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Exact solutions to interfacial flows with kinetic undercooling in a Hele-Shaw cell of time-dependent gap

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. Hele-Shaw cells where the top plate is moving uniformly at a prescribed speed and the bottom plate is fixed have been used to study interface related problems. This paper focuses on interfacial flows with linear and nonlinear kinetic undercooling regularization in a radial Hele-Shaw cell with a time dependent gap. We obtain some exact solutions of the moving boundary problems when the initial shape is a circle, an ellipse or an annular domain. For the nonlinear case, a linear stability analysis is also presented for the circular solutions. The methodology is to use complex analysis and PDE theory.

AMS Subject Classifications: 35Q35, 76S05.

Keywords: Hele-Shaw flow, nonlinear kinetic undercooling, exact solution, Schwarz function, Laplace equation

Contents

1. Introduction and Background

There have been a huge number of studies in the problem of a less viscous fluid displacing a more viscous fluid in a Hele-Shaw cell since Saffman and Taylor's seminal papers [28, 35] in the 1950s. Neglecting the surface tension, Saffman and Taylor [28] found an one-parameter family of exact steady solutions, parameterized by width λ. This theoretical shape are usually referred to in the literature as the Saffman-Taylor finger. It was found [28] experimentally that an unstable planar interface evolves through finger competition to a steady translating finger, with relative finger width λ close to one half. However, in the zero-surface-tension steady-state theory, λ remained arbitary in the (0, 1) interval. The selection problem of λ was solved by incorporating the surface tension regularization, numerical and formal asymptotic calculations [22], [39],

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[19],[7, 8], [31], [17], [32, 33], [4] Rigorous results were later obtained in [34, 40].

The Hele-Shaw problem is similar to the Stefan problem in the context of melting or freezing. Besides surface tension, the physical effect most commonly incorporated in regularizing the ill-posed Stefan problem is kinetic undercooling [2, 14, 20], where the temperature on the moving interface is proportional to the normal velocity of the interface. For the Hele-Shaw problem, kinetic undercooling regularization first appeared in [26, 27]. Since then, results in different aspects were obtained ([1, 5, 9, 15, 24, 25, 41]).

Besides the classical Hele-Shaw setup, there are several variants related to the viscous fingering problem [3, 11, 12, 16]. One of the variants is interfacial flows in a Hele-Shaw cell where the top plate is lifted uniformly at a prescribed speed and the bottom plate is fixed (lifting plate problems) [6, 13, 23, 29, 30, 36, 37, 44]. In the lifting plate problem, the gap $b(t)$ between the two plates is increasing in time but uniform in space. As the plate is pulled, an inner viscous fluid shrinks in the center plane between the two plates and increases in the z-direction to preserve volume. An outer less viscous fluid invades the cell and generates fingering patterns. The patterns are visually similar to those in the classical radial Hele-Shaw problem, but the driving physics is different. In [29], the authors derived the governing equations for the lifting problem and they established the existence, uniqueness and regularity of solutions for analytic data when the surface tension is zero. Some exact solutions were also constructed, both with or without surface tension. Analytic results were also be generalized to higher dimensions in [38]. Numerical simulation and the pattern formation of the interface were presented in [29, 44].

Very recently, local existence of analytic and classical solutions was obtained in [42, 43] for the Hele-Shaw problem with time dependent gap where the kinetic undercooling regularization is used instead of the surface tension. In this paper we obtain some exact solutions of the moving boundary problem when the initial data is a circle or an ellipse. The methodology is to use complex analysis and Schwarz function.

2. Mathematical formulation

Our studies center on the free interface problem in a Hele-Shaw cell with a time dependent gap $b(t)$; see Figure 1. The upper plate is lifted or compressed perpendicular to the cell, while the lower plate stays fixed. We assume that

$$b(t) \in C^1([0,\infty)),\ \ b(t) \geq b_0 \text{ for some positive constants } b_0.$$

We consider the displacement of a viscous fluid by another fluid of negligible viscosity. Let $\Omega(t)$ in the (x, y) plane be the more viscous fluid domain with free boundary $\partial\Omega$. Following M. Shelley, F. Tian, and K. Wlodarski [29], we have the following governing equations: The fluid velocity is

$$u = -\frac{b^2(t)}{12\mu}\nabla p(x, y, t), \tag{2.1}$$

where p is the pressure and μ is the viscosity of the fluid. Conservation of mass equation is

$$\nabla \cdot u = -\frac{\dot{b}(t)}{b(t)} \text{ in } \Omega(t). \tag{2.2}$$

The kinematic boundary condition is

$$-\frac{b^2(t)}{12\mu}\frac{\partial p}{\partial n} = V_n \text{ on } \partial\Omega(t), \tag{2.3}$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the outward normal $\mathbf{n}$ to $\partial\Omega$, and V_n is the velocity of the $\partial\Omega(t)$ in the direction of outward normal vector $\mathbf{n}$; and the dynamic boundary condition is

$$p = \tau V_n \text{ on } \partial\Omega(t), \tag{2.4}$$

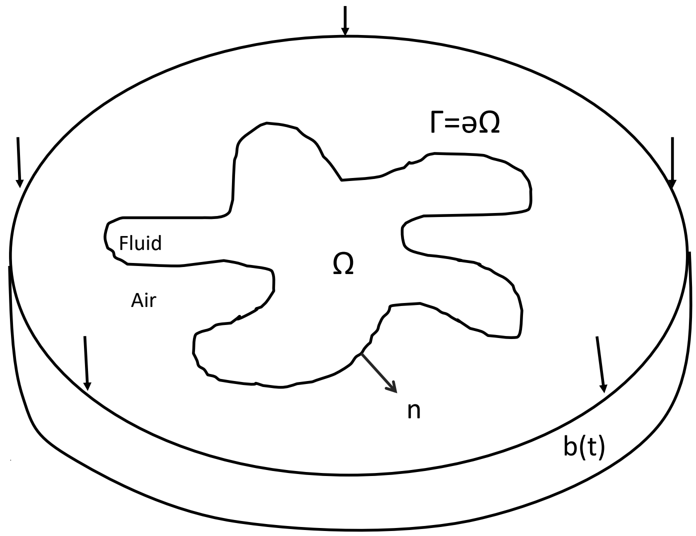

Figure 1: The Hele-Shaw flow with time dependent gap

or

$$p = \tau (V_n)^{\beta} \text{ on } \partial\Omega(t), \tag{2.5}$$

where τ is a kinetic undercooling coefficient and $\beta > 0$ is a real constant.

