta_p) + \mathfrak{d}_\varrho(\zeta_p, \Gamma\eta_p)]\right\} \tag{2.2}$$

and

$$\tilde{\mathcal{N}}(\eta_p, \zeta_p) = \min\{\mathfrak{d}_\varrho^m(\eta_p, \Gamma\eta_p), \mathfrak{d}_\varrho^m(\zeta_p, \Gamma\eta_p)\} \tag{2.3}$$

Theorem 2.2. *Let $(W_p, \mathfrak{d}_\varrho)$ be a complete weak partial metric space and $\Gamma : W_p \to W_p$ be self mapping. Suppose $\hat{\alpha} : W_p \times W_p \to [0, \infty)$ be the mapping satisfying the conditions:*

(i) Γ *is triangular* $\hat{\alpha}$*-admissible;*

(ii) Γ *is almost generalized* $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$*-contractive mapping;*

(iii) There exists $\eta_{p_0} \in W_p$ *such that* $\hat{\alpha}(\eta_{p_0}, \Gamma\eta_{p_0}) \geq 1$*;*

(iv) Γ *is continuous.*

Then Γ has a fixed point in W_p.

Proof. Let there be an arbitrary point η_{p_0} such that $\hat{\alpha}(\eta_{p_0}, \Gamma\eta_{p_0}) \geq 1$. Suppose there is a sequence $\{\eta_{p_i}\}$ in W_p such that $\eta_{p_{i+1}} = \Gamma\eta_{p_i}$ for all $i \in \mathbb{N}_0$.
If $\eta_{p_i} = \eta_{p_{i+1}}$ for some $i \in \mathbb{N}_0$, then η_{p_i} is a fixed point of Γ and then proof of existence part of fixed point is finished. Suppose $\eta_{p_i} \neq \eta_{p_{i+1}}$ for every $i \in \mathbb{N}_0$, Then $\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) > 0$ by Lemma 1.10. Now, since Γ is $\hat{\alpha}$-admissible, so

$$\begin{aligned} \hat{\alpha}(\Gamma\eta_{p_0}, \Gamma\eta_{p_1}) &= \hat{\alpha}(\eta_{p_1}, \eta_{p_2}) \geq 1 \\ \hat{\alpha}(\Gamma\eta_{p_1}, \Gamma\eta_{p_2}) &= \hat{\alpha}(\eta_{p_2}, \eta_{p_3}) \geq 1 \end{aligned}$$

and using induction we have $\hat{\alpha}(\eta_{p_i}, \eta_{p_{i+1}}) \geq 1$ for all $i \in \mathbb{N}$.

Now, from (2.1) we have

$$\begin{aligned} \hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) = \hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i-1}}, \Gamma\eta_{p_i})) &\leq \hat{\alpha}(\eta_{p_{i-1}}, \eta_{p_i})\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i-1}}, \Gamma\eta_{p_i})) \\ &\leq \hat{\varphi}(\tilde{\mathcal{M}}(\eta_{p_{i-1}}, \eta_{p_i})) + L\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_{p_{i-1}}, \eta_{p_i})) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \tilde{\mathcal{N}}(\eta_{p_{i-1}}, \eta_{p_i}) &= \min\{\mathfrak{d}_\varrho^m(\eta_{p_{i-1}}, \Gamma\eta_{p_{i-1}}), \mathfrak{d}_\varrho^m(\eta_{p_i}, \Gamma\eta_{p_{i-1}})\} \\ &= \min\{\mathfrak{d}_\varrho^m(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho^m(\eta_{p_i}, \eta_{p_i})\} \\ &= 0 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\tilde{\mathcal{M}}(\eta_{p_{i-1}}, \eta_{p_i}) &= \max\left\{\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \Gamma\eta_{p_{i-1}}), \mathfrak{d}_\varrho(\eta_{p_i}, \Gamma\eta_{p_i}), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \Gamma\eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \Gamma\eta_{p_{i-1}})]\right\} \\ &= \max\left\{\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_{i+1}}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i})]\right\}\end{aligned} \tag{2.6}$$

Now, using the condition(*WPMS3*) we have

$$\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_{i+1}}) \leq \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) - \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i})$$

