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# Existence and stability analysis of solutions for fractional differential equations with delay

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**This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday**

**Abstract.** In this manuscript, we establish the existence and stability of solutions for fractional differential equations with delay. We utilize the Bielecki Norm and the Ulam-Hyers stability for our results.
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**Keywords**: Existence, Stability, Fractional differential equations, Delay.

## Contents



## 1. Introduction

The concept of the Deformable derivative was introduced by F. Zulfeqarr, A. Ujlayan, and P. Ahuja in 2017 [23]. It continuously deforms a function to a derivative, hence the name deformable derivative. This derivative is linearly related to the usual derivative. There are a few manuscripts pertaining to this fractional derivative. For more information, the reader could consult manuscripts such as [9, 10, 16–18, 23]. In [9], we established the existence and uniqueness of solutions to impulsive Cauchy problems involving the deformable derivative with local and nonlocal conditions.
In [10], we studied the existence of solutions for functional differential equations with infinite delay in the sense of the deformable derivative:

$$\begin{cases} D^{\alpha}y(t) = f(t, y_t), for\ t \in J = [0, b], \alpha \in (0, 1); \\ y(t) = \phi(t), t \in (-\infty, 0] \end{cases}$$

In this paper, we study the existence, uniqueness and the Ulam-Hyers type stability of solutions for the following fractional order differential equation:

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), u(\phi(t))) & t \in J = [0, T] \\ \quad u(t) = \mu(t) & t \in [-h, 0], \end{cases} \tag{1.1}$$

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where $0 < \alpha < 1, D^\alpha$ is the deformable derivative, $f \in C([0,T] \times \mathbb{R}^2, \mathbb{R}), \mu(t) \in C([-h,0], \mathbb{R})$, $\phi \in C([0,T],[-h,T])$; let $\phi(t) \leq t$.
The main motivation for this paper was the work of Develi and Duman (see [8]).

## 2. Preliminaries

In this section, $X := C([-h,T], \mathbb{R})$ stands for the Banach space of all continuous functions with the Bielecki norm:

$$\|u\|_{\mathcal{B}} := max\{|u(t)|e^{-\kappa t} : t \in [-h,T]\}.$$

**Definition 2.1.** *([23]) Let $f$ be a real valued function on [a,b], $\alpha \in$ [0,1]. The Deformable derivative of $f$ of order $\alpha$ at $t \in (a,b)$ is defined as:*

$$D^\alpha f(t) = \lim_{\epsilon \to 0} \frac{(1+\epsilon\beta)f(t+\epsilon\alpha) - f(t)}{\epsilon},$$

*where $\alpha + \beta = 1$. If the limit exists, we say that $f$ is $\alpha$-differentiable at t.*

**Remark 2.2.** *If $\alpha = 1$, then $\beta = 0$, we recover the usual derivative. This shows that the deformable derivative is more general than the usual derivative.*

**Definition 2.3.** *([23]) For $f$ defined on $[a,b], \alpha \in (0,1]$, the $\alpha$-integral of the function $f$ is defined by*

$$I_a^\alpha f(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_a^t e^{\frac{\beta}{\alpha} x} f(x) dx, \quad t \in [a,b],$$

*where $\alpha + \beta = 1$. When $a = 0$ we use the notation*

$$I^\alpha f(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} x} f(x) dx.$$

**Remark 2.4.** *If $\alpha = 1$, then $\beta = 0$, we recover the usual Riemann integral. This also shows that the $\alpha$-integral is more general than the usual Riemann integral.*

**Theorem 2.5.** ([23]) A differentiable function $h$ at a point $t \in (a,b)$ is always $\alpha$-differentiable at that point for any $\alpha$. Moreover, we have

$$D^\alpha h(t) = \beta h(t) + \alpha D h(t).$$

**Corollary 2.6.** *([23]) An $\alpha$ - differentiable function $f$ defined in $(a,b)$ is differentiable as well.*

**Theorem 2.7.** ([17],[23]) *The operators $D^\alpha$ and $I_a^\alpha$ possess the following properties:*
*Let $\alpha, \alpha_1, \alpha_2 \in (0,1]$ such that $\alpha + \beta = 1, \ \alpha_i + \beta_i = 1$ for $i = 1,2$.*