Non-dimensionalizing the length and time and the pressure, the nondimensional version of (2.1), (2.2), (2.3, (2.4) and (2.5) is

$$u = -b^2(t)\nabla p(x,y,t) \text{ in } \Omega(t), \tag{2.6}$$

$$\nabla \cdot u = -\frac{\dot{b}(t)}{b(t)} \text{ in } \Omega(t), \tag{2.7}$$

$$-b^2(t)\frac{\partial p}{\partial n} = V_n \text{ on } \partial\Omega(t), \tag{2.8}$$

$$p = cV_n \text{ on } \partial\Omega(t); \tag{2.9}$$

or

$$p = c(V_n)^{\beta} \text{ on } \partial\Omega(t); \tag{2.10}$$

where c is the nondimensional kinetic undercooling coefficient; β is a positive number and $\beta \neq 1$.
Plugging (2.6) into (2.7), we obtain

$$\nabla^2 p = \frac{\dot{b}(t)}{b^3(t)} \text{ in } \Omega(t). \tag{2.11}$$

We are going to consider the following two problems:
Problem one consists of the equations (2.8), (2.9) and (2.11) and
Problem Two consists of the equations (2.8), (2.10) and (2.11) with $\beta > 0$ and $\beta \neq 1$.

Let $\tilde{p} = p - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2)$, then $\tilde{p}$ satisfies

$$\nabla^2 \tilde{p} = 0, \text{ in } \Omega(t) \tag{2.12}$$

$$V_n = -b^2(t)\frac{\partial \tilde{p}}{\partial n} - \frac{\dot{b}(t)}{2b(t)}(x,y)\cdot \mathbf{n} \text{ on } \partial\Omega(t), \tag{2.13}$$

$$\tilde{p} = cV_n - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2), \text{ on } \partial\Omega(t), \tag{2.14}$$

or

$$\tilde{p} = c(V_n)^\beta - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2), \text{ on } \partial\Omega(t), \tag{2.15}$$

So equivalently, *Problem one* consists of the equations (2.12), (2.13) and (2.14) and *Problem Two* consists of the equations (2.12), (2.13) and (2.15) with $\beta \neq 1$.

3. Exact solutions of Problem One

It is noted in [42] that Problem One has a radially symmetric solution $\Omega(t) = \{(x,y) : r = \sqrt{x^2+y^2} < R(t)\}$, where

$$\frac{\dot{R(t)}}{R(t)} = -\frac{\dot{b(t)}}{2b(t)}, \quad R(t) = \frac{R(0)\sqrt{b(0)}}{\sqrt{b(t)}}; \tag{3.1}$$

$$p(t,r) = -\frac{cR(0)\sqrt{b(0)}\dot{b}(t)}{2b(t)\sqrt{b(t)}} - \frac{R^2(0)b(0)\dot{b}(t)}{4b^4(t)} + \frac{\dot{b}(t)}{4b^3(t)}r^2. \tag{3.2}$$

It was also obtained in [42] that the linear perturbation of solution (3.1) and (3.2) grows when $\dot{b}(t)$ is positive while the perturbation decays when $\dot{b}(t)$ is negative.
Next we are going to re-establish (3.1) and 3.2) using Schwarz function approach. We refer to [10] for properties of Schwarz function, we first give some preliminary lemmas.

Lemma 3.1. *Assume that $\partial\Omega(t)$ is an analytic curve, and $S(t,z)$ is the Schwarz Function of $\partial\Omega(t)$, where $z = x+iy$; then the outward normal velocity is*

$$V_n = -\frac{i\partial_t S(t,z)}{2\sqrt{S_z(t,z)}}. \tag{3.3}$$

The proof of the lemma can be found in [18, 21].
Let $W(z) = \tilde{p} + i\tilde{q}$, where $\tilde{q}$ is a harmonic conjugate of $\widetilde{p}$. The analytic function $W(z)$ is called the complex velocity potential for Problem One. Let us obtain an equation for W_z in terms of the Schwarz function $S(t,z)$.We introduce s to be the arc length variable along $\partial\Omega(t)$. Using the properties of $S(z)$,(2.14) and (3.3), we have

$$\begin{aligned}
\partial_s \tilde{p} &= \frac{\partial \tilde{p}}{\partial z}\cdot\frac{\partial z}{\partial s} + \frac{\partial \tilde{p}}{\partial \bar{z}}\cdot\frac{\partial \bar{z}}{\partial s} \\
&= \frac{\partial \tilde{p}}{\partial z}\cdot\frac{1}{\sqrt{S_z(z)}} + \sqrt{S_z(z)}\frac{d\tilde{p}}{d\bar{z}} \\
&= \frac{1}{\sqrt{S_z(z)}}\left(-\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\bar{z} + \frac{\partial}{\partial z}c\,(V_n)\right) + \sqrt{S_z(z)}\left(-\frac{1}{4}\frac{\dot{b(t)}}{b^3(t)}z + \frac{\partial}{\partial \bar{z}}cV_n\right) \\
&= \frac{1}{\sqrt{S_z(z)}}\left(-\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\bar{z} - c\frac{\partial}{\partial z}\left(\frac{i}{2}\frac{\partial_t S}{\sqrt{S_z}}\right)\right) + \sqrt{S_z(z)}\left(-\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\right) \\
&= -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\left(\frac{S(z)}{\sqrt{S_z(z)}} + z\sqrt{S_z(z)}\right) - \frac{ci}{2}\frac{1}{\sqrt{S_z(z)}}\frac{\partial}{\partial z}\left(\frac{\partial_t S}{\sqrt{S_z}}\right)
\end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
\frac{\partial \tilde{p}}{\partial \vec{n}} &= -\frac{V_n}{b^2(t)} - \frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\frac{\partial}{\partial n}(z\bar{z}) \\
&= +\frac{i}{2}\frac{\partial_t S}{b^2\sqrt{S_z}} - \frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\left(\bar{z}\frac{\partial z}{\partial n} + z\frac{\partial \bar{z}}{\partial n}\right) \\
&= +\frac{i}{2}\frac{\partial_t S}{b^2\sqrt{S}z} - \frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\left(S\cdot\frac{-i}{\sqrt{S_z(z)}} + iz\sqrt{S_z(z)}\right) \\
&= +\frac{i}{2}\frac{\partial_t S}{b^2\sqrt{S_z}} - \frac{i}{4}\frac{\dot{b}(t)}{b^3(t)}\left(\frac{-S}{\sqrt{S_z(z)}} + z\sqrt{S_z(z)}\right)
\end{aligned} \tag{3.5}
$$

