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# Screen invariant lightlike hypersurfaces of almost product-like statistical manifolds and locally product-like statistical manifolds

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**Abstract.** The main formulas and relations are presented for screen invariant lightlike hypersurfaces. Concurrent and recurrent vector fields are investigated and several formulas are obtained for screen invariant lightlike hypersurfaces.

**AMS Subject Classifications**: 53C05, 53C25, 53C55.

**Keywords**: Hypersurface, statistical structure, product manifold.

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## Contents



## 1. Introduction

One of the techniques to characterize a Riemannian manifold is to review the geometry of some appropriate vector fields. The appropriate vector fields that have been widely studied in the literature recently are torse-forming, concircular concurrent, geodesic and recurrent vector fields, etc. The impression of concurrent vector fields is firstly announced by K. Yano [22] in such a way:

Let $(L, h)$ be a Riemannian manifold equipped with a metric $h$ and $D$ be the Riemannian connection on $(L, h)$. A vector field $\zeta$ is entitled concurrent if

$$D_Z \zeta = Z$$

holds for each tangent vector field $Z$.

There exist remarkable applications dealing with concurrent vector fields into submanifolds of Riemannian manifolds admitting differential structures [10, 13, 14, 18, 23, 24], etc. Besides these facts, statistical structures on Riemannian manifolds have been widely studied lately with interesting geometrical properties. The impression of statistical manifolds was initially announced by S. Amari [2] and the basic properties of hypersurfaces were

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revealed by H. Furuhata in [11, 12]. Later, this concept admitting complex, contact and product structures was examined by various authors in [4, 5, 15–17, 21].

An interesting perspective on statistical manifolds came from K. Takano's definition of Hermite-like manifolds, which is a generalization of Hermitian manifolds. A Riemannian manifold $(L, h)$ included two almost complex structures $J$ and $J^\star$ is entitled a Hermite-like manifold [19, 20] if

$$h(JZ_1, Z_2) = -h(Z_1, J^\star Z_2)$$

holds for each tangent vector fields $Z_1$ and $Z_2$. One of the interesting aspects of Hermite-like manifolds is that although there are no examples in classical Euclidean spaces, there are examples of Hermite-like manifolds in non-Euclidean geometry. With a similar idea, product-like manifolds were introduced and the geometry of some special type hypersurfaces of these manifolds was investigated in [1, 7].

The primary objective of this paper is to review screen invariant lightlike hypersurfaces of an almost product-like statistical manifold. With the aid of statistical structures, some main formulas and relations are obtained and concurrent vector fields are examined on these hypersurfaces.

## 2. Almost product-like manifolds and their lightlike hypersurfaces

A differentiable manifold $\widetilde{L}$ is entitled an almost product manifold if it includes a tensor field providing $F^2 = I$, where $I$ expresses the identity transformation. We note that the eigenvalues of $F$ are $+1$ and $-1$. If we put

$$T = \frac{1}{2}(I + F), \;\; Q = \frac{1}{2}(I - F)$$

then we find

$$T + Q = I, \;\; T^2 = T, \;\; Q^2 = Q, \;\; TQ = QT = 0$$

and

$$F = T - Q.$$

If a Riemannian metric $\widetilde{h}$ on $\widetilde{L}$ provides

$$\widetilde{h}(FZ_1, Z_2) = \widetilde{h}(Z_1, FZ_2) \tag{2.1}$$

for each $Z_1, Z_2 \in \Gamma(T\widetilde{L})$, then $(\widetilde{L}, \widetilde{h}, F)$ is called an almost product Riemannian manifold.

Now, we remind the following definition [7]:

**Definition 2.1.** *Let $F$ and $F^\star$ be two almost product structures on $\widetilde{L}$. If the equation*

$$\widetilde{h}(FZ_1, Z_2) = \widetilde{h}(Z_1, F^\star Z_2) \tag{2.2}$$

*is provided then $(\widetilde{L}, \widetilde{h}, F)$ is entitled an almost product-like semi-Riemannian manifold.*

If we indite $FZ_1$ in place of of $Z_1$ in (2.2), we obtain that

$$\widetilde{h}(FZ_1, F^\star Z_2) = \widetilde{h}(Z_1, Z_2) \tag{2.3}$$

is provided.

**Example 2.2.** *Let $F$ be a tensor field on $\mathbb{R}^4_1$ such that*

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

*Then we find* $(\mathbb{R}^4_1, F)$ *is an almost product manifold. If we write*

$$\widetilde{h} = \begin{bmatrix} -e^{x_1} & 0 & 0 & 0 \\ 0 & e^{x_1} & 0 & 0 \\ 0 & 0 & e^{x_1} & 0 \\ 0 & 0 & 0 & e^{x_1} \end{bmatrix} \quad and \quad F^\star = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

*then we obtain* $(\mathbb{R}^4_1, \widetilde{h}, F)$ *is provided (2.2).*

Presume that $\widetilde{D}$ is a torsion-free connection on $(\widetilde{L}, \widetilde{h}, F)$. If $\widetilde{D}g$ is symmetric, then $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is entitled an almost product-like statistical manifold. For each $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$, we indite another torsion-free connection satisfying

$$Z_3\widetilde{h}(Z_1, Z_2) = \widetilde{h}(\widetilde{D}_{Z_3}Z_1, Z_2) + \widetilde{h}(Z_1, \widetilde{D}^\star_{Z_3}Z_2) \tag{2.4}$$

for each $Z_1, Z-2, Z_3 \in \Gamma(T\widetilde{L})$. $\widetilde{D}^\star$ is called the dual connection of $\widetilde{D}$. In addition, we indite

$$\widetilde{D}^0_{Z_1}Z_2 = \frac{1}{2}(\widetilde{D}_{Z_1}Z_2 + \widetilde{D}^\star_{Z_1}Z_2), \tag{2.5}$$

where $\widetilde{D}^0$ is the Levi-Civita connection of $(\widetilde{L}, \widetilde{h}, F)$.