Therefore

$$\begin{aligned}\frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_{i+1}}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i})] &\leq \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) - \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i})] \\ &= \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}) + \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})] \\ &\leq \max\{\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})\}\end{aligned} \tag{2.7}$$

By (2.6) and (2.7) we get that

$$\tilde{\mathcal{M}}(\eta_{p_{i-1}}, \eta_{p_i}) \leq \max\{\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})\} \tag{2.8}$$

Now, using (2.5) and (2.8) in (2.4) and the fact that and $\hat{\vartheta}(u) = 0 \Leftrightarrow u = 0$, we get that

$$\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) \leq \hat{\varphi}(\max\{\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}), \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i}\}) \tag{2.9}$$

Now, if $\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) > \mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i})$ using definition that $\hat{\varphi}(u) < \hat{\psi}(u)$ for $u > 0$ we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) < \hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}))$$

which is a contradiction. Hence

$$\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}})) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i})) < \hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_{i-1}}, \eta_{p_i})) \tag{2.10}$$

We get a sequence of non-negative real numbers $\{\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) : i \in \mathbb{N}\}$ that decreases. Therefore there exists $\lambda_0 \geq 0$ such that

$$\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) = \lambda_0$$

Let $\lambda_0 > 0$. Then taking limit $i \to \infty$ in (2.10) we get

$$\hat{\psi}(\lambda_0) \leq \hat{\varphi}(\lambda_0) < \hat{\psi}(\lambda_0)$$

This is contradiction. Hence

$$\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_{i+1}}) = 0 \tag{2.11}$$

We now show that $\{\eta_{p_i}\}$ is a Cauchy sequence in W_p. i.e. $\lim_{i,j\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) = 0$.
By contradiction, we prove it.
Let

$$\lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) \neq 0$$

Then, with reference to lemma 1.9 all sequences tends to $\mu_p > 0$, when $u \to \infty$.
So we can see that

$$\lim_{u\to\infty} \mathfrak{d}_\varrho(\eta_{p_{j(u)}}, \eta_{p_{i(u)}}) = \mu_p \tag{2.12}$$

Further corresponding to $j(u)$, we can choose $i(u)$ in such a way that it is smallest integer with $i(u) > j(u) > u$. Then

$$\lim_{u\to\infty} \mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)}}) = \mu_p \tag{2.13}$$

Again,

$$\mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)-1}}) \le \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)}}) + \mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \eta_{p_{i(u)-1}}) - \mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \eta_{p_{i(u)}})$$

Letting $u \to \infty$ and using lemma 1.9 we get

$$\lim_{u\to\infty} \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)-1}}) = \mu_p \tag{2.14}$$

Again note that

Now, since Γ is triangular $\hat{\alpha}$-admissible, from Lemma 1.4 we derive that $\hat{\alpha}(\eta_{p_i}, \eta_{p_j}) \ge 1$ for all $i > j \in \mathbb{N}_0$.
Replacing η_p by $\eta_{p_{i(u)}}$ and ζ_p by $\eta_{p_{j(u)}}$ in (2.1) respectively, we get

$$\begin{aligned}\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \eta_{p_{j(u)}})) = \hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{j(u)-1}})) &\le \hat{\alpha}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}})\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{j(u)-1}})) \\ &\le \hat{\varphi}(\tilde{\mathcal{M}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}})) + L(\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}})))\end{aligned} \tag{2.15}$$

Where

$$\begin{aligned}\tilde{\mathcal{M}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}) &= \max\Big\{\mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}), \mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{i(u)-1}}), \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \Gamma\eta_{p_{j(u)-1}}), \\ &\qquad \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{j(u)-1}}) + \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \Gamma\eta_{p_{i(u)-1}})]\Big\} \\ &= \max\Big\{\mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}), \mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{i(u)}}), \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{j(u)}}), \\ &\qquad \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)}}) + \mathfrak{d}_\varrho(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)}})]\Big\}\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}) &= \min\{\mathfrak{d}_\varrho^m(\eta_{p_{i(u)-1}}, \Gamma\eta_{p_{i(u)-1}}), \mathfrak{d}_\varrho^m(\eta_{p_{j(u)-1}}, \Gamma\eta_{p_{i(u)-1}})\} \\ &= \min\{\mathfrak{d}_\varrho^m(\eta_{p_{i(u)-1}}, \eta_{p_{i(u)}}), \mathfrak{d}_\varrho^m(\eta_{p_{j(u)-1}}, \eta_{p_{i(u)}})\}\end{aligned} \tag{2.16}$$