1. *Let $f$ be differentiable at a point t for some $\alpha$. Then it is continuous there.*

2. *Suppose f and g are $\alpha$-differentiable. Then*

$$\begin{aligned} D^\alpha (f \circ g)(t) &= \beta (f \circ g)(t) + \alpha D(f \circ g)(t) \\ &= \beta (f \circ g)(t) + \alpha f'(g(t)) g'(t). \end{aligned}$$

3. *Let $f$ be continuous on [a,b]. Then $I_a^\alpha f$ is $\alpha$- differentiable in (a,b), and we have*

$$D^\alpha (I_a^\alpha f(t)) = f(t), and$$

$$I_a^\alpha (D^\alpha f(t)) = f(t) - e^{\frac{\beta}{\alpha}(a-t)} f(a).$$

4. $D^{\alpha}\left(\dfrac{f}{g}\right) = \dfrac{gD^{\alpha}(f) - \alpha f Dg}{g^2}$.

5. *Linearity :* $D^{\alpha}(af + bg) = aD^{\alpha}f + bD^{\alpha}g$.

6. *Commutativity :* $D^{\alpha_1} \cdot D^{\alpha_2} = D^{\alpha_2} \cdot D^{\alpha_1}$.

7. *For a constant* $c, D^{\alpha}(c) = \beta c$.

8. $D^{\alpha}(fg) = (D^{\alpha}f)g + \alpha f Dg$.

9. *Linearity :* $I_a^{\alpha}(bf + cg) = bI_a^{\alpha}f + cI_a^{\alpha}g$.

10. *Commutativity :* $I_a^{\alpha_1} I_a^{\alpha_2} = I_a^{\alpha_2} I_a^{\alpha_1}$.

**Definition 2.8.** *Problem (1.1) is Ulam-Hyers stable if there exists a real number* $\zeta > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $\theta \in C([-h, T], \mathbb{R})$ *of the inequality*

$$|D^{\alpha}\theta(t) - f(t, \theta(t), \theta(\phi(t)))| \leq \epsilon, \ \ t \in [0, T], \tag{2.1}$$

*there exists a solution* $u$ *in* $C([-h, T], \mathbb{R})$ *to problem (1.1) with*

$$|\theta(t) - u(t)| \leq \zeta\epsilon, \ \ t \in [-h, T].$$

**Remark 2.9.** *A function* $\theta \in C([0, T], \mathbb{R})$ *is a solution of the inequality (2.1) if and only if there exists a function* $\Omega \in C([0, T], \mathbb{R})$ *such that*
$(i)|\Omega(t)| \leq \epsilon$ *for all* $t \in [0, T]$,
$(ii)D^{\alpha}\theta(t) = f(t, \theta(t), \theta(\phi(t))) + \Omega(t)$ *for all* $t \in [0, T]$.

**Remark 2.10.** *It can readily be seen that using Definition 2.3 and Theorem 2.7, a solution* $\theta \in C([0, T], \mathbb{R})$ *of inequality (2) is also a solution to the following integral inequality:*

$$\left|\theta(t) - \theta(0)e^{\frac{-\beta}{\alpha}t} - \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, \theta(s), \theta(\phi(s)))\right| ds \leq \frac{\epsilon}{\beta}$$

*for all* $t \in [0, T]$.

We derive the following inequality for our subsequent results:
For $\kappa > 0, 0 \leq s \leq t, t \in [0, T]$,

$$\begin{aligned}\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}e^{\kappa s}ds &= \frac{1}{\alpha}\int_0^t e^{\frac{-\beta}{\alpha}(t-s)}e^{\kappa s}ds \\ &\leq \frac{1}{\alpha}\int_0^t e^{\kappa s}ds \\ &\leq \frac{e^{\kappa t}}{\kappa\alpha}.\end{aligned}$$

**Definition 2.11.** *[21, 22]Let* $(X, d)$ *be a metric space. An operator* $\mathcal{A} : X \to X$ *is said to be a Picard operator if there exists* $x^* \in X$ *such that*
*(i)* $F_{\mathcal{A}} = \{x^*\}$ *where* $F_{\mathcal{A}} = \{x \in X : \mathcal{A}(x) = x\}$ *is the fixed point set of* $\mathcal{A}$*;*
*(ii) The sequence* $(\mathcal{A}^n(x_0))_{n\in N}$ *converges to* $x^*$ *for all* $x_0 \in X$.