Using (3.4) and (3.5), we have

$$
\begin{aligned}
W_z =& \frac{\partial_s W}{\partial_s z} = \frac{\partial_s(\tilde{p}+i\tilde{q})}{\partial_s z} \\
&= \frac{\partial_s\tilde{p}+i\partial_s\tilde{q}}{\partial_s z} = \frac{\partial_s\tilde{p}+i\partial_n\tilde{p}}{\partial_s z} \\
&= -\frac{1}{4}\frac{\dot{b}}{b^3}\left(S(z)+zS_z(z)\right) - \frac{ci}{2}\frac{\partial}{\partial z}\left(\frac{\partial_t S}{\sqrt{S_z}}\right) \\
&+ i\left(\frac{i}{2}\frac{\partial_t S}{b^2} - \frac{i}{4}\frac{b(t)}{b^3(t)}\left(-S+zS_z(z)\right)\right) \\
=& -\frac{ci}{2}\frac{\partial}{\partial z}\left(\frac{\partial_t S}{\sqrt{S_z}}\right) - \frac{\partial_t S}{2b^2} - \frac{1}{2}\frac{\dot{b}}{b^3}S
\end{aligned} \tag{3.6}
$$

From (2.13), (2.14) and (3.3), we have on $\partial\Omega(t)$:

$$
-b^2(t)\frac{\partial \tilde{p}}{\partial n} = \frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}zS(z) - \frac{i\partial_t S(t,z)}{2\sqrt{S_z(t,z)}} \tag{3.7}
$$

and

$$
\tilde{p} = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\bar{z} + cV_n = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\bar{z} - \frac{ic\partial_t S(t,z)}{2\sqrt{S_z(t,z)}}. \tag{3.8}
$$

Now we look at the following cases:

(1) Circular solutions: If the initial shape is circle, $\Omega(0) = \{(x,y) : x^2+y^2 \le a_0^2\}$, then we are seeking circular solution for all $t>0$, $\Omega(t) = \{(x,y) : x^2+y^2 \le a(t)^2\}$. The Schwarz function is $S(z,t) = \frac{a^2(t)}{z}$, so $\partial_t S = 2a\frac{\dot{a}(t)}{z}, S_z = \frac{-a^2(t)}{z^2}$. From (3.6), we have

$$
W_z = -\frac{a(t)}{b^2 z}\left(\dot{a} + \frac{\dot{b}a}{2b}\right); \tag{3.9}
$$

Let $A(t)$ be the area of $\Omega(t)$, then

$$
\begin{aligned}
\frac{d}{dt}A(t) &= \frac{d}{dt}\iint\limits_{\Omega(t)} 1dxdy = \oint_{\partial\Omega t} V_n ds \\
&= \oint_{\partial\Omega(t)} -b^2(t)\frac{\partial p}{\partial n}ds = -b^2(t)\oint_{\partial\Omega(t)}\frac{\partial p}{\partial n}ds \\
&= -b^2(t)\iint\limits_{\Omega(t)}\Delta p dxdy = -b^2(t)\cdot\frac{\dot{b}(t)}{b^3(t)}A(t) = \frac{-\dot{b}(t)}{b(t)}A(t).
\end{aligned} \tag{3.10}
$$

Since $A(t) = \pi a^2(t)$, we have

$$2\dot{a} = -\frac{a(t)}{b(t)}\dot{b}(t) \tag{3.11}$$

(3.9) implies $W_z = 0$, so $W(t,z) = W(t)$ is independent of z, consequently $\tilde{p}(t,z) = \tilde{p}(t)$ is independent of z. Using (3.8) we have

$$\begin{aligned}\tilde{p} &= -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)} \cdot a^2(t) + cV_n = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}a^2(t) \\ &= -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}a^2(t) + c\dot{a}(t) = -\frac{1}{4}\frac{\dot{b}(t)a^2(t)}{b^3(t)} - \frac{ca(t))\dot{b}(t)}{2b(t)}.\end{aligned} \tag{3.12}$$

We note that (3.11) and (3.12) are equivalent to (3.1) and (3.2) respectively.

(2) Solutions of Ellipse shape: If $\Omega(0) = \{(x,y) : \frac{x^2}{a_0^2} + \frac{y^2}{h_0^2} \leqslant 1\}$, where $a_0 > h_0$.
Let us seek ellipse solution for $t > 0$, $\Omega(t) = \{(x,y) : \frac{x^2}{a^2(t)} + \frac{y^2}{h^2(t)} \leqslant 1\}$, then the Schwarz function is

$$S(z,t) = \frac{a^2 + h^2}{d^2(t)}z - \frac{2a(t)h(t)}{d(t)^2}\sqrt{z^2 - d^2}, \tag{3.13}$$

where $d(t) = \sqrt{a(t)^2 - h(t)^2}$.
Taking derivative in above, we have

$$\partial_t S(z,t) = z\frac{\partial}{\partial t}\left(\frac{a^2 + h^2}{d^2}\right) - \sqrt{z^2 - d^2}\frac{\partial}{\partial t}\left(\frac{2ah}{d^2}\right) + \frac{1}{\sqrt{z^2 - d^2}}\frac{ah}{d^2}\frac{\partial}{\partial t}\left(d^2\right) \tag{3.14}$$

and

$$S_z(z,t) = \frac{\left(a^2 + h^2\right)\sqrt{z^2 - d^2} - 2ahz}{d^2\sqrt{z^2 - d^2}} \tag{3.15}$$