Letting $u \to \infty$ in (**??**) and (2.16) and using (2.11), (2.12), (2.13), (2.14) and lemma 1.9 we get

$$\lim_{u\to\infty} \tilde{\mathcal{M}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}) = \max\{\mu_p, 0, 0, \mu_p\} = \mu_p \tag{2.17}$$

and

$$\lim_{u\to\infty} \tilde{\mathcal{N}}(\eta_{p_{i(u)-1}}, \eta_{p_{j(u)-1}}) = 0. \tag{2.18}$$

Now Letting $u \to \infty$ in (2.15) and using (2.17) and (2.18) we get

$$\hat{\psi}(\mu_p) \leq \hat{\varphi}(\mu_p) < \hat{\psi}(\mu_p)$$

This is a contradiction, Therefore

$$\lim_{i,j\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) = 0 \tag{2.19}$$

This implies that $\{\eta_{p_i}\}$ is a Cauchy sequence in $(W_p, \mathfrak{d}_\varrho)$. On the other hand, since

$$\begin{aligned} \mathfrak{d}_\varrho^m(\eta_{p_i}, \eta_{p_j}) &= \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) - \min\{\mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i}), \mathfrak{d}_\varrho(\eta_{p_j}, \eta_{p_j})\} \\ &\leq \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) \end{aligned}$$

Now, taking the limit as $j, i \to \infty$ and using (2.19) we get that

$$\lim_{i,j\to\infty} \mathfrak{d}_\varrho^m(\eta_{p_i}, \eta_{p_j}) = 0 \tag{2.20}$$

This shows that $\{\eta_{p_i}\}$ is also a Cauchy sequence in the metric space $(W_p, \mathfrak{d}_\varrho^m)$. Since $(W_p, \mathfrak{d}_\varrho)$ is complete, then from Lemma 1.12, the sequence $\{\eta_{p_i}\}$ converges in the metric space $(W_p, \mathfrak{d}_\varrho^m)$, say $\lim_{i\to\infty} \mathfrak{d}_\varrho^m(\eta_{p_i}, \delta_p) = 0$. Again from Lemma 1.12 we have

$$\mathfrak{d}_\varrho(\delta_p, \delta_p) = \lim_{i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \delta_p) = \lim_{j,i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_j}) \tag{2.21}$$

Therefore, from (2.21) and (2.19) we get that

$$\mathfrak{d}_\varrho(\delta_p, \delta_p) = \lim_{n\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \delta_p) = \lim_{j,i\to\infty} \mathfrak{d}_\varrho(\eta_{p_i}, \eta_{p_i}) = 0 \tag{2.22}$$

Moreover, As Γ is continuous, we have

$$\delta_p = \lim_{i\to\infty} \eta_{p_{i+1}} = \lim_{i\to\infty} \Gamma\eta_{p_i} = \Gamma\delta_p$$

■

In the following, we omit the continuity assumption of Γ in Theorem 2.2.

Theorem 2.3. *Let $(W_p, \mathfrak{d}_\varrho)$ be a complete weak partial metric space and $\Gamma : W_p \to W_p$ be self mapping. Suppose $\hat{\alpha} : W_p \times W_p \to [0, \infty)$ be the mappings satisfying the conditions:*

(i) Γ is triangular $\hat{\alpha}$-admissible;

(ii) Γ is almost generalized $(\hat{\alpha}, \hat{\psi}, \hat{\varphi}, \hat{\vartheta})$-contractive mapping;

(iii) There exists $\eta_{p_0} \in W_p$ such that $\hat{\alpha}(\eta_{p_0}, \Gamma\eta_{p_0}) \geq 1$;

(iv) If $\{\eta_{p_i}\}$ is a sequence in W_p such that $\eta_{p_i} \to \eta_p \in W_p$, $\hat{\alpha}(\eta_{p_i}, \eta_{p_{i+1}}) \geq 1$ for all i, there exists a subsequence $\{\eta_{p_{i(u)}}\}$ of $\{\eta_{p_i}\}$ such that $\hat{\alpha}(\eta_{p_{i(u)}}, \eta_p) \geq 1$ for all u.

