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On solutions of the Diophantine equation $L_n \pm L_m = p^a$

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Abstract. Lucas sequence is one of the most studied binary recurrence sequence defined by the relation $L_{n+2} = L_{n+1} + L_n;\ L_0 = 2, L_1 = 1$. In this paper, we investigate all the sums and differences of two Lucas numbers that are powers of a odd prime p satisfying $p < 10^3$.

AMS Subject Classifications: 11B39, 11D72, 11J86.

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Contents

1. Introduction

The Fibonacci and Lucas numbers are always being studied by many researchers whether as solutions of Diophantine equations or their existence in the nature. The Fibanacci (F_n) and Lucas (L_n) numbers are the most common binary recurrence sequences definded by the relations:

$$F_{n+2} = F_{n+1} + F_n;\ L_{n+2} = L_{n+1} + L_n$$

with the initial values $F_0 = 0, L_0 = 2, F_1 = L_1 = 1$. Both the sequences have the characteristic equation $x^2 - x - 1 = 0$ with the characteristic roots $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The closed form or the binet form of these numbers are given by:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n; \quad n \geq 0. \tag{1.1}$$

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Bugeaud et al. [5] investigated the Diophantine equations $F_n = y^p$ and $L_n = y^p$ and determined all perfect powers in Fibonacci and Lucas sequences. Similar Diophantine equations have been tackled by many researcher involving powers of $2, 3, 5$ and the recurrence sequences such as Fibonacci, Lucas, Pell and k-Fibonacci numbers (see [3, 4, 8, 11–13]) .

In this paper, we explore the solutions of the Diophantine equation:

$$L_n \pm L_m = p^a, \tag{1.2}$$

where p is any odd prime and n, m, a are nonnegative integers satisfying $n \geq m$.

2. Preliminaries

This section deals with the basic concepts of algebraic numbers, some results concerning the bounds of linear forms in logarithms and reduction methods from the theory of continued fractions, which plays a vital role during the proof of our main result.

Let γ be an algebraic number of degree d having the minimal polynomial

$$a_0 \prod_{i=1}^{d} (x - \gamma_i) \in \mathbb{Z}[x],$$

where γ^i are conjugates of γ and $a_0 > 0$. If $\gamma \neq 0$, then its absolute logarithmic height is defined as

$$h(\gamma) = \frac{1}{d}(\log |a_0| + \sum_{i=1}^{d} \log max\{1, |\gamma_i|\}).$$

The following properties of the logarithmic height holds, which will be used in the forthcoming sections as and when necessary with or without any further references:

- $h(\gamma \pm \eta) \leq h(\gamma) + h(\eta) + \log 2$
- $h(\gamma\eta^{\pm 1}) \leq h(\gamma) + h(\eta)$
- $h(\gamma^s) = |s|h(\gamma); \quad s \in \mathbb{Z}.$

2.1. Inequalities involving the Lucas numbers

Inequalities involving the Lucas numbers In this section, we state and prove important inequalities associated with the Lucas numbers that will be used in solving the equation 1.2

Proposition 2.1 (P. Tiebekabe and I. Diouf [12])**.**
For $n \geq 2$, we have

$$0.94\alpha^n < (1 - \alpha^{-6})\alpha^n \leq L_n \leq (1 + \alpha^{-4})\alpha^n < 1.15\alpha^n \tag{2.1}$$

Proof.
This follows directly from the formula $L_n = \alpha^n + (-1)^n \alpha^{-n}$. ∎

Proposition 2.2. *[5]*
The only prime powers in Fibonacci and Lucas sequences are

$$F_1 = F_2 = 1,\ F_6 = 2^3,\ L_1 = 1, L_3 = 2^2.$$

2.2. Linear forms in logarithms and continued fractions

In order to prove our main result, we have to use a Baker-type lower bound several times for a non-zero linear forms of logarithms in algebraic numbers. There are many of these methods in the literature like that of Baker and Wüstholz in [1]. We recall the result of Bugeaud, Mignotte, and Siksek which is a modified version of the result of Matveev [10]. With the notation of section 2, Laurent, Mignotte, and Nesterenko [9] proved the following theorem:

Theorem 2.3.
Let γ_1, γ_2 be two non-zero algebraic numbers, and let $\log \gamma_1$ and $\log \gamma_2$ be any determination of their logarithms. Put $D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]/[\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}]$, and

$$\Gamma := b_2 \log \gamma_2 - b_1 \log \gamma_1,$$

where b_1 and b_2 are positive integers. Further, let A_1, A_2 be real numbers > 1 such that

$$\log A_i \geq max \left\{ h\left(\gamma_i\right), \frac{|\log \gamma_i|}{D}, \frac{1}{D} \right\}, (i = 1, 2).$$

Then assuming that γ_1 and γ_2 are multiplicatively independent, we have

$$\log |\Gamma| > -30.9 \cdot D^4 \left(max\{\log b', \frac{21}{D}, \frac{1}{2}\} \right)^2 \log A_1 \cdot \log A_2,$$

where

$$b^{'} = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

We shall also need the following theorem due to Mantveev, Lemma due to Dujella and Pethő and Lemma due to Legendre [7, 10].