**Lemma 2.12.** *[21, 22]Let* $(X, d, \leq)$ *be an ordered metric space and* $\mathcal{A} : X \to X$ *be an increasing Picard operator* $(F_{\mathcal{A}} = \{x^*\})$*. Then, for* $x \in X, x \leq \mathcal{A}(x) \implies x \leq x^*$ *while* $x \geq \mathcal{A}(x) \implies x \geq x^*$.

**Lemma 2.13.** *[6] If*

$$x(t) \le h(t) + \int_{t_0}^{t} k(s)x(s)ds, \ \ t \in [t_0, T),$$

*where all functions involved are continuous on* $[t_0, T), T \le +\infty$, *and* $k(t) \ge 0$, *then* $x(t)$ *satisfies*

$$x(t) \le h(t) + \int_{t_0}^{t} h(s)k(s)\ exp\left[\int_{s}^{t} k(\omega)d(\omega)\right] ds,\ t \in [t_0, T).$$

## 3. Existence and Uniqueness

In this section, we prove the existence and uniqueness of solutions for problem (1.1).

**Definition 3.1.** *A function* $u \in C([-h, T], \mathbb{R})$ *is said to be a mild solution of problem (1.1) if*

$$u(t) = \begin{cases} \mu(t), & t \in [-h, 0] \\ \mu(0)e^{\frac{-\beta}{\alpha}t} + \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s} f(s, u(s), u(\phi(s)))ds, & t \in [0, T]. \end{cases}$$

We investigate problem (1.1) with the following assumptions:
(H1) $f \in C([0, T] \times \mathbb{R}^2, \mathbb{R}), \phi \in C([0, T], [-h, T])$ and $\phi(t) \le t$ on $[0, T]$,
(H2) There is a constant $L > 0$ such that
$|f(t, u_1, \theta_1) - f(t, u_2, \theta_2)| \le L(|u_1 - u_2| + |\theta_1 - \theta_2|)$ for all $u_i, \theta_i \in C([-h, T], \mathbb{R})$ and $t \in [0, T]$.

**Theorem 3.2.** *Under the asumptions (H1)-(H2), if* $\kappa > \frac{2L}{\alpha}$, *then problem (1.1) has a unique mild solution.*

**Proof**. We first transform problem (1.1) into a fixed point problem.
Define $\mathcal{F} : C([-h, T], \mathbb{R}) \to C([-h, T], \mathbb{R})$ such that

$$\mathcal{F}u(t) = \begin{cases} \mu(t), & t \in [-h, 0] \\ \mu(0)e^{\frac{-\beta}{\alpha}t} + \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s} f(s, u(s), u(\phi(s)))ds, & t \in [0, T]. \end{cases} \tag{3.1}$$

Then we find a unique fixed point of $\mathcal{F}$, which is the unique solution. We consider the Banach space $X := C([-h, T], \mathbb{R})$ endowed with following norm

$$\|u\|_{\mathcal{B}} = \max_{t \in [-h, T]} |u(t)|e^{-\kappa t}. \tag{3.2}$$

Using Remark 2.10, we show that $\mathcal{F}$ is a contraction mapping on $(X, \|\cdot\|_{\mathcal{B}})$. For all $u(t), \theta(t) \in X, \mathcal{F}u(t) = \mathcal{F}\theta(t)$ if $t \in [-h, 0]$. For $t \in [0, T]$, we have

$$\begin{aligned} |\mathcal{F}u(t) - &\mathcal{F}\theta(t)| \\ &\le \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}|f(s, u(s), u(\phi(s))) - f(s, \theta(s), \theta(\phi(s)))|ds \\ &\le \frac{L}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}e^{\kappa s}\left(\max_{-h \le s \le T}|u(s) - \theta(s)|e^{-\kappa s} + \max_{-h \le s \le T}|u(\phi(s)) - \theta(\phi(s))|e^{-\kappa s}\right) \\ &\le \frac{2L}{\alpha}e^{\frac{-\beta}{\alpha}t}\|u - \theta\|_{\mathcal{B}}\int_0^t e^{\frac{\beta}{\alpha}s}e^{\kappa s}ds \\ &\le \frac{2L}{\kappa\alpha}\|u - \theta\|_{\mathcal{B}}e^{\kappa t}. \end{aligned}$$