Then Γ has a fixed point in W_p. Further if δ_p, δ_q are fixed points of Γ such that $\hat{\alpha}(\delta_p, \delta_q) \geq 1$ then $\delta_p = \delta_q$.

Proof. From the proof of the Theorem 2.2, the sequence η_{p_i} defined by $\eta_{p_{i+1}} = \Gamma\eta_{p_i}$ is Cauchy in W_p and converges to $\delta_p \in W_p$. According to the assumptions, there is a subsequence of $\{\eta_{p_{i(u)}}\}$ of $\{\eta_{p_i}\}$ such that $\hat{\alpha}(\eta_{p_{i(u)}}, \delta_p) \geq 1$ for all u. We will now demonstrate that δ_p is a fixed point of Γ. Consider the alternative, then $\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p) > 0$.
Now in (2.1) replacing η_p by $\eta_{p_{i(u)}}$ and ζ_p by δ_p we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\eta_{p_{i(u)+1}}, \Gamma\delta_p)) = \hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i(u)}}, \Gamma\delta_p)) \leq \hat{\alpha}(\eta_{p_{i(u)}}, \delta_p)\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_{p_{i(u)}}, \Gamma\delta_p))$$
$$\leq \hat{\varphi}(\tilde{\mathcal{M}}(\eta_{p_{i(u)}}, \delta_p)) + L(\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_{p_{i(u)}}, \delta_p))) \tag{2.23}$$

Where

$$\begin{aligned}\tilde{\mathcal{M}}(\eta_{p_{i(u)}}, \delta_p) &= \max\left\{\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \delta_p), \mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \Gamma\eta_{p_{i(u)}}), \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \Gamma\delta_p) + \mathfrak{d}_\varrho(\delta_p, \Gamma\eta_{p_{i(u)}})]\right\} \\ &= \max\left\{\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \delta_p), \mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \eta_{p_{i(u)+1}}), \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_{p_{i(u)}}, \Gamma\delta_p) + \mathfrak{d}_\varrho(\delta_p, \eta_{p_{i(u)+1}})]\right\}\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(\eta_{p_{i(u)}}, \delta_p) &= \min\{\mathfrak{d}^m_\varrho(\eta_{p_{i(u)}}, \Gamma\eta_{p_{i(u)}}), \mathfrak{d}^m_\varrho(\delta_p, \Gamma\eta_{p_{i(u)}})\} \\ &= \min\{\mathfrak{d}^m_\varrho(\eta_{p_{i(u)}}, \eta_{p_{i(u)+1}}), \mathfrak{d}^m_\varrho(\delta_p, \eta_{p_{i(u)+1}})\}\end{aligned} \tag{2.25}$$

Now, taking $u \to \infty$ in (2.24) and ((2.25) and using the fact that due to (2.22) we have $\mathfrak{d}_\varrho(\delta_p, \delta_p) = 0$, we get

$$\lim_{u\to\infty} \tilde{\mathcal{M}}(\eta_{p_{i(u)}}, \delta_p) = \max\{0, 0, \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p) + 0]\} = \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p) \tag{2.26}$$

and

$$\lim_{u\to\infty} \tilde{\mathcal{N}}(\eta_{p_{i(u)}}, \delta_p) = 0 \tag{2.27}$$