Plugging (3.13), (3.14) and (3.15) into (3.6) and integrating we obtain,

$$\begin{aligned}W(t,z) = &\left(-\frac{ci}{2}\right)\frac{d\left[z\frac{\partial}{\partial t}\left(\frac{a^2+h^2}{d^2}\right)\sqrt{z^2 - d^2} - \left(z^2 - d^2\right)\frac{\partial}{\partial t}\left(\frac{2ah}{d^2}\right) + \frac{ah}{d^2}\frac{\partial}{\partial t}\left(d^2\right)\right]}{\left(z^2 - d^2\right)^{\frac{1}{4}}\left(\left(a^2 + h^2\right)\sqrt{z^2 - d^2} - 2ahz\right)^{1/2}} \\ &- \frac{z^2}{4b^2}\left[\partial_t\left(\frac{a^2 + h^2}{d^2}\right) - \frac{\left(a^2 + h^2\right)\partial_t(ah)}{ahd^2}\right] \\ &- z\sqrt{z^2 - d^2}\frac{ah\partial_t(d^2)}{2b^2d^4} + q(t)\end{aligned} \tag{3.16}$$

We examine the singularity of $W(t,z)$ in (3.16), W has to be analytic in $\Omega(t) = \frac{x^2}{a^2} + \frac{y^2}{h^2} \leqslant 1$.
We note that the zero of $\left(a^2 + h^2\right)\sqrt{z^2 - d^2} - 2ahz = 0$ are $z = \pm\frac{a^2+h^2}{d(t)} = \pm\frac{a^2+h^2}{\sqrt{a^2-h^2}}$, which are outside of $\Omega(t)$. So $\left(\left(a^2 + h^2\right)\sqrt{z^2 - d^2} - 2ahz\right)^{1/2}$ is analytic in $\Omega(t)$. That means that the singularities of the first term and the third term can not be canceled out to make $W(t,z)$ analytic. Hence we have the following result:

Proposition 3.2. *If the initial shape of Problem One is an ellipse, then for any $T > 0$, there is no solution of Problem One on $[0,T]$ which is of ellipse shape.*

(3) Solutions on annular domain:
If $\Omega(t) = \{(x, y) : r(t) < \sqrt{x^2 + y^2} < R(t)\}$ is an annular domain, then the problem will consist of the following equations:

$$\Delta p = \frac{\dot{b}}{b^3} \text{ in } \Omega(t); \tag{3.17}$$

$$p = c\dot{R} \; on \; x^2 + y^2 = R^2(t) \tag{3.18}$$

$$p = -c\dot{r} \; on \; x^2 + y^2 = r^2(t) \tag{3.19}$$

$$-b^2 \frac{\partial p}{\partial r} = \dot{R} \; on \; x^2 + y^2 = R^2(t) \tag{3.20}$$

$$-b^2 \frac{\partial p}{\partial r} = \dot{r} \; on \; x^2 + y^2 = r^2(t). \tag{3.21}$$

The general solution of (3.17) is given by

$$p(r, \theta) = \frac{1}{4} \frac{b}{b^3}(x^2 + y^2) + \alpha(t) \log \sqrt{x^2 + y^2} + \gamma(t) \tag{3.22}$$

where $\alpha(t)$ and $\gamma(t)$ with $R(t)$ and $r(t)$ are to be determined. Plugging above into (3.18)- (3.21) to obtain

$$c\dot{R} = \frac{1}{4} \frac{\dot{b}}{b^3} R^2 + \alpha(t) \log R + \gamma(t), \tag{3.23}$$

$$-c\dot{r} = \frac{1}{4} \frac{\dot{b}}{b^3} r^2 + \alpha(t) \log r + \gamma(t), \tag{3.24}$$

$$-\frac{1}{2} \frac{\dot{b}}{b} R - \frac{\alpha(t) b^2}{R} = \dot{R}, \tag{3.25}$$

$$-\frac{1}{2} \frac{\dot{b}}{b} r - \frac{\alpha(t) b^2}{r} = \dot{r}. \tag{3.26}$$

From (3.25) and (3.26) we have

$$-\frac{1}{2} \frac{\dot{b}}{b}(R^2 - r^2) = \frac{1}{2} \frac{d}{dt}(R^2 - r^2). \tag{3.27}$$

Solving (3.27) we derive

$$R^2 - r^2 = \frac{b(0)\big(R^2(0) - r^2(0)\big)}{b(t)} \tag{3.28}$$

which implies

$$R(t) = \sqrt{r(t)^2 + \frac{b(0)\big(R^2(0) - r^2(0)\big)}{b(t)}} \tag{3.29}$$

To solve $\alpha(t)$ we subtract (3.24) from (3.23), then use (3.25) and (3.26) to derive

$$\alpha(t) = \frac{-\frac{c}{2} \frac{\dot{b}}{b}(R + r) - \frac{1}{4} \frac{\dot{b}}{b^3} \frac{b(0)\left(R^2(0) - r^2(0)\right)}{b}}{\left[\log\left(\frac{R}{r}\right) + c\left(\frac{1}{R} + \frac{1}{r}\right)\right]} \tag{3.30}$$

Now plugging (3.30) into (3.26), we derive

$$\dot{r} = -\frac{1}{2} \frac{\dot{b}}{b} r + \frac{\frac{c}{2} \dot{b} b(R + r) + \frac{1}{4} \frac{\dot{b} b(0)\left(R^2(0) - r^2(0)\right)}{b^2}}{r\left[\log\left(\frac{R}{r}\right) + c\left(\frac{1}{R} + \frac{1}{r}\right)\right]} \tag{3.31}$$

Plugging (3.29) into (3.31), we obtain an ODE for $r(t)$

$$\dot{r} = -\frac{1}{2}\frac{\dot{b}}{b}r + \frac{\frac{cb\dot{b}}{2}\left(\sqrt{r^2+B(t)}+r\right)+\frac{1}{4}\frac{\dot{b}B(t)}{b}}{r\left[\frac{1}{2}\log\left(1+\frac{B(t)}{r^2}\right)+c\left(\frac{1}{\sqrt{r^2+B(t)}}+\frac{1}{r}\right)\right]}, \tag{3.32}$$

Where

$$B(t) = \frac{b(0)\big(R^2(0)-r^2(0)\big)}{b(t)}. \tag{3.33}$$

Once $r(t)$ is solved from (3.32), we solve $R(t)$ from (3.29), $\alpha(t)$ from (3.30) and $\gamma(t)$ from (3.24). In Figure 2, using MATLAB, we have numerically solved $r(t)$, $R(t)$, $\alpha(t)$ and $\gamma(t)$ from (3.32), (3.29), (3.30) and (3.24) when we take $b(t) = (1+t)/(2+t), c = 0.1, r(0) = 1, R(0) = 2$.