**Definition 2.3.** *Let* $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ *be an almost product-like statistical manifold. If* $F$ *is parallel with regard to* $\widetilde{D}$*, then* $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ *is entitled a locally product-like statistical manifold.*

In view of (2.4), we find the following equation is satisfied:

$$\widetilde{h}((\widetilde{D}_{Z_1}F)Z_2, Z_3) = \widetilde{h}(Z_2, (\widetilde{D}^\star_{Z_1}F^\star)Z_3). \tag{2.6}$$

From (2.6), it is clear that

$$\widetilde{D}F = 0 \Leftrightarrow \widetilde{D}^\star F^\star = 0.$$

Therefore, $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is a locally product-like statistical manifold if and only if so is $(\widetilde{L}, \widetilde{h}, \widetilde{D}^\star, F^\star)$.

**Example 2.4.** *Let* $(\mathbb{R}^4_1, \widetilde{h}, F)$ *be an almost product-like Lorentzian manifold of Example 2.2. By a straightforward computation, we put*

$$\begin{aligned}
&\widetilde{D}_{\partial_1}\partial_1 = \widetilde{D}_{\partial_4}\partial_4 = \frac{1}{2}\partial_1, \\
&\widetilde{D}_{\partial_1}\partial_2 = \widetilde{D}_{\partial_2}\partial_1 = \widetilde{D}_{\partial_3}\partial_4 = \widetilde{D}_{\partial_4}\partial_3 = \frac{1}{2}\partial_2, \\
&\widetilde{D}_{\partial_1}\partial_3 = \widetilde{D}_{\partial_3}\partial_1 = \widetilde{D}_{\partial_2}\partial_4 = \widetilde{D}_{\partial_4}\partial_2 = \frac{1}{2}\partial_3, \\
&\widetilde{D}_{\partial_1}\partial_4 = \widetilde{D}_{\partial_4}\partial_1 = \frac{1}{2}\partial_4, \\
&\widetilde{D}_{\partial_2}\partial_2 = \widetilde{D}_{\partial_3}\partial_3 = \frac{1}{2}\partial_1 + \Gamma^2_{22}\partial_2 + \Gamma^3_{22}\partial_3, \\
&\widetilde{D}_{\partial_2}\partial_3 = \widetilde{D}_{\partial_3}\partial_2 = \Gamma^3_{22}\partial_2 + \Gamma^2_{22}\partial_3 + \frac{1}{2}\partial_4
\end{aligned}$$

*and*

$$\widetilde{D}^{\star}_{\partial_1}\partial_1 = \widetilde{D}^{\star}_{\partial_4}\partial_4 = \frac{1}{2}\partial_1,$$
$$\widetilde{D}^{\star}_{\partial_1}\partial_2 = \widetilde{D}^{\star}_{\partial_2}\partial_1 = -\widetilde{D}^{\star}_{\partial_3}\partial_4 = -\widetilde{D}^{\star}_{\partial_4}\partial_3 = \frac{1}{2}\partial_2,$$
$$\widetilde{D}^{\star}_{\partial_1}\partial_3 = \widetilde{D}^{\star}_{\partial_3}\partial_1 = -\widetilde{D}^{\star}_{\partial_2}\partial_4 = -\widetilde{D}^{\star}_{\partial_4}\partial_2 = \frac{1}{2}\partial_3,$$
$$\widetilde{D}^{\star}_{\partial_1}\partial_4 = \widetilde{D}^{\star}_{\partial_4}\partial_1 = \frac{1}{2}\partial_4,$$
$$\widetilde{D}^{\star}_{\partial_2}\partial_2 = \widetilde{D}^{\star}_{\partial_3}\partial_3 = \frac{1}{2}\partial_1 - \Gamma^2_{22}\partial_2 - \Gamma^3_{22}\partial_3,$$
$$\widetilde{D}^{\star}_{\partial_2}\partial_3 = \widetilde{D}^{\star}_{\partial_3}\partial_2 = -\Gamma^3_{22}\partial_2 - \Gamma^2_{22}\partial_3 - \frac{1}{2}\partial_4,$$

*where $\Gamma^2_{22}$ and $\Gamma^3_{22}$ are any functions on $\mathbb{R}^4_1$ and $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is the natural basis of $\mathbb{R}^4_1$. Then we obtain that $(\mathbb{R}^4_1, \widetilde{h}, \widetilde{D}, F)$ is a locally product-like statistical manifold.*

Let $(L, h)$ be a hypersurface of $(\widetilde{L}, \widetilde{h}, F)$ with the induced metric $h$ from $\widetilde{h}$. If $h$ is degenerate on $L$, then $(L, h)$ is entitled a lightlike hypersurface. For any lightlike hypersurface, the radical distribution $Rad(TL)$ is given as follows:

$$Rad(TL) = \text{span}\{\xi : h(\xi, Z) = 0, \forall Z \in \Gamma(TL)\}.$$

Denote a complementary vector bundle of $Rad(TL)$ in $TL$ by $S(TL)$. The distribution $S(TL)$ is called a screen distribution of $(L, h)$ and thus we write

$$TL = Rad(TL) \oplus_{orth} S(TL),$$

where $\oplus_{orth}$ stands for the orthogonal direct sum. It is known that the screen distribution is not unique since $h$ is degenerate. There is a unique null section $N$ providing

$$\widetilde{h}(\xi, N) = 1, \ \ \widetilde{h}(N, N) = \widetilde{h}(N, Z) = 0$$

for any $Z \in \Gamma(S(TL))$. We note that the vector bundle $ltr(TL) = \text{span}\{N\}$ is called the transversal bundle of $(L, h, S(TL))$ [8, 9].