Now, taking $u \to \infty$ in (2.23) and using (2.26), (2.27) and definitions of $\hat{\psi}$, $\hat{\varphi}$ and $\hat{\vartheta}$ we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p)) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p)) < \hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p))$$

which is a contradiction. Therefore $\Gamma\delta_p = \delta_p$ i.e. δ_p is a fixed point.
Further, suppose δ_p and δ_q be two fixed point of Γ such that $\mathfrak{d}_\varrho(\delta_p, \delta_q) > 0$ and $\hat{\alpha}(\delta_p, \delta_q) \geq 1$ then replacing η_p by δ_p and ζ_p by δ_q in (2.1) we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \delta_q)) = \hat{\psi}(\mathfrak{d}_\varrho(\Gamma\delta_p, \Gamma\delta_q)) \leq \hat{\alpha}(\delta_p, \delta_q)\mathfrak{d}_\varrho(\Gamma\delta_p, \Gamma\delta_q)$$
$$\leq \hat{\varphi}(\tilde{\mathcal{M}}(\delta_p, \delta_q)) + L(\hat{\vartheta}(\tilde{\mathcal{N}}(\delta_p, \delta_q))) \tag{2.28}$$

Where

$$\begin{aligned}\tilde{\mathcal{M}}(\delta_p, \delta_q) &= \max\left\{\mathfrak{d}_\varrho(\delta_p, \delta_q), \mathfrak{d}_\varrho(\delta_p, \Gamma\delta_p), \mathfrak{d}_\varrho(\delta_q, \Gamma\delta_q), \frac{1}{2}[\mathfrak{d}_\varrho(\delta_p, \Gamma\delta_q) + \mathfrak{d}_\varrho(\delta_q, \Gamma\delta_p)]\right\} \\ &= \max\left\{\mathfrak{d}_\varrho(\delta_p, \delta_q), \mathfrak{d}_\varrho(\delta_p, \delta_p), \mathfrak{d}_\varrho(\delta_q, \delta_q), \frac{1}{2}[\mathfrak{d}_\varrho(\delta_p, \delta_q) + \mathfrak{d}_\varrho(\delta_q, \delta_p)]\right\} \\ &= \max\left\{\mathfrak{d}_\varrho(\delta_p, \delta_q), 0, 0, \frac{1}{2}[\mathfrak{d}_\varrho(\delta_p, \delta_q) + \mathfrak{d}_\varrho(\delta_p, \delta_q)]\right\} \textit{by(WPMS2)} \\ &= \mathfrak{d}_\varrho(\delta_p, \delta_q)\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(\delta_p, \delta_q) &= \min\{\mathfrak{d}_\varrho^m(\delta_p, \Gamma\delta_p), \mathfrak{d}_\varrho^m(\delta_q, \Gamma\delta_p)\} \\ &= \min\{\mathfrak{d}_\varrho^m(\delta_p, \delta_p), \mathfrak{d}_\varrho^m(\delta_q, \delta_p)\} \\ &= 0 \end{aligned} \tag{2.30}$$

By putting (2.29), (2.30) in (2.28) and using the definitions of $\hat{\psi}, \hat{\varphi}$ and $\hat{\vartheta}$ we get

$$\hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \delta_q)) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\delta_p, \delta_q)) < \hat{\psi}(\mathfrak{d}_\varrho(\delta_p, \delta_q))$$

This is contradictory. As a result, Γ has a unique fixed point. The evidence is now complete.

■

The theorems' consequences are given below.

Corollary 2.4. *Let $(W_p, \mathfrak{d}_\varrho)$ be a complete weak partial metric space. $\Gamma : W_p \to W_p$ satisfy the criterion by self-mapping with*

$$\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p)) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\eta_p, \zeta_p)) + L(\tilde{\mathcal{N}}(\eta_p, \zeta_p)) \tag{2.31}$$

For all $\eta_p, \zeta_p \in W_p$, $\hat{\psi} \in \Psi$, $\hat{\varphi} \in \Phi$ and $L \geq 0$. Then Γ has a unique fixed point in W_p.

Corollary 2.5. *Let $(W_p, \mathfrak{d}_\varrho)$ be a complete weak partial metric space. A self-mapping $\Gamma : W_p \to W_p$ be such that*

$$\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p) \leq k(\tilde{\mathcal{M}}(\eta_p, \zeta_p))$$

For all $\eta_p, \zeta_p \in W_p$, $k \in (0, 1)$, where

$$\tilde{\mathcal{M}}(\eta_p, \zeta_p) = \max\left\{\mathfrak{d}_\varrho(\eta_p, \zeta_p), \mathfrak{d}_\varrho(\eta_p, \Gamma\eta_p), \mathfrak{d}_\varrho(\zeta_p, \Gamma\zeta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p, \Gamma\zeta_p) + \mathfrak{d}_\varrho(\zeta_p, \Gamma\eta_p)]\right\} \tag{2.32}$$

Then Γ has a unique fixed point in W_p.