Theorem 2.4 (Matveev [10])**.**
Let $n \geq 1$ an integer. Let $\mathbb{L}$ a field of algebraic number of degree D. Let $\eta_1, \eta_2, ..., \eta_l$ non-zero elements of $\mathbb{L}$ and let $b_1, b_2, ..., b_l$ integers,

$$B := max \left\{ |b_1|, |b_2|, ..., |b_l| \right\},$$

and

$$\Lambda := \eta_1^{b_1} ... \eta_l^{b_l} - 1 = \left(\prod_{i=1}^{l} \eta_i^{b_i} \right) - 1.$$

Let $A_1, A_2, ..., A_l$ reals numbers such that

$$A_j \geq max \left\{ Dh(\eta_j), |\log(\eta_j)|, 0.16 \right\}, 1 \leq j \leq l.$$

Assume that $\Lambda \neq 0$, so we have

$$\log |\Lambda| > -3 \times 30^{l+4} \times (l+1)^{5.5} \times d^2 \times A_1 ... A_l (1 + \log D)(1 + \log nB)$$

Further, if $\mathbb{L}$ is real, then

$$\log |\Lambda| > -1.4 \times 30^{l+3} \times (l)^{4.5} \times d^2 \times A_1 ... A_l (1 + \log D)(1 + \log B).$$

During our calculations, we get upper bounds on our variables which are too large, so we have to reduce them. To do this, we use some results from the theory of continued fractions. In particular, for a non-homogeneous linear form with two integer variables, we use a slight variation of a result due to Dujella and Pethő, (1998) which is in itself a generalization of the result of Baker and Davemport [2].

For a real number X, we write $||X|| := min \left\{ |X - n| : n \in \mathbb{Z} \right\}$ for the distance of X to the nearest integer.

Lemma 2.5 (Dujella and Pethő, [7])**.**
Let M a positive integer, let p/q the convergent of the continued fraction expansion of k such that $q > 6M$ and let A, B, μ real numbers such that $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\kappa q||$. If $\epsilon > 0$ then there is no solution of the inequality

$$0 < m\kappa - n + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log{(Aq/\epsilon)}}{\log B} \leq m \leq M.$$

Lemma 2.6 (Legendre)**.**
Let τ real number such that x, y are integers such that

$$|\tau - \frac{x}{y}| < \frac{1}{2y^2}.$$

then $\frac{x}{y} = \frac{p_k}{q_k}$ is the convergence of τ.

Further

$$|\tau - \frac{x}{y}| < \frac{1}{(q_{k+1}+2)y^2}.$$

3. Main Results

This section deals with the main findings of the following Diophantine equation.

Theorem 3.1.
The only solutions (n, m, a) of the exponential Diophantine equation (1.2) in non negative integers n, m, a and odd prime p are listed in Table 1 and Table 2.

p	(n, m, a)
3	$(3, 1, 1), (4, 3, 1)(5, 0, 2)$
5	$(4, 0, 1), (7, 3, 2), (8, 6, 2)$
7	$(5, 3, 1), (6, 5, 1)$
11	$(6, 4, 1), (7, 6, 1)$
29	$(8, 6, 1), (9, 8, 1)$
47	$(9, 7, 1), (10, 9, 1)$
199	$(12, 10, 1), (13, 12, 1)$
521	$(14, 12, 1), (15, 14, 1)$

Table 1: $L_n - L_m = p^a$.

p	(n, m, a)
3	$(1, 0, 1), (4, 0, 2)$
5	$(2, 0, 1), (3, 1, 1), (6, 4, 2)$
7	$(3, 2, 1)$
11	$(4, 3, 1)$
29	$(6, 5, 1)$
47	$(7, 6, 1)$
199	$(10, 9, 1)$
521	$(12, 11, 1)$

Table 2: $L_n + L_m = p^a$.

Corollary 3.1.
The only solutions (p, a) of the double exponential Diophantine equations $L_u - L_v = L_s + L_t$ in non negative intergers $u,\ v,\ s$ and t with $u > v$ are liste in Table 3.