4. Exact solutions of Problem Two

Parallel to (3.6), (3.8) and Problem One, we have the following equations for Problem Two:

$$W_z = c\left[-\frac{i}{2}\frac{\partial}{\partial z}\left(\frac{\partial_t S}{\sqrt{S_z}}\right)\right]^\beta - \frac{\partial_t S}{2b^2} - \frac{1}{2}\frac{\dot{b}}{b^3}S \tag{4.1}$$

$$\tilde{p} = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\bar{z} + c(V_n)^\beta == -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}z\bar{z} + c\left[-\frac{i\partial_t S(t,z)}{2\sqrt{S_z(t,z)}}\right]^\beta. \tag{4.2}$$

(1) Circular solutions: If the initial shape is circle, $\Omega(0) = \{(x,y) : x^2+y^2 \le a_0^2\}$, then we are seeking circular solution for all $t > 0$, $\Omega(t) = \{(x,y) : x^2+y^2 \le a(t)^2\}$. Parallel to (3.9), (3.11) and (3.12) we have the following radially symmetric solution for Problem Two:

$$\frac{\dot{a(t)}}{a(t)} = -\frac{\dot{b(t)}}{2b(t)}, \quad a(t) = \frac{a_0\sqrt{b(0)}}{\sqrt{b(t)}}; \tag{4.3}$$

$$\begin{aligned}\tilde{p} &= -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}\cdot a^2(t) + c[V_n]^\beta = -\frac{1}{4}\frac{\dot{b}(t)}{b^3(t)}a^2(t) + c[\dot{a}(t)]^\beta \\ &= -\frac{1}{4}\frac{\dot{b}(t)a^2(t)}{b^3(t)} - c\left[\frac{a(t))\dot{b}(t)}{2b(t)}\right]^\beta.\end{aligned} \tag{4.4}$$

Now we will discuss the linear stability of the circular solution (4.3) and (4.4). The perturbed interface is $\Big\{r(\theta,t) = a(t) + \varepsilon\delta(t)\cos(k\theta)\Big\} = \partial\Omega(t)$, then we have

$$\begin{aligned}\overrightarrow{n} &= \frac{<\partial_\theta r\ \sin\theta + r\cos\theta, -r_\theta\cos\theta + r\sin\theta>}{\sqrt{r_\theta^2+r^2}} \\ &= \frac{<a(t)\cos\theta, a(t)\sin\theta>}{a(t)\sqrt{1+\frac{2\varepsilon\delta(t)\cos(kt)}{a(t)}+O(\varepsilon^2)}} \\ &+ \frac{\varepsilon\delta(t)\langle -k\sin\theta\sin(k\theta)+\cos(k\theta)\cos\theta, k\cos\theta\sin(k\theta)+\cos(k\theta)\sin\theta\rangle}{a(t)\sqrt{1+\frac{2\varepsilon\delta(t)\cos(kt)}{a(t)}+O(\varepsilon^2)}} \\ &=<\cos\theta,\sin\theta> - \frac{\varepsilon\delta(t)k\sin(k\theta)}{a(t)} <\sin\theta, -\cos\theta> + O(\varepsilon^2)\end{aligned}$$

and

$$\begin{aligned} V_n &= < \dot{x}(t), \dot{y}(t) > \cdot \overrightarrow{n} \\ &= \big(a\dot{(t)} + \varepsilon\dot{\delta}(t)\cos(k\theta)\big) < \cos\theta, \sin\theta > \\ &\cdot \Big[< \cos\theta, \sin\theta > - \frac{\varepsilon\delta(t)k\sin(k\theta)}{a(t)} < \sin\theta, -\cos\theta > + O(\varepsilon^2) \Big] \\ &= \dot{a}(t) + \varepsilon\dot{\delta}(t)\cos(k\theta) + O(\varepsilon^2) \end{aligned} \tag{4.5}$$

Let

$$\widetilde{p}(t,r,\theta) = \widetilde{p}_0(t,r) + \varepsilon\widetilde{p}_1(t,r,\theta) + ... \tag{4.6}$$

then

$$\widetilde{p}(t,r,\theta)\Big|_{\partial\Omega(t)} = \widetilde{p}(t,r,\theta)\Big|_{r=a(t)} + \Big(\frac{\partial\widetilde{p}}{\partial r}\Big|_{r=a(t)}\Big)(\varepsilon\delta\cos(k\theta)) + O(\varepsilon^2) \tag{4.7}$$

from (2.15) and (4.5), we have on $\partial\Omega(t)$

$$\begin{aligned} \widetilde{p} &= c(\dot{a} + \varepsilon\dot{\delta}\cos(k\theta))^{\beta} - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2) \\ &= c(\dot{a})^{\beta} + c\beta(\dot{a})^{\beta-1}(\varepsilon\dot{\delta}\cos(k\theta)) - \frac{\dot{b}(t)}{4b^3(t)}(x^2+y^2) + O(\varepsilon^2) \end{aligned} \tag{4.8}$$

From (4.6),(4.7) and (4.8) , we have

$$\widetilde{p}_1\Big|_{r=a(t)} = c\beta(\dot{a})^{\beta-1}\dot{\delta}\cos(k\theta) - \frac{\dot{b}}{2b^3(t)}a(t)\delta(t)\cos(k\theta); \tag{4.9}$$

From (2.13) and (4.5) we have

$$\dot{\delta}(t)\cos(k\theta) = -b^2(t)\frac{\partial\widetilde{p}_1}{\partial r}\Big|_{r=a(t)} = \frac{\delta(t)\dot{b}}{2b}\cos(k\theta) \tag{4.10}$$

$\Big(Note: \ \frac{\dot{b}}{2b} < x, y > \cdot \overrightarrow{n} = \frac{\dot{b}(t)}{2b(t)}a(t) + \frac{\varepsilon\delta(t)\dot{b}}{2b}\cos(k\theta) + O(\varepsilon^2)\ using\ (36).\Big)$ From (12), we have

$$\Delta\widetilde{p}_1 = 0 \text{ in } \{r < a(t)\} \tag{4.11}$$

Now using separation of variables. We solve the boundary value problem of (4.9) and (4.11), we have

$$\widetilde{p}_1(r,\theta,t) = \frac{c\beta(\dot{a})^{\beta-1}\dot{\delta} - \frac{\dot{b}}{2b^3}a(t)\delta(t)}{(a(t))^k}r^k\cos(k\theta) \tag{4.12}$$