Thus

$$\|\mathcal{F}u - \mathcal{F}\theta\|_{\mathcal{B}} \leq \eta\|u - \theta\|_{\mathcal{B}}, \text{where } \eta = \frac{2L}{\kappa\alpha}.$$

Since $\eta < 1$, we find a unique fixed point $\mathcal{F}$ by the Banach contraction principle. ■

**Remark 3.3.** *For a constant delay $\tau > 0,$ and $\phi(t) = t - \tau$, problem (1.1) becomes*

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), u(t-\tau)), & t \in [0, T] \\ u(t) = \mu(t), & t \in [-\tau, 0]. \end{cases} \tag{3.3}$$

*The proof for the existence and uniqueness of solutions for the above fractional differential equation is obtained using the following three steps. To that end, we introduce the following Lipschitz condition.*

**Theorem 3.4.** *Let $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume that there exists a positive constant L such that*

$$|f(t, u_1, \theta) - f(t, u_2, \theta)| \leq L|u_1 - u_2|$$

*for all $u_i, \theta \in C([0, T], \mathbb{R}), (i = 1, 2, ...)$ and $t \in [0, T]$. And in addition, assume that $\kappa > \frac{L}{\alpha}$.Then* (3.3) *has a unique solution.*

**Proof**. Problem (3.3) is equivalent to:

$$u(t) = \begin{cases} \mu(t), & -\tau \leq t \leq 0 \\ \mu(0)e^{\frac{-\beta}{\alpha}t} + \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s} f(s, u(s), u(s-\tau))ds, & 0 \leq t \leq T. \end{cases}$$

We partition the interval [0,T] into $n$ sub-intervals of equal length $S$. And have the following for $0 < S < \tau$ and $nS = T$: $0 = S_0 < S_1 < \cdots < S_n = T, S_i - S_{i-1} = S$.
We see that $t \leq S_{i+1} \implies t - \tau \leq S_i$ using this argument:
$t \leq S_{i+1} \implies t - \tau \leq S_{i+1} - \tau \leq S_{i+1} - S = S_i$.
Step 1. let $(\mathcal{E}_1, \|\cdot\|_1)$ be a Banach space of continuous functions $u : [-\tau, S_1] \to \mathbb{R}$ with the following norm :

$$\|u\|_1 = \max_{t\in[-\tau,S_1]} |u(t)|e^{-\kappa t},$$

and $u(t) = \mu(t)$ for $-\tau \leq t \leq 0$. Define a mapping $\mathcal{F}_1 : \mathcal{E}_1 \to \mathcal{E}_1$ by:

$$\mathcal{F}_1u(t) = \begin{cases} \mu(t), & -\tau \leq t \leq 0 \\ \mu(0)e^{\frac{-\beta}{\alpha}t} + \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s} f(s, u(s), u(s-\tau))ds, & 0 \leq t \leq S_1. \end{cases}$$

For $u(t), \theta(t) \in \mathcal{E}_1, \mathcal{F}_1u(t) = \mathcal{F}_1\theta(t)$ if $t \in [-\tau, 0]$, For $t \in [0, S_1]$, we have

$$|\mathcal{F}_1u(t) - \mathcal{F}_1\theta(t)| \leq \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, u(s), u(s-\tau) - f(s, \theta(s), \theta(s-\tau)))|ds.$$