Example 2.6. *Let $W_p = [0, 1]$ and $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \frac{1}{2}(\eta_p + \zeta_p)$, Then $\mathfrak{d}_\varrho^m(\eta_p, \zeta_p) = \frac{1}{2}|\eta_p - \zeta_p|$. Therefore, since $(W_p, \mathfrak{d}_\varrho^m)$ is complete, the by Lemma 1.12 $(W_p, \mathfrak{d}_\varrho)$ is a complete weak partial metric space (WPMS).*

Consider the mapping $\Gamma : W_p \to W_p$ defined by $\Gamma(\eta_p) = \frac{\eta_p}{3}$ and let $\hat{\psi}, \hat{\varphi}, \hat{\vartheta} : [0, \infty) \to [0, \infty)$ be such that $\hat{\psi}(u) = 2u$, $\hat{\varphi}(u) = \frac{2u}{3}$ and $\hat{\vartheta}(u) = u$ for all $u \geq 0$. If we define the functions $\hat{\alpha} : W_p \times W_p \to [0, \infty)$ as

$$\hat{\alpha}(\eta_p, \zeta_p) = \begin{cases} 1 & \eta_p, \zeta_p \in [0, \frac{1}{2}] \\ o & \eta_p, \zeta_p \in (\frac{1}{2}, 1] \end{cases} \tag{2.33}$$

We show that contractive condition of Theorem 2.2 is satisfied.
Let $\eta_p, \zeta_p \in [0, \frac{1}{2}]$ we get

$$\begin{aligned}\hat{\alpha}(\eta_p, \zeta_p)\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p)) &= \hat{\alpha}(\eta_p, \zeta_p)\hat{\psi}(\mathfrak{d}_\varrho(\frac{\eta_p}{3}, \frac{\zeta_p}{3})) \\ &= \hat{\psi}(\frac{1}{2}(\frac{\eta_p + \zeta_p}{3})) \\ &= \frac{2}{3}\mathfrak{d}_\varrho(\eta_p, \zeta_p) \end{aligned} \tag{2.34}$$

On the other side

$$\begin{aligned}\tilde{\mathcal{M}}(\eta_p,\zeta_p) =& \max\left\{\mathfrak{d}_\varrho(\eta_p,\zeta_p), \mathfrak{d}_\varrho(\eta_p,\Gamma\eta_p), \mathfrak{d}_\varrho(\zeta_p,\Gamma\zeta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p,\Gamma\zeta_p)+\mathfrak{d}_\varrho(\zeta_p,\Gamma\eta_p)]\right\}\\ =& \max\left\{\mathfrak{d}_\varrho(\eta_p,\zeta_p), \mathfrak{d}_\varrho(\eta_p,\frac{\eta_p}{3}), \mathfrak{d}_\varrho(\zeta_p,\frac{\zeta_p}{3}), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p,\frac{\zeta_p}{3})+\mathfrak{d}_\varrho(\zeta_p,\frac{\eta_p}{3})]\right\}\\ =& \max\left\{\frac{\eta_p+\zeta_p}{2}, \frac{2\eta_p}{3}, \frac{2\zeta_p}{3}, \frac{\eta_p+\zeta_p}{3}\right\}\\ =& \frac{\eta_p+\zeta_p}{2} = \mathfrak{d}_\varrho(\eta_p,\zeta_p)\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(\eta_p,\zeta_p) &= \min\{\mathfrak{d}^m_\varrho(\eta_p,\Gamma\zeta_p), \mathfrak{d}^m_\varrho(\zeta_p,\Gamma\eta_p)\}\\ &= \min\{\mathfrak{d}^m_\varrho(\eta_p,\frac{\eta_p}{3}), \mathfrak{d}^m_\varrho(\zeta_p,\frac{\eta_p}{3})\}\end{aligned} \tag{2.36}$$