From (4.12) we have

$$\frac{\partial\widetilde{p}_1}{\partial r}\Big|_{r=a(t)} = \frac{k\Big(c\beta(\dot{a})^{\beta-1}\dot{\delta} - \frac{\dot{b}}{2b^3}a(t)\delta(t)\delta(t)\Big)\cos(k\theta)}{a(t)} \tag{4.13}$$

Now using (4.10) and (4.13) we obtain

$$\dot{\delta}(t) = -\frac{kb^2\big[c\beta(\dot{a})^{\beta-1}\dot{\delta} - \frac{\dot{b}}{2b^3}a(t)\delta(t)\big]}{a(t)} - \frac{\delta\dot{b}}{2b} \tag{4.14}$$

We rewrite (4.14) as

$$\frac{\dot{\delta}(t)}{\delta(t)} = \frac{\dot{b}(t)}{2b(t)}\frac{(k-1)a(t)}{\big[a(t) + kc\beta b^2(\dot{a})^{\beta-1}\big]} \tag{4.15}$$

Using $\dot{a} = -\dfrac{\dot{b}}{2b}a$, we can rewrite (4.15) as

$$\frac{\dot{\delta}(t)}{\delta(t)} = \frac{\dot{b}(t)}{2b(t)} \frac{(k-1)a(t)}{\Big[a(t) + kc\beta b^2\big(-\frac{\dot{b}}{2b}a\big)^{\beta-1}\Big]} \tag{4.16}$$

So when $\dot{b} < 0$, $\dfrac{\dot{\delta}(t)}{\delta(t)} < 0$, the solution is stable

If $\dot{b} > 0$, when $\beta = 2m+1$ is positive odd numbers then $\dfrac{\dot{\delta}(t)}{\delta(t)} > 0$, the solution is unstable

If $\dot{b} > 0$, and $\beta = 2m$ is positive even numbers , then

$$\begin{aligned}\frac{\dot{\delta}(t)}{\delta(t)} &= \frac{\dot{b}(t)}{2b(t)} \frac{(k-1)a(t)}{\Big[a - 2kcmb^2\big(\frac{\dot{b}}{2b}\big)^{2m-1}a^{2m-1}\Big]} \\ &= \frac{\dot{b}(t)}{2b(t)} \frac{(k-1)}{\Big[1 - 2kcmb^2\big(\frac{\dot{b}}{2b}\big)^{2m-1}\big(\frac{a_0\sqrt{b(0)}}{\sqrt{b}}\big)^{2m-2}\Big]} \\ &= \frac{\dot{b}(t)}{2b(t)} \frac{(k-1)}{\Big[1 - 2^{2-2m}kcm(a_0\sqrt{b(0)})^{2m-2}(\dot{b})^{2m-1}b^{-(2m-4)}\Big]}\end{aligned}$$

the linear stability in this case will depend on the wave number k and the gap function $b(t)$.

(2) ellipse shape case: If there is an elliptic shape solution $\Omega(t) = (x,y) : \frac{x^2}{a^2(t)} + \frac{y^2}{h^2(t)} = 1$, then from (3.10) with $A(t) = \pi a(t)h(t)$

$$a(t)h(t) = \frac{\pi a(0)h(0)b(0)}{b(t)}. \tag{4.17}$$

Parallel to (3.13)-(3.16), we have

$$\begin{aligned}W(t,z) = c&\left(\left(-\frac{i}{2}\right)\frac{d\left[z\frac{\partial}{\partial t}\left(\frac{a^2+h^2}{d^2}\right)\sqrt{z^2-d^2} - \left(z^2-d^2\right)\frac{\partial}{\partial t}\left(\frac{2ah}{d^2}\right) + \frac{ah}{d^2}\frac{\partial}{\partial t}\left(d^2\right)\right]}{(z^2-d^2)^{\frac{1}{4}}\left((a^2+h^2)\sqrt{z^2-d^2} - 2ahz\right)^{1/2}}\right)^{\beta} \\ &- \frac{z^2}{4b^2}\left[\partial_t\left(\frac{a^2+h^2}{d^2}\right) - \frac{\left(a^2+h^2\right)\partial_t(ah)}{ahd^2}\right] \\ &- z\sqrt{z^2-d^2}\frac{ah\partial_t(d^2)}{2b^2d^4} + q(t)\end{aligned} \tag{4.18}$$

It is clear that singularities at $z = \pm d$ can not be removed in the case where β is an odd integer. It looks that from (4.18) that singularities can be removed for the case where β is positive even integer if $a(t)$ and $h(t)$ are properly chosen. However, by examining the case $\beta = 2$, we found that $a(t)$ and $h(t)$ must satisfies two independent ODEs which have no solution that also satisfies (4.18). Hence we have the following result.

Proposition 4.1. *If the initial shape of Problem Two is an ellipse, then for any $T > 0$, there is no solution of Problem Two which is of ellipse shape on $[0,T]$.*

(3) Solution on annular domain: For the nonlinear problem 2 on a annular domain, we have the same equations (32) (35) and (36) as for the linear problem 1, but instead of (33) and (34), the pressure conditions are

$$p = c\dot{R}^{\beta} \ on \ x^2 + y^2 = R^2, \tag{4.19}$$

and

$$p = c(-\dot{r})^{\beta} \ on \ x^2 + y^2 = r^2. \tag{4.20}$$

We also note that (40)-(47) also hold for Problem 2. To solve for $\alpha(t)$, we use (37), (4.19) and (4.20) to obtain:

$$c(\dot{R})^{\beta} = \frac{1}{4}\frac{\dot{b}}{b^3}R^2 + \alpha(t)\log R + \gamma(t), \tag{4.21}$$

$$c(-\dot{r})^{\beta} = \frac{1}{4}\frac{\dot{b}}{b^3}r^2 + \alpha(t)\log r + \gamma(t), \tag{4.22}$$

Subtracting (4.21) and (4.22), and using (35)-(37), we have

$$c\Big(-\frac{1}{2}\frac{\dot{b}}{b}R - \frac{\alpha(t)b^2}{R}\Big)^{\beta} - c\Big(\frac{1}{2}\frac{\dot{b}}{b}r + \frac{\alpha(t)b^2}{r}\Big)^{\beta} = \frac{1}{4}\frac{\dot{b}}{b^3}B(t) + \alpha(t)\log\frac{R}{r}; \tag{4.23}$$