The Gauss and Weingarten formulas with regard to the Levi-Civita connection $\widetilde{\nabla}^0$ are formulated by

$$\widetilde{D}^0_{Z_1} Y = D^0_{Z_1} Y + B^0(Z_1, Y) N \tag{2.7}$$

and

$$\widetilde{D}^0_{Z_1} N = -A^0_N Z_1 + \tau^0(Z_1) N, \tag{2.8}$$

where $D^0$ is the induced connection, $A^0_N$ is the shape operator and $\tau^0$ is a $1-$form.

The hypersurface $(L, h, S(TL))$ is called

i) totally geodesic if $B^0 = 0$,

ii) totally umbilical if there is a differentiable function $\mu$ such that $B^0(Z_1, Z_2) = \mu h(Z_1, Z_2)$,

iii) minimal if $trace_{S(TL)} B^0 = 0$, where $trace_{S(TM)}$ is the trace with regard to $S(TL)$.

Similar formulas and definitions could be given with regard to $\widetilde{D}$.

The Gauss and Weingarten type formulas with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$ is written by

$$\widetilde{D}_{Z_1} Y = D_{Z_1} Y + B(Z_1, Z_2) N, \tag{2.9}$$
$$\widetilde{D}_{Z_1} N = -A^{\star}_N Z_1 + \tau^{\star}(Z_1) N \tag{2.10}$$

and

$$\widetilde{D}^\star_{Z_1} Z_2 = D^\star_{Z_1} Z_2 + B^\star(Z_1, Z_2)N, \tag{2.11}$$
$$\widetilde{D}^\star_{Z_1} N = -A_N Z_1 + \tau(Z_1)N, \tag{2.12}$$

where $D_X Y, DZ_1{}^\star Z_2, A_N Z_1, A^\star_N Z_1 \in \Gamma(TL)$, $\tau$ and $\tau^\star$ are 1−forms. Also, the Gauss and Weingarten type formulas on $S(TL)$ could be given as follows:

$$D_{Z_1} PZ_2 = \overline{D}_{Z_1} PZ_2 + C(Z_1, PZ_2)\xi, \tag{2.13}$$
$$D_{Z_1} \xi = -\overline{A}_\xi Z_1 - \tau(Z_1)\xi \tag{2.14}$$

and

$$D^\star_{Z_1} PZ_2 = \overline{D}^\star_{Z_1} PZ_2 + C^\star(Z_1, PZ_2)\xi, \tag{2.15}$$
$$D^\star_{Z_1} \xi = -\overline{A}^\star_\xi Z_1 - \tau^\star(Z_1)\xi, \tag{2.16}$$

where $P$ is the projection morphism from $\Gamma(TL)$ onto $\Gamma(S(TL))$, $\overline{D}_{Z_1} PZ_2, \overline{D}^\star_{Z_1} PZ_2 \in \Gamma(S(TL))$ and $\overline{A}_\xi, \overline{A}^\star_\xi \in \Gamma(S(TL))$.

A lightlike hypersurface $(L, h, S(TL))$ is called screen conformal with regard to $\widetilde{D}$ if there exists a smooth function $\alpha$ satisfying

$$A_N = \alpha A_\xi \tag{2.17}$$

and it is called screen conformal with regard to $\widetilde{D}^\star$ if there exists a smooth function $\alpha^\star$ satisfying

$$A^\star_N = \alpha^\star \overline{A}^\star_\xi. \tag{2.18}$$

Furthermore, the following concepts could be given:

A lightlike hypersurface $(L, h, S(TL))$ of $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is called

i) totally geodesic with regard to $\widetilde{D}$ if $B = 0$,

ii) totally geodesic with regard to $\widetilde{D}^\star$ if $B^\star = 0$,

iii) $S(TL)-$geodesic with regard to $\widetilde{D}$ if $C = 0$,

iv) $S(TL)-$geodesic with regard to $\widetilde{D}^\star$ if $C^\star = 0$,

v) totally tangential umbilical with regard to $D$ if $B(Z_1, Z_2) = kh(Z_1, Z_2)$,

vi) totally tangential umbilical with regard to $D^\star$ if $B^\star(Z_1, Z_2) = k^\star h(Z_1, Z_2)$,

vii) totally normally umbilical with regard to $D$ if $A^\star_N Z_1 = kZ_1$,

viii) totally normally umbilical with regard to $D^\star$ if $A_N Z_1 = k^\star Z_1$,

where $k$ and $k^\star$ are smooth functions on $L$.

For any lightlike hypersurface $(M, g, S(TM))$, the following equalities are satisfied [6]:

$$B(Z_1, \xi) + B^\star(Z_1, \xi) = 0, \quad h(A_N Z_1 + A^\star_N Z_1, Z_2) = 0, \tag{2.19}$$
$$C(Z_1, PZ_2) = h(A_N Z_1, PZ_2), \quad C^\star(Z_1, PZ_2) = h(A^\star_N Z_1, PZ_2), \tag{2.20}$$
$$B(Z_1, Z_2) = h(\overline{A}^\star_\xi Z_1, Z_2) + B^\star(Z_1, \xi)\widetilde{h}(Z_2, N), \tag{2.21}$$
$$B^\star(Z_1, Z_2) = h(\overline{A}_\xi Z_1, Z_2) + B(Z_1, \xi)\widetilde{h}(Z_2, N). \tag{2.22}$$

## 3. Screen invariant lightlike hypersurfaces

**Definition 3.1.** *Let $(L, h, S(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, F)$. If $F(S(TL))$ belongs to $S(TL)$, then $(L, h, S(TL))$ is called a screen invariant lightlike hypersurface.*