Since $0 \leq s \leq S_1$ implies $(s-\tau) \in [-\tau, 0]$, and the definition of $\mathcal{E}_1$, we have

$$\begin{aligned}
&|\mathcal{F}_1 u(t) - \mathcal{F}_1 \theta(t)| \\
&\qquad \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, u(s), u(s-\tau)) - f(s, \theta(s), \theta(s-\tau))| ds \\
&\qquad \leq \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} e^{\kappa s} [\max_{-h \leq s \leq s_1} |u(s) - \theta(s)| e^{-\kappa s}] ds \\
&\qquad \leq \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t} \|u - \theta\|_1 \int_0^t e^{\frac{\beta}{\alpha} s} e^{\kappa s} ds \\
&\qquad \leq \frac{L}{\kappa \alpha} \|u - \theta\|_1 e^{\kappa s}.
\end{aligned}$$

Therefore,

$$\|\mathcal{F}_1 u - \mathcal{F}_1 \theta\|_1 \leq \eta \|u - \theta\|_1.$$

Since $\eta = \frac{L}{\kappa \alpha} < 1$, we get that $\mathcal{F}_1$ is a contraction mapping, and so there exists a unique fixed point $\mu_1 \in \mathcal{E}_1$ that satisfies (3.3) on $[-\tau, s_1]$.

Step 2: In this step, we extend the interval of step 1 into $[-\tau, S_2]$. Let $(\mathcal{E}_2, \|\cdot\|_2)$ be a complete normed space of continuous functions $u : [-\tau, S_2] \to \mathbb{R}$ with the following norm

$$\|u\|_2 = \max_{t \in [-r, S_2]} |u(t)| e^{-\kappa s}.$$

Let $u(t) = \mu_1(t)$ for $-\tau \leq t \leq S_1$. Continuing in like manner, define a mapping $\mathcal{F}_2 : \mathcal{E}_2 \to \mathcal{E}_2$ by

$$\mathcal{F}_2 u(t) = \begin{cases} \mu_1(t), & -\tau \leq t \leq S_1 \\ \mu(0) e^{\frac{-\beta}{\alpha} t} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, u(s), u(s-\tau)) ds, & S_1 \leq t \leq S_2. \end{cases}$$

For $u(t), \theta(t) \in \mathcal{E}_2, F_2 u(t) = F_2 \theta(t)$ if $t \in [-\tau, S_1]$; else we take $t \in [S_1, S_2]$. Thus

$$|F_2 u(t) - F_2 \theta(t)| \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, u(s), u(s-\tau)) - f(s, \theta(s), \theta(s-\tau))| ds.$$

Observe that $0 \leq s \leq S_2 \implies (s - r) \in [-\tau, S_1]$. Based on the the definition of $\mathcal{E}_2$, we may derive the following inequality:

$$\begin{aligned}
&|\mathcal{F}_2 u(t) - \mathcal{F}_2 \theta(t)| \\
&\qquad \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, u(s), \mu_1(s-\tau)) - f(s, \theta(s), \mu_1(s-\tau))| ds \\
&\qquad \leq \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} e^{\kappa s} (\max_{-h \leq s \leq S_2} |u(s) - \theta(s)| e^{-\kappa s}) ds \\
&\qquad \leq \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t} \|u - \theta\|_2 \int_0^t e^{\frac{\beta}{\alpha} s} e^{\kappa s} ds \\
&\qquad \leq \frac{L}{\kappa \alpha} \|u - \theta\|_2 e^{\kappa t}.
\end{aligned}$$

Thus, $\|\mathcal{F}_2 u - \mathcal{F}_2 \theta\|_2 \leq \eta \|u - \theta\|_2$, where $\eta < 1$ as aforementioned. Therefore, $\mathcal{F}_2$ has a unique fixed point $\mu_2$ in $\mathcal{E}_2$ that satisfies (3.3) on $[-\tau, S_2]$.

Step 3: By following this method up to the the $n$th step, we can find that $\mathcal{F}_n$ has a unique fixed point $\mu_n$ in $\mathcal{E}_n$ satisfying (3.3) on $[-\tau, S_n] = [-\tau, T]$. ■

4. Ulam-Hyers stability.

**Theorem 3.5.** *Assume that conditions H1 and H2 are fulfilled. Then the first equation of problem (1.1) is Ulam-Hyers stable.*

**Proof**. Let $\theta$ be a solution to (2.1) and $u$ be a unique solution to the following problem:

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), u(\phi(t)) & t \in [0, T] \\ u(t) = \theta(t) & t \in [-h, 0]. \end{cases}$$