Where $B(t)$ is given by (3.33). We need to solve for $\alpha(t)$ from (4.23). We consider the case $\beta = 2$, then (4.23) becomes a quadratic equation

$$\alpha^2(t)cb^4\left(\frac{1}{r^2} - \frac{1}{R^2}\right) + \alpha(t)\left[\log\frac{R}{r}\right] + \frac{\dot{b}}{4b^2}\left(-c\dot{b} + \frac{1}{b}\right)B(t) = 0 \tag{4.24}$$

Solving (4.24) we have

$$\alpha(t) = \frac{-\log\frac{R}{r} + \sqrt{\log^2\frac{R}{r} - cb\dot{b}\left(\frac{1}{r^2} - \frac{1}{R^2}\right)B(t) + c^2b^2\dot{b}^2\left(\frac{1}{r^2} - \frac{1}{R^2}\right)B(t)}}{2cb^4\left(\frac{1}{r^2} - \frac{1}{R^2}\right)} \tag{4.25}$$

In (4.25), the expression under the square root is positive for sufficiently small c, and we take the positive square root because of

$$\lim_{c\to 0}\alpha(t) = -\frac{\dot{b}}{4b^3}\frac{B(t)}{\log\frac{R}{r}}. \tag{4.26}$$

Plugging the above into (3.26), we obtain

$$\dot{r} = -\frac{1}{2}\frac{\dot{b}}{b}r + \frac{b^2}{r}\frac{\log\frac{R}{r} - \sqrt{\log^2\frac{R}{r} - cb\dot{b}\left(\frac{1}{r^2} - \frac{1}{R^2}\right)B(t) + c^2b^2\dot{b}^2\left(\frac{1}{r^2} - \frac{1}{R^2}\right)B(t)}}{2cb^4\left(\frac{1}{r^2} - \frac{1}{R^2}\right)} \tag{4.27}$$

Now using (3.28), (3.29) and (3.33) we have

$$\dot{r} = -\frac{1}{2}\frac{\dot{b}}{b}r + \frac{b^2}{r}\frac{\log\sqrt{1 + \frac{B(t)}{r^2}}}{2cb^4\left(\frac{1}{r^2} - \frac{1}{r^2+B(t)}\right)} - \frac{b^2}{r}\frac{\sqrt{\log^2\sqrt{1 + \frac{B(t)}{r^2}} - cb\dot{b}\left(\frac{1}{r^2} - \frac{1}{r^2+B(t)}\right)B(t) + c^2b^2\dot{b}^2\left(\frac{1}{r^2} - \frac{1}{r^2+B(t)}\right)B(t)}}{2cb^4\left(\frac{1}{r^2} - \frac{1}{r^2+B(t)}\right)} \tag{4.28}$$

Once we have solved $r(t)$ from (4.28), we obtain $R(t)$ from (3.29), $\alpha(t)$ from (4.25) and $\gamma(t)$ from (4.22), so we have the problem for the case $\beta = 2$. In Figure 3, using MATLAB, we have numerically solved $r(t)$, $R(t)$, $\alpha(t)$ and $\gamma(t)$ from nonlinear problem (4.28), (3.29), (4.22) and (4.25) when we take $b(t) = (1+t)/(2+t), c = 0.1, r(0) = 1, R(0) = 2$.

Comparing Figure 2 to Figure 3, we note that the solution to the nonlinear problem 2 is qualitatively very similar to that to the linear problem 1, only the limit values of $R(t)$ and $r(t)$ as $t \to \infty$ are slightly different.

5. Conclusion and discussion

In this paper, we are concerned with exact solutions to some interfacial problems with kinetic undercooling regularization in a Hele-Shaw cell with time-dependent gap $b(t)$. For both linear and nonlinear regularization, using Schwarz function, we first recovered the circular solutions for the linear case obtained in [42], and we did linear stability analysis of circular solution for the nonlinear case. Then we found that the solution of the initial ellipse shape of the free boundary could not keep elliptic shape for any small time interval for both linear and nonlinear regularization. Using PDE theory, we obtained exact solutions for the linear case and some nonlinear cases.

In [42], the existence of analytic solution of the problem with linear regularization was obtained for any initial analytic shape of simply connected domain. We are going to study the existence and uniqueness of analytic solution to the problem with nonlinear regularization when the initial free boundary is analytic.

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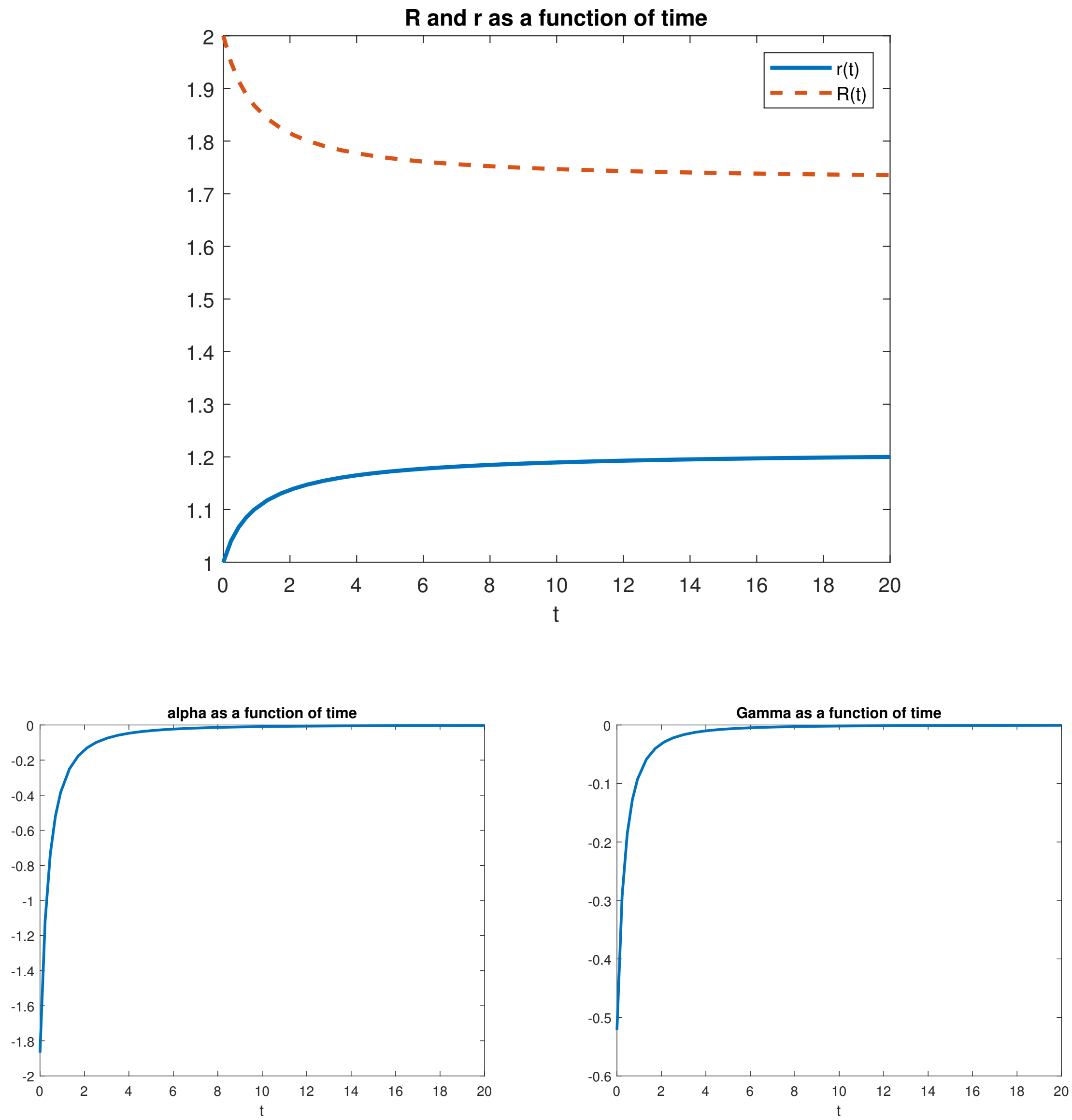