In view of (2.2), we obtain that if $(L, h, S(TL))$ is a screen invariant lightlike hypersurface, then $F^{\star}(S(TL))$ belongs to $S(TL)$. Thus, we can write

$$F\xi = \lambda_1\xi + \mu_1 N, \quad F^{\star}\xi = \mu_2\xi + \mu_1 N, \tag{3.1}$$

$$FN = \lambda_2\xi + \mu_2 N, \quad F^{\star}N = \lambda_2\xi + \lambda_1 N, \tag{3.2}$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are smooth functions. Using the fact that $F^2\xi = \xi$, we find

$$\begin{aligned} \xi &= F(\lambda_1\xi + \mu_1 N) \\ &= \lambda_1^2\xi + \lambda_1\mu_1 N + \mu_1\lambda_2\xi + \mu_1\mu_2 N \end{aligned}$$

which yields

$$\lambda_1^2 + \mu_1\lambda_2 = 1 \text{ and } \lambda_1\mu_1 + \mu_1\mu_2 = 0. \tag{3.3}$$

Moreover, using the fact that $(F^{\star})^2\xi = \xi$, we find

$$\mu_2^2 + \mu_1\lambda_2 = 1 \text{ and } \mu_2\mu_1 + \mu_1\lambda_1 = 0. \tag{3.4}$$

Now, we write a tangent vector field $Z$ in $\Gamma(TL)$ by

$$Z = PZ + \eta(Z)\xi, \tag{3.5}$$

where $\eta(Z) = \widetilde{g}(Z, N)$ and $P$ is the projection morphism from $\Gamma(TL)$ onto $\Gamma(S(TL))$.

In view of (3.1), (3.2) and (3.5), we put

$$\begin{aligned} FZ &= FPZ + \eta(Z)F\xi \\ &= \varphi Z + \eta(Z)\lambda_1\xi + \eta(Z)\mu_1 N \end{aligned} \tag{3.6}$$

and

$$F^{\star}Z = \varphi^{\star}Z + \eta(Z)\mu_1\xi + \eta(Z)\mu_1 N, \tag{3.7}$$

where $\varphi Z$ and $\varphi^{\star}Z$ belong to $\Gamma(S(TM))$. Using (2.2), (3.6) and (3.7), we find

$$h(\varphi Z_1, Z_2) = h(Z_1, \varphi^{\star}Z_2) \tag{3.8}$$

for any $Z_1, Z_2 \in \Gamma(TM)$.

**Example 3.2.** *Let $(\mathbb{R}_1^4, \widetilde{h}, F)$ be an almost product-like Lorentzian manifold of Example 2.2. Consider a hypersurface $M$ given by*

$$L = \{(x_1, x_2, x_3, x_4) : x_1 = x_4\}.$$

*Then the induced metric of $M$ becomes*

$$h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{x_1} \end{bmatrix}.$$

*By a straightforward computation, we obtain*

$$\begin{aligned} Rad(TL) &= span\{\xi = \partial_1 + \partial_4\}, \\ S(TL) &= span\{e_1 = \partial_2, \ e_2 = \partial_3\} \end{aligned}$$

*and*

$$ltr(TL) = span\left\{N = \frac{1}{2e^{x_1}}(-\partial_1 + \partial_4)\right\}.$$

*Then, we find $F(S(TL)) \subset S(TL)$, which yields to $(L, h, S(TL))$ is a screen invariant lightlike hypersurface of $(\mathbb{R}^4_1, \widetilde{h}, \widetilde{D}, F)$.*

**Proposition 3.3.** *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$. Then we have the following equalities:*

$$(\widetilde{D}_Z\lambda_2)\xi + \lambda_2 D_Z\xi - \mu_2 A^\star_N Z = -\varphi A^\star_N Z - \eta(A^\star_N Z)\lambda_1\xi + \tau^\star(Z)\lambda_2\xi \tag{3.9}$$

*and*

$$\lambda_2 B(Z,\xi) + \widetilde{D}_Z\mu_2 = -\eta(A^\star_N Z)\mu_1. \tag{3.10}$$

**Proof**. From (3.2), we have

$$\begin{aligned}\widetilde{D}_Z FN &= \widetilde{D}_Z(\lambda_2\xi + \mu_2 N)\\ &= (\widetilde{D}_Z\lambda_2)\xi + \lambda_2\widetilde{D}_Z\xi + (\widetilde{D}_Z\mu_2)N + \mu_2\widetilde{D}_Z N.\end{aligned} \tag{3.11}$$

Putting (2.9) in (3.11), we obtain

$$\begin{aligned}\widetilde{D}_Z FN = (\widetilde{D}_Z\lambda_2)\xi + \lambda_2\nabla_Z\xi + \lambda_2 B(Z,\xi)N + (\widetilde{D}_Z\mu_2)N - \mu_2 A^\star_N Z\\ + \mu_2\tau^\star(Z)N.\end{aligned} \tag{3.12}$$

Besides this fact, using (2.10) we have

$$\begin{aligned}F\widetilde{D}_Z N &= F(-A^\star_N Z + \tau^\star(Z)N)\\ &= -FA^\star_N Z + \tau^\star(Z)FN.\end{aligned} \tag{3.13}$$

Putting (3.2) and (3.6) in (3.13), we find

$$\begin{aligned}F\widetilde{D}_Z N = -\varphi A^\star_N Z - \eta(A^\star_N Z)\lambda_1\xi - \eta(A^\star_N Z)\mu_1 N + \tau^\star(Z)\lambda_2\xi\\ + \tau^\star(Z)\mu_2 N.\end{aligned} \tag{3.14}$$

Using the fact that $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is a locally product-like statistical manifold in (3.14), we get (3.9) and (3.10). ■

As a result of Proposition 3.3, we find

**Theorem 3.4.** *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. If $A^\star_N = 0$, then $\xi$ is a recurrent vector field with regard to $D$ and*

$$B(Z,\xi) = -\frac{1}{\lambda_2}\widetilde{D}_Z\mu_2 \tag{3.15}$$

*is satisfied.*

**Proof**. Under the assumption, if we write $A^\star_N Z = 0$ in (3.9), we obtain

$$D_Z\xi = \frac{1}{\lambda_2}(\tau^\star(Z)\lambda_2 - \widetilde{D}_Z\lambda_2)\xi, \tag{3.16}$$

which shows that $\xi$ is a recurrent vector field. Putting $A^\star_N Z = 0$ in (3.10), we obtain (3.15). ■

**Corollary 3.5.** *Let $(L, h, S(TH))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. If $\xi$ is a recurrent vector field, then one of the following situations occurs:*

*i)* $A^\star_N Z$ *is in the direction of* $\xi$.