Then

$$u(t) = \begin{cases} \theta(t) & t \in [-h, 0] \\ \theta(0)e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, u(s), u(\phi(s))ds & t \in [0, T]. \end{cases}$$

Observe that we have the following inequality from Remark 2.10:

$$|\theta(t) - \theta(0)e^{\frac{-\beta}{\alpha}t} - \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, \theta(s), \theta(\phi(s)))ds| \leq \frac{\epsilon}{\beta}$$

for all $t \in [0, T]$, and $|\theta(t) - u(t)| = 0$ for all $t \in [-h, 0]$. For $t \in [0, T]$ we obtain from $H2$ that

$$\begin{aligned} |\theta(t) - u(t)| \leq & |\theta(t) - \theta(0)e^{\frac{-\beta}{\alpha}t} - \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} f(s, \theta(s), \theta(\phi(s)))ds| \\ & + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}|f(s, \theta(s), \theta(\phi(s)) - f(s, u(s), u(\phi(s))|ds \\ \leq & \frac{\epsilon}{\beta} + \frac{L}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}(|\theta(s) - u(s)| + |\theta(\phi(s)) - u(\phi(s))|)ds. \end{aligned} \tag{3.4}$$

We define an operator for $v \in C([-h, T], \mathbb{R}^+)$:

$$\mathcal{A} := C([-h, T], \mathbb{R}^+) \to C([-h, T], \mathbb{R}^+),$$

given by

$$\mathcal{A}(v)(t) = \begin{cases} 0 & t \in [-h, 0] \\ \frac{\epsilon}{\beta} + \frac{L}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\left(v(s) + v(\phi(s))\right)ds & t \in [0, T]. \end{cases}$$

We show that $\mathcal{A}$ is a Picard operator via the contraction mapping principle. For $v, \tilde{v} \in C([-h, T], \mathbb{R}^+)$, one estimates

$$\begin{aligned} |\mathcal{A}v - \mathcal{A}\tilde{v}| &\leq \frac{L}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}(|v(s) - \tilde{v}(s)| + |v(\phi(s)) - \tilde{v}(\phi(s))|)ds \\ &\leq \frac{2L}{\alpha}\|v - \tilde{v}\|_{\mathcal{B}}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}e^{\kappa s}ds \\ &\leq \frac{2L}{\kappa\alpha}\|v - \tilde{v}\|_{\mathcal{B}}e^{\kappa t}ds, \end{aligned}$$

which means

$\|\mathcal{A}v - \mathcal{A}\tilde{v}\|_{\mathcal{B}} \leq \eta\|v - \tilde{v}\|_{\mathcal{B}}$ where $\eta = \frac{2L}{\kappa\alpha}$.

For $\kappa > \frac{2L}{\alpha} > 0$ , we observe that $\eta < 1$, and consequently we get that $\mathcal{A}$ is a contraction mapping with respect to the Bielecki norm $\|\cdot\|_{\mathcal{B}}$ on $C([-h,T],\mathbb{R}^+)$. Thus, $\mathcal{A}$ is a Picard operator such that $F_{\mathcal{A}} = \{v^*\}$ and the Banach Contraction principle gives the equality:

$$v^*(t) = \frac{\epsilon}{\beta} + \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} (v^*(s) + v^*(\phi(s))) ds$$