Figure 2: Numerical solution of $r(t)$, $R(t)$, $\alpha(t)$ and $\beta(t)$ from the linear problem 1 (3.23) - (3.26), when we take $b(t) = (1+t)/(2+t)$, $c = 0.1$, $r(0) = 1$, $R(0) = 2$.

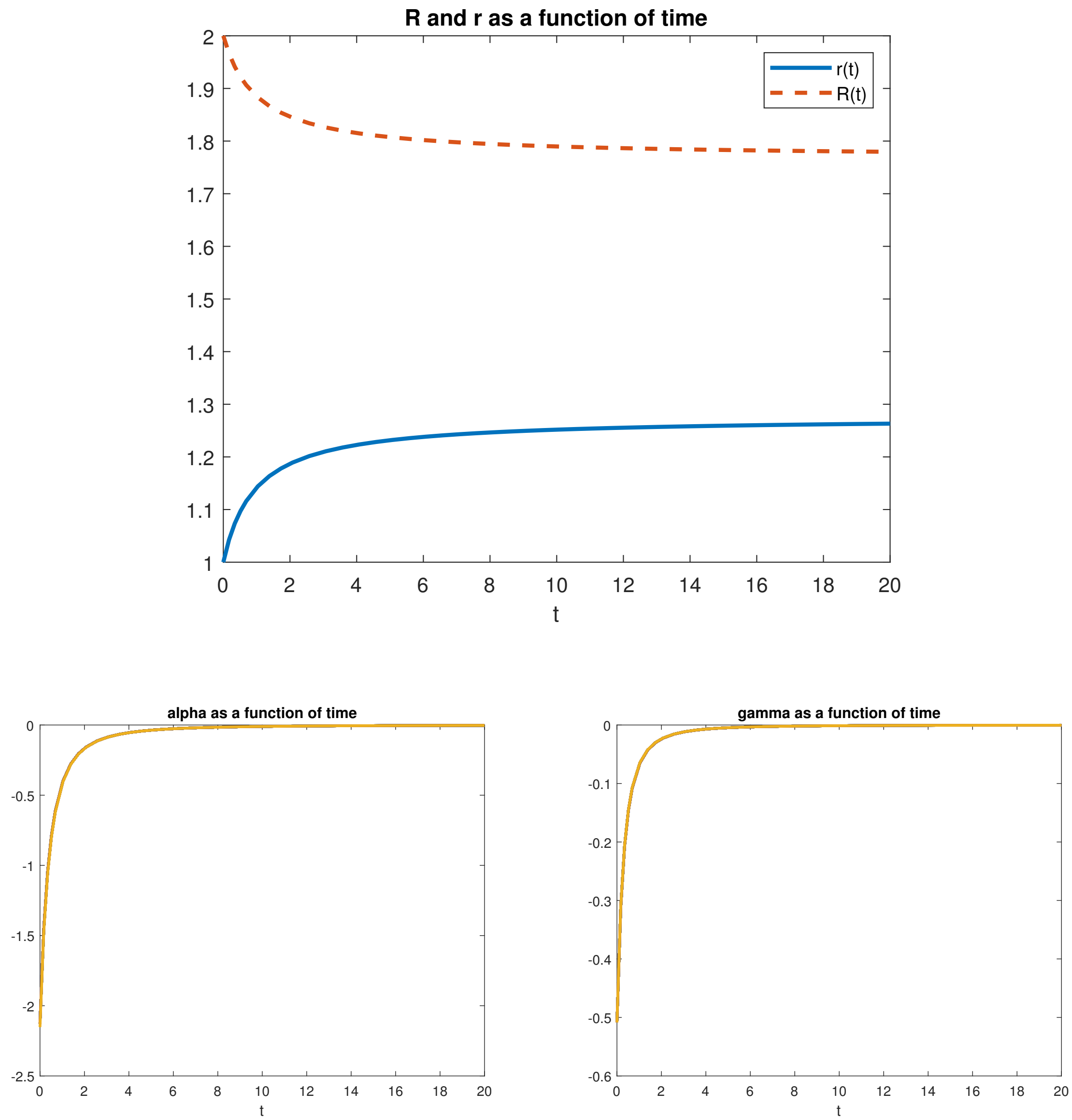

Figure 3: Numerical solution of $r(t)$, $R(t)$, $\alpha(t)$ and $\gamma(t)$ from the nonlinear problem 2 in a annular domain, when we take $\beta = 2, b(t) = (1+t)/(2+t), c = 0.1, r(0) = 1, R(0) = 2$.