(p,a)	$L_n \pm L_m$
$(3,1)$	$L_1 + L_0,\ L_3 - L_1,\ L_4 - L_3$
$(5,1)$	$L_2 + L_0,\ L_3 + L_1,\ L_4 - L_0$
$(7,1)$	$L_3 + L_2,\ L_5 - L_3,\ L_6 - L5$
$(3,2)$	$L_4 + L_0,\ L_5 - L_0$
$(11,1)$	$L_4 + L_3,\ L_6 - L_4,\ L_7 - L_6$
$(5,2)$	$L_6 + L_4,\ L_7 - L_3$
$(29,1)$	$L_6 + L_5,\ L_8 - L_6,\ L_9 - L_8$
$(47,1)$	$L_7 + L_6,\ L_9 - L_7,\ L_{10} - L_9$
$(199,1)$	$L_{10} + L_9,\ L_{12} - L_{10}, L_{13} - L_{12}$
$(521,1)$	$L_{12} + L_{11},\ L_{14} - L_{12},\ L_{15} - L_{14}$

Table 3: $L_n \pm L_m = p^a$.

Solutions in Table 3 are intersections of those in Table 1 and Tabler 2.

Proof of theorem 3.1.
It is obvious that, the case $n = m$ is not possible. Therefore, we assume that $n > m$. A computation using *SageMath* in the range $0 \le m < n \le 200$ reveals that there does not exist any solution of (1.2) other than the solutions listed in Table 1. Furthermore, it is easy to observe that when $1 \le (n - m) \le 3$, $L_n \pm L_m$ results either in $L_k, 2L_k$ or $5F_k$ for some values of k and hence, using Proposition 2.2 we obtain the solutions of (1.2). So from now on, we assume that $n > 200$ and $(n - m) \ge 4$.

Combining (1.1), (1.2) and (2.1) we get:

$$p^a = L_n \pm L_m \le L_n + L_m \le \alpha^{n+1} + \alpha^{m+1} < 2\alpha^{n+1} < 2^{n+2}.$$

Applying logarithms on both sides of the above inequality, we obtain

$$a \log p \le (n + 2) \log 2 \implies a \le (n + 2)\frac{\log 2}{\log p}.$$

It is easy to observe that for any prime p, $0 < \frac{\log 2}{\log p} < 4/5$ and hence, $a \le n+1$. Indeed, for all $n > 200$ and any prime p, $a < n$. Using (1.1) in (1.2) we can obtain the inequality:

$$L_n \pm L_m = \alpha^n + \beta^n \pm L_m = p^a \implies \alpha^n - p^a = -\beta^n \pm L_m.$$

Taking absolute value both sides, we get

$$|\alpha^n - p^a| = |\beta^n \pm L_m| \le |\beta|^n + L_m < \frac{1}{2} + 2\alpha^m$$

$\because |\beta|^n < \frac{1}{2}$, and $L_m < 2\alpha^m$. Dividing both sides by α^n and considering that $n > m$, we get:

$$|1 - \alpha^{-n} \cdot p^a| < \frac{\alpha^{-n}}{2} + 2\alpha^{m-n} < \frac{1}{\alpha^{n-m}} + \frac{2}{\alpha^{n-m}} \because \frac{1}{2\alpha^n} < \frac{1}{\alpha^{n-m}}; n > m.$$

Hence

$$|1 - \alpha^{-n} \cdot p^a| < \frac{3}{\alpha^{n-m}} \tag{3.1}$$

To apply Theorem 2.4, we take $\Gamma := \alpha^{-n} \cdot p^a - 1$ with $\eta_1 = \alpha, \eta_2 = p, b_1 = -n, b^2 = a$. The logarithm heights of η_1 and η_2 are:
$h(\eta_1) = \frac{1}{2}\log\alpha = 0.2406\ldots, h(\eta_2) = \log p$, thus we can choose

$$A_1 := 0.5 \text{ and } A_2 := 2\log p, B := \max\{1, n, a\} = n$$

Using Theorem 2.4, we have

$$\log|\Gamma| > -1.4 \times 30^{2+3} \times 2^{4.5} \times 2^2 \times 0.5 \cdot 2\log p \cdot (1+\log 2)(1+\log n),$$

which when combined with (3.1) gives

$$(n-m)\log\alpha < 6.23 \cdot 10^9 \log p \cdot (1+\log n). \tag{3.2}$$

We define a second linear form in logarithm by rewriting (1.2) as follows:

$$\alpha^n\left(1 \pm \alpha^{m-n}\right) - p^a = -\beta^n \mp \beta^m.$$

Taking absolute values in the above relation with the fact that $|\beta| < 1$, we get

$$|\alpha^n\left(1 \pm \alpha^{m-n}\right) - p^a| < 2, \quad \forall n > 200, m \geq 0.$$