*ii)* $A^\star_N Z = 0$.

**Corollary 3.6.** *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then,* $A^\star_N = 0$ *and* $\lambda_2$ *is constant if and only if* $B$ *vanishes on* $Rad(TL)$.

With similar arguments in the proof of Proposition 3.3, we find

**Proposition 3.7.** *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then the following equalities hold for any* $Z \in \Gamma(TL)$*:*

$$(\widetilde{D}^\star_Z \lambda_2)\xi + \lambda_2 D^\star_Z \xi - \lambda_1 A_N Z = -\varphi^\star A_N Z - \eta(A_N Z)\mu_1 \xi + \tau(Z)\lambda_2 \xi \tag{3.17}$$

*and*

$$\lambda_2 B^\star(Z, \xi) + \widetilde{D}^\star_Z \lambda_1 = -\eta(A_N Z)\mu_1. \tag{3.18}$$

**Theorem 3.8.** *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. If* $A_N X = 0$*, then* $\xi$ *is a recurrent vector field with regard to* $D^\star$ *and the following equality is satisfied:*

$$B^\star(Z, \xi) = -\frac{1}{\lambda_2}(\widetilde{D}^\star_Z \lambda_1). \tag{3.19}$$

**Corollary 3.9.** *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. If* $\xi$ *is a recurrent vector field, then one of the following situations occurs:*

*i)* $A_N$ *is in the direction of* $\xi$.

*ii)* $A_N = 0$.

**Corollary 3.10.** *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface of a locally product-like statistical manifold.* $A_N = 0$ *and* $\lambda_2$ *is constant if and only if* $B^\star$ *vanishes on* $Rad(TL)$.

**Proposition 3.11.** *Let* $(L, h, S(TL))$ *be a screen invariant lightlike hypersurface. Then the following relations are satisfied:*

$$(\widetilde{D}_Z \lambda_1)\xi + \lambda_1 D_Z \xi - \mu_1 A^\star_N Z = \varphi D_Z \xi + \eta(D_Z \xi)\lambda_1 \xi + B(Z, \xi)\lambda_2 \xi \tag{3.20}$$

*and*

$$\lambda_1 B(Z, \xi) + \widetilde{D}_Z \mu_1 + \mu_1 \tau^\star(Z) = \eta(D_Z \xi)\mu_1 + B(Z, \xi)\mu_2. \tag{3.21}$$

**Proof**. From (3.1), we have

$$\widetilde{D}_Z F\xi = \widetilde{D}_Z(\lambda_1 \xi + \mu_1 N). \tag{3.22}$$

Using (2.9) and (2.10) in (3.22), we obtain

$$\begin{aligned}\widetilde{D}_Z F\xi = {} & (\widetilde{D}_Z \lambda_1)\xi + \lambda_1 D_Z \xi + \lambda_1 B(Z, \xi)N + (\widetilde{D}_Z \mu_1)N - \mu_1 A^\star_N Z \\ & + \mu_1 \tau^\star(Z)N.\end{aligned} \tag{3.23}$$

On the other hand, using (2.9), (2.10) and (3.1), we have

$$\begin{aligned}F\widetilde{D}_Z \xi = {} & \varphi \nabla_Z \xi + \eta(D_Z \xi)\lambda_1 \xi + \eta(D_Z \xi)\mu_1 N + B(Z, \xi)\lambda_2 \xi \\ & + B(Z, \xi)\mu_2 N.\end{aligned} \tag{3.24}$$

Using the fact that $(\widetilde{L}, \widetilde{h}, \widetilde{D}, F)$ is a locally product-like statistical manifold, we find (3.20) and (3.21) immediately. ■

As a result of (3.20), we find

**Theorem 3.12.** *If $\xi$ is a parallel vector field with regard to $D$, then one of the following relations holds:*

*i)* $A^{\star}_{N}$ *is in the direction of* $\xi$.

*ii)* $A^{\star}_{N} = 0$.

**Proof.** Under the assumption, if $\xi$ is a parallel vector field with regard to $D$, we obtain from (3.20) that

$$A^{\star}_{N} Z = \frac{1}{\mu_1}(\widetilde{D}_Z \lambda_1 + B(Z,\xi)\lambda_2)\xi,$$

which shows that $A^{\star}_{N}$ is in the direction of $\xi$ or $A^{\star}_{N} = 0$. ■

With similar arguments as in the proof of Proposition 3.11, we get the followings:

**Proposition 3.13.** *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then the following relations hold:*

$$(\widetilde{D}^{\star}_{Z}\mu_2)\xi + \mu_2 D^{\star}_{Z}\xi - \mu_1 A_N Z = \varphi^{\star} D^{\star}_{Z}\xi + \eta(D^{\star}_{Z}\xi)\mu_1\xi + B^{\star}(Z,\xi)\lambda_2\xi \tag{3.25}$$

*and*

$$\mu_2 B^{\star}(Z,\xi) + \widetilde{D}^{\star}_{Z}\mu_1 + \mu_1\tau(Z) = \eta(D^{\star}_{Z}\xi)\mu_1 + B^{\star}(Z,\xi)\lambda_1. \tag{3.26}$$

**Theorem 3.14.** *If $\xi$ is a parallel vector field with regard to $D^{\star}$, then one of the following situations holds:*

*i)* $A_N$ *is in the direction of* $\xi$.