for $t \in [0,T]$. To show that $v^*$ is increasing, let $m := \min_{t\in[0,T]}[v^*(t) + v^*(\phi(t))] \in \mathbb{R}^+$. For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
&v^*(t_2) - v^*(t_1)\\
&= \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t_2} \int_0^{t_2} e^{\frac{\beta}{\alpha} s}(v^*(s) + v^*(\phi(s)))ds - \frac{L}{\alpha} e^{\frac{-\beta}{\alpha} t_1} \int_0^{t_1} e^{\frac{\beta}{\alpha} s}(v^*(s) + v^*(\phi(s)))ds\\
&= \frac{L}{\alpha} \int_0^{t_2} e^{\frac{-\beta}{\alpha} t_2} e^{\frac{\beta}{\alpha} s}(v^*(s) + v^*(\phi(s)))ds - \frac{L}{\alpha} \int_0^{t_1} e^{\frac{-\beta}{\alpha} t_1} e^{\frac{\beta}{\alpha} s}(v^*(s) + v(\phi(s)))ds\\
&= \frac{L}{\alpha} \int_0^{t_1} (e^{\frac{-\beta}{\alpha} t_2} - e^{\frac{-\beta}{\alpha} t_1}) e^{\frac{\beta}{\alpha} s}(v^*(s) + v^*(\phi(s)))ds + \frac{L}{\alpha} \int_{t_1}^{t_2} e^{\frac{-\beta}{\alpha} t_2} e^{\frac{\beta}{\alpha} s}(v^*(s) + v^*(\phi(s)))ds\\
&\geq \frac{mL}{\alpha} \int_0^{t_1} (e^{\frac{-\beta}{\alpha} t_2} - e^{\frac{-\beta}{\alpha} t_1}) e^{\frac{\beta}{\alpha} s} ds + \frac{mL}{\alpha} \int_{t_1}^{t_2} e^{\frac{-\beta}{\alpha} t_2} e^{\frac{\beta}{\alpha} s} ds\\
&= \frac{mL}{\alpha} (e^{\frac{-\beta}{\alpha} t_2} - e^{\frac{-\beta}{\alpha} t_1}) \int_0^{t_1} e^{\frac{\beta}{\alpha} s} ds + \frac{mL}{\alpha} e^{\frac{-\beta}{\alpha} t_2} \int_{t_1}^{t_2} e^{\frac{\beta}{\alpha} s} ds\\
&= \frac{mL}{\beta} \left[ e^{\frac{-\beta}{\alpha}(t_2-t_1)} - e^{\frac{-\beta}{\alpha} t_2} - 1 + e^{\frac{-\beta}{\alpha} t_1} \right] + \frac{mL}{\beta} \left[ 1 - e^{\frac{-\beta}{\alpha}(t_2-t_1)} \right]\\
&= \frac{mL}{\beta} \left[ e^{\frac{-\beta}{\alpha} t_1} - e^{\frac{-\beta}{\alpha} t_2} \right] > 0.
\end{aligned}$$

Therefore, $v^*$ is an increasing function, and so $v^*(\phi(t)) \leq v^*(t)$ because $\phi(t) \leq t$. It follows that

$$v^*(t) \leq \frac{\epsilon}{\beta} + \frac{2L}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}(v^*(s))ds.$$

Using Lemma 2.13, one derives the following inequality

$$\begin{aligned}
v^*(t) &\leq \frac{\epsilon}{\beta} + \frac{2L}{\alpha} \int_0^t e^{\frac{-\beta}{\alpha} t} e^{\frac{\beta}{\alpha} s} \frac{\epsilon}{\beta}\, exp \left[ \int_s^t e^{\frac{-\beta}{\alpha} t} e^{\frac{\beta}{\alpha} \omega} d\omega \right] ds\\
&\leq \frac{\epsilon}{\beta} + \frac{2L\epsilon}{\alpha\beta} \int_0^t e^{\frac{-\beta}{\alpha} t} e^{\frac{\beta}{\alpha} s}\, exp \left[ \frac{\alpha}{\beta} \right] ds\\
&\leq \frac{\epsilon}{\beta} + exp \left[ \frac{\alpha}{\beta} \right] \frac{2L\epsilon}{\beta^2} \left[ 1 - e^{\frac{-\beta}{\alpha} t} \right]\\
&\leq \frac{\epsilon}{\beta} \left[ 1 + \frac{2e^{\frac{\alpha}{\beta}}}{\beta} L \right]
\end{aligned}$$

for $t \in [-h,T]$. If $v = |\theta - u|$ in (3.4), then $v \leq \mathcal{A}v$. So, we have $v < v^*$ because $\mathcal{A}$ is an increasing Picard operator. Consequently, we have

$$|\theta(t) - u(t)| \leq \zeta\epsilon$$

where

$$\zeta = \frac{1}{\beta} \left[ 1 + \frac{2}{\beta} L e^{\frac{\alpha}{\beta}} \right].$$

Thus the first equation of problem (1.1) is Ulam-Hyers stable. The proof is complete. ■

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