Dividing both sides of the above inequality by $\alpha^n\left(1 + \alpha^{m-n}\right)$, we obtain

$$|1 - p^a\alpha^{-n}\left(1 \pm \alpha^{m-n}\right)^{-1}| < \frac{2}{\alpha^n}\ . \tag{3.3}$$

We define

$$\Lambda := p^a\alpha^{-n}\left(1 \pm \alpha^{m-n}\right)^{-1} - 1$$

and take

$$t := 3, \gamma_1 := p, \gamma_2 := \alpha, \gamma_3 := 1 + \alpha^{m-n}, b_1 := a; b_2 := -n, b_3 = -1.$$

As before, $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ contains $\gamma_1, \gamma_2, \gamma_3$ and has $D := [\mathbb{K} : \mathbb{Q}] = 2$. If $\Lambda = 0$, then

$$p^a = \alpha^n \pm \alpha^m,$$

which is not possible for $n > m$. Therefore $\Lambda \neq 0$.

Let us now estimate $h(\gamma_3)$ where $\gamma_3 = 1 \pm \alpha^{m-n}$

$$\gamma_3 = 1 \pm \alpha^{m-n} < 2 \ \ and \ \ \gamma_3^{-1} = \frac{1}{1+\alpha^{m-n}} < \frac{5}{2}$$

so $|\log\gamma_3| < 1$. Notice that

$$h(\gamma_3) \leq |m-n|\left(\frac{\log\alpha}{2}\right) + \log 2 = \log 2 + (n-m)\left(\frac{\log\alpha}{2}\right)$$

Proceeding as before, we take

$$A_1 := 2\log p, \quad A_2 := 0.5$$

and we can take

$$A_3 := 2 + (n-m)\log\alpha \text{ since } h(\gamma_3) := \log 2 + (n-m)\left(\tfrac{\log\alpha}{2}\right)$$

Recalling, $a < (n+2)\frac{\log 2}{\log p} < n$, it follows that, $B = \max\{1, n, a\}$. Thus we can take $B = n$. The Matveev's theorem gives the lower bound on the left hand side of (3.3) by replacing the data. We get:

$$\exp\left(-C\left(1+\log n\right)\cdot 2\log p\cdot 0.5\cdot\left(2+(n-m)\log\alpha\right)\right) < |\Lambda| < \frac{2}{\alpha^n}$$

which leads to

$$n\log\alpha - \log 2 < C((1+\log n)\cdot\log p\cdot(2+(n-m)\log\alpha) < 2C\log n\cdot\log p\cdot(2+(n-m)\log\alpha),$$

where $C := 1.4\times 30^{3+3}\times 3^{4.5}\times 2^2(1+\log 2) < 9.7\times 10^{11}$. Then

$$n\log\alpha - \log 2 < 1.94\times 10^{12}\log n\log p\cdot(2+(n-m)\log\alpha) \tag{3.4}$$

where we used inequality $1+\log n < 2\log n$, which holds for $n > 200$. Now, using (3.2) in the right term of the above inequality (3.4) and doing the related calculations, we get

$$n < 5.05\times 10^{22}\log^2 n\log^2 p. \tag{3.5}$$

Hence,

$$n < 2.1\times 10^{26}\log^2 p.$$

All the calculations done so far can be summarized in the following lemma.

Lemma 3.2.
If (n, m, p, a) *is a solution in positive integers of* (1.2) *with conditions* $n > m$ *and* $n > 200$*, then inequalities*

$$a \le n+2 < 2.11\times 10^{26}\log^2 p$$

hold.

4. Reducing of the bound on n

Rewriting (3.1) as

$$|1 - e^{a\log p - n\log\alpha}| < \frac{3}{\alpha^{n-m}}$$

and using the fact that $|\Lambda| < 2|e^{\Lambda}-1|$ whenever $|e^{\Lambda}-1| < \frac{1}{2}$, we obtain the inequality

$$0 < |a\log p - n\log\alpha| < \frac{3}{\alpha^{n-m}}$$

for all $(n-m)\ge 4$. Dividing the above inequality by $\log\alpha$, we get

$$0 < |a\gamma_p - n| < \frac{7}{\alpha^{n-m}};\ \text{where } \gamma_p := \frac{\log p}{\log\alpha} \tag{4.1}$$

We run a computer program to find the continued fraction $[a_0, a_1, a_2, ...]$ of the irrational number γ_p. Let p_k/q_k denotes the k^{th} convergent of γ_p. For each prime p, we compute the denominators $q_k(p)$ and $q_{k+1}(p)$ of the convergents of γ_p such that $q_k(p) < 2.11\times 10^{26}\log^2 p < q_{k+1}(p)$ and find $a_M(p) := \max\{a_i | i = 0, 1, ..., k+1\}$. Therefore, taking a_M to be the maximum of all $a_M(p)$, we get $a_M = 130620$.