*ii)* $A_N = 0$.

**Proposition 3.15.** *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then we have the following formulas:*

$$\begin{aligned}(\nabla_{Z_1}\varphi)Z_2 &= \eta(D_{Z_1}Z_2)\lambda_1\xi + B(Z_1,Z_2)\lambda_2\xi - C(Z_1,PZ_2)\lambda_1\xi + g(A^{\star}_{N}Z_1,Z_2)\lambda_1\xi\\ &\quad - \eta(Z_2)\lambda_1 D_{Z_1}\xi + \eta(Z_2)\mu_1 A^{\star}_{N} Z_1\end{aligned} \tag{3.27}$$

*and*

$$\begin{aligned}&B(Z_1,\varphi Z_2) + \eta(Z_2)\lambda_1 B(Z_1,\xi) + C(Z_1,PZ_2)\mu_1 - g(A^{\star}_{N}Z_1,Z_2)\mu_1 + \widetilde{D}_{Z_1}\mu_1\eta(Z_2)\\ &+ \eta(Z_2)\mu_1\tau^{\star}(Z_1) = \eta(D_{Z_1}Z_2)\mu_1 + B(Z_1,Z_2)\mu_2.\end{aligned} \tag{3.28}$$

**Proof.** Using (3.6), we have

$$\begin{aligned}\widetilde{D}_{Z_1}FZ_2 &= \widetilde{D}_{Z_1}\varphi Z_2 + Z_1 g(Z_2,N)\lambda_1\xi + \widetilde{D}_{Z_1}\lambda_1\eta(Z_2)\xi + \eta(Z_2)\lambda_1\widetilde{D}_{Z_1}\xi\\ &\quad + Z_1 g(Z_2,N)\mu_1 N + \widetilde{D}_{Z_1}\mu_1\eta(Z_2)N + \eta(Z_2)\mu_1\widetilde{D}_{Z_1}N.\end{aligned} \tag{3.29}$$

Considering (2.4), (2.9) and (2.10) in (3.29), it follows that

$$\begin{aligned}\widetilde{D}_{Z_1}FZ_2 &= D_{Z_1}\varphi Z_2 + B(Z_1,\varphi Z_2)N + C(Z_1,PZ_2)\lambda_1\xi - g(A^{\star}_{N}Z_1,Z_2)\lambda_1\xi\\ &\quad + \widetilde{D}_{Z_1}\lambda_1\eta(Z_2)\xi + \eta(Y)\lambda_1 D_{Z_1}\xi + \eta(Z_2)\lambda_1 B(Z_1,\xi)N\\ &\quad + C(Z_1,PZ_2)\mu_1 N - g(A^{\star}_{N}Z_1,Z_2)\mu_1 N + \widetilde{D}_{Z_1}\mu_1\eta(Z_2)N\\ &\quad - \eta(Z_2)\mu_1 A^{\star}_{N}Z_1 + \eta(Z_2)\mu_1\tau^{\star}(Z_1)N.\end{aligned} \tag{3.30}$$

Besides the above fact, we have from (3.6) that

$$\begin{aligned} F\widetilde{D}_{Z_1}Z_2 &= FD_{Z_1}Z_2 + B(Z_1,Z_2)FN \\ &= \varphi D_{Z_1}Z_2 + \eta(D_{Z_1}Z_2)\lambda_1\xi + \eta(D_{Z_1}Z_2)\mu_1 N + B(Z_1,Z_2)\lambda_2\xi \\ &\quad + B(Z_1,Z_2)\mu_2 N. \end{aligned} \tag{3.31}$$

Considering the tangential and transversal parts of (3.30), (3.31) and using the fact that $\widetilde{D}_{Z_1}FZ_2 = F\widetilde{D}_{Z_1}Z_2$, we get (3.27) and (3.28) immediately. ■

As a result of (3.27), we get the following theorem:

**Theorem 3.16.** *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold. Then $\varphi$ is parallel with regard to $D$.*

**Proof**. For a special case, if we choose $Z_2 \in \Gamma(TL)$ in (3.27), then we get

$$(D_{Z_1}\varphi)Z_2 = [\eta(D_{Z_1}Z_2)\lambda_1 + B(Z_1,Z_2)\lambda_2 - C(Z_1,Z_2)\lambda_1 + h(A^{\star}_N Z_1, Z_2)\lambda_1]\xi,$$

which is a contradiction to $Z_2$ belonging $\Gamma(S(TL))$. Thus, $\varphi$ is parallel with regard to $D$. ■

**Proposition 3.17.** *Let $(L, h, S(TL))$ be a totally geodesic screen invariant lightlike hypersurface with regard to $\widetilde{D}$. Then the following relation is satisfied:*

$$C(Z_1, PZ_2) = \eta(D_{Z_1}Z_2) + h(A^{\star}_N Z_1, Z_2). \tag{3.32}$$

**Proof**. Putting $B(Z_1, Z_2) = 0$ for any $Z_1, Z_2 \in \Gamma(TL)$, the proof is easy to follow from (3.27) or (3.28). ■

## 4. Concurrent vector fields

Let $(\widetilde{L}, \widetilde{h}, \widetilde{D})$ be a statistical manifold. A vector field $\zeta$ is called a concurrent vector field with regard to $\widetilde{D}$ (resp. $\widetilde{D}^{\star}$) if $\widetilde{D}_Z\zeta = Z$ (resp. $\widetilde{D}^{\star}_Z\zeta = Z$) for each $Z \in \Gamma(T\widetilde{L})$.