Therefore from (2.35) we get

$$\begin{aligned}\hat{\varphi}(\tilde{\mathcal{M}}(\eta_p,\zeta_p)) + L(\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_p,\zeta_p))) &= \hat{\varphi}(\frac{\eta_p+\zeta_p}{2}) + L(\hat{\vartheta}(\tilde{\mathcal{N}}(\eta_p,\zeta_p)))\\ &= \frac{2}{3}(\frac{\eta_p+\zeta_p}{2}) + L(\tilde{\mathcal{N}}(\eta_p,\zeta_p))\\ &= \frac{2}{3}\mathfrak{d}_\varrho(\eta_p,\zeta_p) + L(\tilde{\mathcal{N}}(\eta_p,\zeta_p))\end{aligned} \tag{2.37}$$

Now since $L(\tilde{\mathcal{N}}(\eta_p,\zeta_p)) = L(\min\{\mathfrak{d}^m_\varrho(\eta_p,\frac{\eta_p}{3}), \mathfrak{d}^m_\varrho(\zeta_p,\frac{\eta_p}{3})\}) \geq 0$ for all $\eta_p,\zeta_p \in W_p$, and from (2.34) and (2.37) we get

$$\frac{2}{3}\mathfrak{d}_\varrho(\eta_p,\zeta_p) \leq \frac{2}{3}\mathfrak{d}_\varrho(\eta_p,\zeta_p) + L(\tilde{\mathcal{N}}(\eta_p,\zeta_p)) \tag{2.38}$$

for all $\eta_p,\zeta_p \in W_p$.

Now, let $\eta_p,\zeta_p \in (\frac{1}{2},1]$, in this case the contractive conditions of theorem 2.2 is already satisfied since $\hat{\alpha}(\eta_p,\zeta_p) = 0$. It is clear that all the conditions of Theorem 2.2 hold. Hence Γ has a fixed point, which in this case is 0.

Example 2.7. *Let $W_p = [0,1]$ and $\mathfrak{d}_\varrho(\eta_p,\zeta_p) = \frac{1}{2}(\eta_p+\zeta_p)$, Then $\mathfrak{d}^m_\varrho(\eta_p,\zeta_p) = \frac{1}{2}|\eta_p-\zeta_p|$. Therefore, since $(W_p,\mathfrak{d}^m_\varrho)$ is complete, the by lemma 1.12 $(W_p,\mathfrak{d}_\varrho)$ is a complete weak partial metric space (WPMS).*

Consider the mapping $\Gamma : W_p \to W_p$ defined by $\Gamma(\eta_p)$= $\begin{cases}\eta_p^2 & \eta_p \in [0,\frac{1}{2}]\\ o & \eta_p \in (\frac{1}{2},1]\end{cases}$ and let $\hat{\psi},\hat{\varphi} : [0,\infty) \to [0,\infty)$ be such that $\hat{\psi}(u) = u$, $\hat{\varphi}(u) = \frac{u}{2}$ for all $u \geq 0$.
Now, we show that contractive condition of corollary 2.4 is satisfied for $L = 1$, i.e.,

$$\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p,\Gamma\zeta_p)) \leq \hat{\varphi}(\mathfrak{d}_\varrho(\eta_p,\zeta_p)) + L(\tilde{\mathcal{N}}(\eta_p,\zeta_p)) \tag{2.39}$$

for all $\eta_p,\zeta_p \in W_p$. Let $\eta_p,\zeta_p \in [0,\frac{1}{2}]$, then

$$\begin{aligned}\hat{\psi}(\mathfrak{d}_\varrho(\Gamma\eta_p,\Gamma\zeta_p)) =& \hat{\psi}(\frac{\eta_p^2+\zeta_p^2}{2}) = \frac{\eta_p^2+\zeta_p^2}{2} \leq \frac{1}{2}(\frac{\eta_p+\zeta_p}{2}) = \frac{1}{2}\mathfrak{d}_\varrho(\eta_p,\zeta_p)\\ &\leq \frac{1}{2}\mathfrak{d}_\varrho(\eta_p,\zeta_p) + \min\{\mathfrak{d}^m_\varrho(\eta_p,\Gamma\eta_p), \mathfrak{d}^m_\varrho(\zeta_p,\Gamma\eta_p)\}\\ &= \hat{\varphi}(\mathfrak{d}_\varrho(\eta_p,\zeta_p)) + \min\{\mathfrak{d}^m_\varrho(\eta_p,\Gamma\eta_p), \mathfrak{d}^m_\varrho(\zeta_p,\Gamma\eta_p)\}\end{aligned}$$