Now applying Lemma 2.6 and properties of continued fractions, we obtain

$$|a\gamma_p - n| > \frac{1}{(a_M+2)a}. \tag{4.2}$$

Combining equation (4.1) and (4.2), we get

$$\frac{1}{(a_M+2)a} < |a\gamma_p - n| < \frac{7}{\alpha^{n-m}} \implies \frac{1}{(a_M+2)a} < \frac{7}{\alpha^{n-m}}$$
$$\implies \alpha^{n-m} < 7\cdot(a_M+2)a < 1.93\times 10^{32}\log^2 p < 9.21\times 10^{33}.$$

Applying $\log$ above and divide by $\log\alpha$, we get:

$$(n-m) \leq \frac{\log(9.21\times 10^{33})}{\log\alpha} < 163.$$

To improve the upper bound on n, let consider

$$z := a\log p - n\log\alpha - \log\rho(u) \text{ where } \rho = 1\pm\alpha^{-u}. \tag{4.3}$$

From (3.3), we have

$$|1-e^z| < \frac{2}{\alpha^n}. \tag{4.4}$$

Since $\Lambda \neq 0$, then $z \neq 0$. Two cases arise: $z<0$ and $z>0$. for each case, we will apply Lemma 2.5.

- Case 1: $z>0$ From (4.4) , we obtain $0<z\leq e^z-1<\frac{2}{\alpha^n}$. Replacing (4.3) in the above inequality, we get :

  $$0 < a\log p - n\log\alpha - \log\rho(n-m) \leq p^a\alpha^{-n}\rho(n-m)^{-1} - 1 < 2\alpha^{-n}$$

  hence

  $$0 < a\log 3 - n\log\alpha - \log\rho(n-m) < 2\alpha^{-n}$$

  and by dividing above inequality by $\log\alpha$

  $$0 < a\left(\frac{\log p}{\log\alpha}\right) - n - \frac{\log\rho(n-m)}{\log\alpha} < 5\cdot\alpha^{-n}. \tag{4.5}$$

  Taking, $\gamma_p := \frac{\log p}{\log\alpha}$, $\mu := -\frac{\log\rho(n-m)}{\log\alpha}$, $A := 5$, $B := \alpha$, inequality (4.5) becomes

  $$0 < a\gamma_p - n + \mu < AB^{-n}.$$

  Since γ_p is irrational, we are now ready to apply Lemma 2.5 of Dujella and Petho on (4.5) for $n-m \in \{4,5,...,163\}$. Since $a \leq 2.11\times 10^{26}\log^2 p$ from Lemma 3.2, we can take $M = 2.55\times 10^{27}$, and we get

  $$n < \frac{\log(Aq_p/\epsilon)}{\log B}$$

  where $q_p > 6M$ and q_p is the denominator of the convergent of the irrational number γ_p such that $\epsilon_p := ||\mu q_p|| - M||\gamma q_p|| > 0$.

  With the help of *SageMath*, with conditions $z>0$, and (n,m,a) a possible zero of (1.2) , we get $n<143$ which contradicts our assumption $n>200$. Then it is false.

- Case 2: $z<0$ Since $n>200$, then $\frac{2}{\alpha^n} < \frac{1}{2}$. Hence (4.4) implies that $|1-e^{|z|}| < 2$. Also, since $z<0$, we have

  $$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^{|z|}-1| < \frac{4}{\alpha^n}.$$

  Replacing (4.3) in the above inequality and dividing by $\log p$, we get:

  $$0 < n\left(\frac{\log\alpha}{\log p}\right) - a + \frac{\rho(n-m)}{\log p} < \frac{4}{\log p}\cdot\alpha^{-n} < 4\cdot\alpha^{-n} \tag{4.6}$$

In order to apply Lemma 3.2 on (4.6) for $n-m \in \{4,5,...,111\}$, we take $M = 2.55\times 10^{27}$. With the help of *SageMath*, with conditions $z<0$, and (n,m,a) a possible zero of (1.2), we get $n<143$ which contradicts our assumption $n>200$. Then it is false.

This completes the proof of our main result. ■

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