If $\zeta$ is a concurrent vector field with respect to $\widetilde{D}$ and $\widetilde{D}^{\star}$, we obtain from (2.4) that

$$\widetilde{h}(\widetilde{D}_{Z_2}Z_1, \zeta) = \widetilde{h}(\widetilde{D}^{\star}_{Z_2}Z_1, \zeta)$$

is satisfied for each $Z_1, Z_2 \in \Gamma(T\widetilde{L})$. Also, we get from (2.5) that if $\zeta$ is a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$, then it is also concurrent with regard to the Levi-Civita connection $\widetilde{D}^0$.

Now, we recall the definition of rigged metric for lightlike hypersurfaces [3]:

**Definition 4.1.** *Let $(L, h, S(TL))$ be a lightlike hypersurface and $\psi$ be a vector field such that $\psi_p \notin T_pL$ for any $p \in L$. If we define a $1-$form $\eta$ satisfying*

$$\eta(X) = \widetilde{h}(Z, \psi)$$

*then $\psi$ is called a rigging vector field.*

If we choose $\psi = N$, then a rigged metric $\overline{h}$ with regard to $N$ is defined by

$$\overline{h}(Z_1, Z_2) = h(Z_1, Z_2) + \eta(Z_1)\eta(Z_2) \tag{4.1}$$

for each $Z_1, Z_2 \in \Gamma(TL)$. It is easy to see that $\overline{h}$ is non-degenerate and the following relations are satisfied:

$$\overline{h}(N, Z) = \eta(Z), \;\; \overline{h}(\xi, \xi) = 1 \tag{4.2}$$

and

$$\overline{h}(Z_1, Z_2) = h(Z_1, Z_2),\ \ \forall Z_1, Z_2 \in \Gamma(TL) \tag{4.3}$$

It is known that the gradient of a smooth function could not be defined on a degenerate metric $h$ since the inverse of $h$ does not exist. But the gradient of a function $f$ could be defined by using a rigged metric as follows:

$$gradf = \sum_{i=1}^{n} h^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j},$$

where $[h^{ij}]$ is the inverse of $\overline{h}$. We note that $[h^{ij}]$ is also known as the pseudo-inverse of $h$ [3].

Let $(L, h, S(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{D})$ and $\zeta$ be a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$. Then we can write

$$\zeta = \zeta^T + \zeta^N, \tag{4.4}$$

where $\zeta^T$ is the tangential part, while $\zeta^N$ is the transversal part of $\zeta$. In view of (4.1) and (4.4), we obtain

$$\begin{aligned} \overline{h}(\zeta^T, \xi) &= h(\zeta^T, \xi) + \eta(\zeta^T)\eta(\xi). \\ &= \widetilde{h}(\zeta, N). \end{aligned}$$

In view of (2.4), we find

$$\begin{aligned} Z\overline{h}(\zeta^T, \xi) &= X\widetilde{h}(\zeta, N) \\ &= \widetilde{h}(\widetilde{D}_Z\zeta, N) + \widetilde{h}(\widetilde{D}^{\star}_Z N, \zeta) \\ &= \eta(Z) - \widetilde{h}(A_N Z, \zeta) + \tau(Z)\eta(\zeta). \end{aligned} \tag{4.5}$$

Moreover, (4.5) could be written as

$$\begin{aligned} Z\overline{h}(\zeta^T, \xi) &= \widetilde{h}(\widetilde{D}^{\star}_Z\zeta, N) + \widetilde{h}(\widetilde{D}_Z N, \zeta) \\ &= \eta(Z) - \widetilde{g}(A^{\star}_N Z, \zeta) + \tau^{\star}(Z)\eta(\zeta). \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we find

**Proposition 4.2.** *Let $(L, h, S(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{D})$. If $\zeta$ is a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$, then*

$$\widetilde{h}(A_N Z, \zeta) - \tau(Z)\eta(\zeta) = \widetilde{h}(A^{\star}_N Z, \zeta) - \tau^{\star}(Z)\eta(\zeta) \tag{4.7}$$

*is satisfied for any $Z \in \Gamma(TL)$. In particular, if $\eta(\zeta) = 0$ then*

$$\widetilde{h}(A_N Z, \zeta) = \widetilde{h}(A^{\star}_N Z, \zeta) \tag{4.8}$$

*is satisfied.*

As a result of (2.20) and Proposition 4.2, we have

**Corollary 4.3.** *Let $(L, h, S(TL))$ be an $S(TL)-$geodesic lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{D})$. If $\zeta$ is a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$, then*

$$\tau(Z) = \tau^{\star}(Z) \tag{4.9}$$

*is satisfied for any $Z \in \Gamma(TL)$.*

**Proposition 4.4.** *Let $(L, h, S(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{D})$. Then we have the following situations:*

*i) If $\zeta$ is a concurrent vector field with regard to $\widetilde{D}$, then $B(Z,\zeta)=0$ is satisfied for any $Z\in\Gamma(TL)$.*

*ii) If $\zeta$ is a concurrent vector field with regard to $\widetilde{D}^{\star}$, then $B^{\star}(Z,\zeta)=0$ is satisfied for any $Z\in\Gamma(TL)$.*

As a result of Proposition 4.4, (2.9) and (2.11), we see that if $v$ is a concurrent vector field with regard to $\widetilde{D}$ (resp. $\widetilde{D}^{\star}$), then it is also concurrent with regard to $D$ (resp $D^{\star}$). We note that the converse part of this claim is not correct in general.