Now, let $\eta_p, \zeta_p \in (\frac{1}{2}, 1]$, then result is clear since in this case $\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p) = 0$. As a result, all requirements of corollary 2.4 are completely satisfied. As a result, it has a fixed point, which in this instance is 0.

Now, we demonstrate that the contractive requirement of Corollary 2.5 is met.

Example 2.8. *Let* $W_p = [0, 1]$ *and* $\mathfrak{d}_\varrho(\eta_p, \zeta_p) = \frac{1}{2}(\eta_p + \zeta_p)$*, Then* $\mathfrak{d}^m_\varrho(\eta_p, \zeta_p) = \frac{1}{2}|\eta_p - \zeta_p|$*. Therefore, since* $(W_p, \mathfrak{d}^m_\varrho)$ *is complete, the by lemma 1.12* $(W_p, \mathfrak{d}_\varrho)$ *is a complete weak partial metric space (WPMS).*

Consider the mapping $\Gamma : W_p \to W_p$ defined by $\Gamma(\eta_p) = \frac{\eta_p}{3}$ Then

$$\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p) = \mathfrak{d}_\varrho(\frac{\eta_p}{3}, \frac{\zeta_p}{3}) = \frac{1}{3}\mathfrak{d}_\varrho(\eta_p, \zeta_p) \tag{2.40}$$

On the other hand side

$$\begin{aligned}
\tilde{\mathcal{M}}(\eta_p, \zeta_p) &= \max\big\{\mathfrak{d}_\varrho(\eta_p, \zeta_p), \mathfrak{d}_\varrho(\eta_p, \Gamma\eta_p), \mathfrak{d}_\varrho(\zeta_p, \Gamma\zeta_p), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p, \Gamma\zeta_p) + \mathfrak{d}_\varrho(\zeta_p, \Gamma\eta_p)]\big\} \\
&= \max\big\{\mathfrak{d}_\varrho(\eta_p, \zeta_p), \mathfrak{d}_\varrho(\eta_p, \frac{\eta_p}{3}), \mathfrak{d}_\varrho(\zeta_p, \frac{\zeta_p}{3}), \frac{1}{2}[\mathfrak{d}_\varrho(\eta_p, \frac{\zeta_p}{3}) + \mathfrak{d}_\varrho(\zeta_p, \frac{\eta_p}{3})]\big\} \\
&= \max\big\{\frac{\eta_p + \zeta_p}{2}, \frac{2\eta_p}{3}, \frac{2\zeta_p}{3}, \frac{\eta_p + \zeta_p}{3}\big\} \\
&= \frac{\eta_p + \zeta_p}{2} = \mathfrak{d}_\varrho(\eta_p, \zeta_p)
\end{aligned} \tag{2.41}$$

From (2.40) and (2.41) we get

$$\frac{1}{3}\mathfrak{d}_\varrho(\eta_p, \zeta_p) \le k\mathfrak{d}_\varrho(\eta_p, \zeta_p) \tag{2.42}$$

for $k \in [\frac{1}{3}, 1)$. i.e.

$$\mathfrak{d}_\varrho(\Gamma\eta_p, \Gamma\zeta_p) \le k(\tilde{\mathcal{M}}(\eta_p, \zeta_p))$$

for $k \in [\frac{1}{3}, 1)$.It is evident from (2.42) that it satisfies the requirement of Corollary 2.5. As a result, it has a fixed point, which in this instance is 0.

3. Conclusion

In this study, we proved certain fixed point theorems in the context of complete weak partial metric spaces using triangular $\hat{\alpha}$-admissible mappings and provided some implications of the main findings. We included some examples to support our results. The results in this article expand upon and generalise several results from the existing literature.

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