**Example 4.5.** *Let $(L,h,S(TL))$ be a screen invariant lightlike hypersurface of Example 3.2. In view of Example 2.4, it is clear that $\zeta=\partial_1$ is a concurrent vector field with regard to $\widetilde{D}$ and $\widetilde{D}^{\star}$.*

Now, suppose that $\zeta$ belongs to $\Gamma(TL)$. Then we write

$$\zeta=P\zeta+a\xi, \tag{4.10}$$

where $P\zeta\in\Gamma(S(TL))$ and $\eta(\zeta)=a$. Thus, we find

**Proposition 4.6.** *Let $\zeta$ be a concurrent vector field with regard to $\widetilde{D}$. Then we have*

$$h(A_N\zeta,\zeta)=\frac{1}{2}a\tau(\zeta). \tag{4.11}$$

**Proof**. From (4.5), it follows that

$$\begin{aligned}\zeta\overline{h}(\zeta,\xi)&=\eta(\zeta)-g(A_N\zeta,\zeta)+\tau(\zeta)\eta(\zeta)\\&=a-g(A_N\zeta,\zeta)+a\tau(\zeta).\end{aligned} \tag{4.12}$$

Now, we compute the left-hand side of (4.12). From (4.1), we find

$$\zeta\overline{h}(\zeta,\xi)=\eta(\zeta)+h(A_N\zeta,\zeta). \tag{4.13}$$

The proof is easy to follow from (4.12) and (4.13). ■

In a similar way to Proposition 4.6, we find

**Proposition 4.7.** *Let $\zeta$ be a concurrent vector field with regard to $\widetilde{D}$. Then we have*

$$h(A_N^{\star}\zeta,\zeta)=\frac{1}{2}a\tau^{\star}(\zeta). \tag{4.14}$$

**Theorem 4.8.** *Let $(L,h,S(TL))$ be an $S(TL)-$geodesic lightlike hypersurface with regard to $\widetilde{D}$ and $\zeta$ be a concurrent vector field with regard to $\widetilde{D}$ such that $\zeta\in\Gamma(TL)$. Then $\zeta$ could not be concurrent with regard to $\widetilde{D}^{\star}$.*

**Proof**. Under the assumption and from (4.13), we get $A_N\zeta=\tau(\zeta)=0$. If we put this equation in (2.12), we find $\widetilde{D}_\zeta N=0$, which shows that $\zeta$ could not be concurrent with regard to $\widetilde{D}^{\star}$. ■

In a similar way to Theorem 4.8, we obtain

**Theorem 4.9.** *Let $(L,h,S(TL))$ be an $S(TL)-$geodesic lightlike hypersurface with regard to $\widetilde{D}^{\star}$ and $\zeta$ be a concurrent vector field with regard to $\widetilde{D}^{\star}$ such that $\zeta\in\Gamma(TL)$. Then $\zeta$ could not be concurrent with regard to $\widetilde{D}$.*

Now, we shall investigate concurrent vector fields in screen invariant lightlike hypersurfaces.

**Proposition 4.10.** *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold and $\zeta$ be a concurrent vector field with regard to $\widetilde{D}$. Then we have the following relations:*

$$A^{\star}_N(\zeta) = \frac{1}{\mu_1}(\widetilde{D}_v\lambda_1)\xi \tag{4.15}$$

*and*

$$\tau^{\star}(v) = 1 - \frac{1}{\mu_1}(\widetilde{D}_\zeta\mu_1). \tag{4.16}$$

**Proof**. Since $\zeta$ is concurrent with regard to $\widetilde{D}$, we write

$$\widetilde{D}_\zeta F\xi = F\xi. \tag{4.17}$$

Using the fact that if $\zeta$ is a concurrent vector field with regard to $\widetilde{D}$, then it is also concurrent with regard to $D$ and using (3.23), we obtain

$$\widetilde{D}_\zeta F\xi = (\widetilde{D}_\zeta\lambda_1)\xi + \lambda_1\xi + (\widetilde{D}_\zeta\mu_1)N - \mu_1 A^{\star}_N\zeta + \mu_1\tau^{\star}(\zeta)N. \tag{4.18}$$

From (3.1), (4.17) and (4.18), the proof is easy to follow. ■

As a result of Proposition 4.10, we obtain

**Corollary 4.11.** *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface of a locally product-like statistical manifold and $\zeta$ be a concurrent with regard to $\widetilde{D}$, then the following situations occur:*

*i) If $\widetilde{D}_\zeta\lambda_1 = 0$, then $A^{\star}_N(\zeta) = 0$.*

*ii) If $\widetilde{D}_\zeta\mu_1 = 0$, then $\tau^{\star}(\zeta) = 1$.*

*iii) If $\mu_1 = 0$, then $\widetilde{D}_\zeta\lambda_1 = 0$ and $\tau^{\star}(\zeta) = 1$.*

By a similar arguments to Proposition 4.10, we get

**Proposition 4.12.** *Let $(L, h, S(TL))$ be a screen invariant lightlike hypersurface and $\zeta$ be a concurrent vector field with regard to $\widetilde{D}^{\star}$. Then we have the following relations:*

$$A_N\zeta = \frac{1}{\mu_1}(\widetilde{D}_\zeta\mu_2)\xi \tag{4.19}$$

*and*

$$\tau^{\star}(\zeta) = 1 - \frac{1}{\mu_1}(\widetilde{D}_\zeta\mu_1). \tag{4.20}$$

**Corollary 4.13.** *Let $\zeta$ be concurrent with regard to $\widetilde{D}^{\star}$. Then the following situations occur:*

*i) If $\widetilde{D}_\zeta\mu_2 = 0$, then $A_N\zeta = 0$.*

*ii) If $\widetilde{D}_\zeta\mu_1 = 0$, then $\tau(\zeta) = 1$.*

*iii) If $\mu_1 = 0$, then $\widetilde{D}_\zeta\mu_2 = 0$ and $\tau(\zeta) = 1$.*

**Corollary 4.14.** *Let $(L, h, S(TL))$ be an $S(TL)-$umbilical screen invariant lightlike hypersurface with regard to $\widetilde{D}^{\star}$ (resp. $\widetilde{D}$) and $\zeta \notin \Gamma(Rad(TL))$. Then $\zeta$ is not concurrent with regard to $\widetilde{D}$ (resp $\widetilde{D}^{\star}$).*

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