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Polynomial stability of nonlinear Timoshenko beam with distributed delay

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Abstract. In this work, we consider a Nonlinear Timoshenko system with distributed delay-time. We prove the polynomial stability of the system for the case of nonequal speeds of wave propagation. This is after verifying the exponential stability in the opposite one.

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Contents

1. Introduction

In this work, we consider a nonlinear Timoshenko system with distributed delay term,

$$\begin{cases} \rho_1\phi_{\mathbf{tt}} - k(\phi_\varkappa + \psi)_\varkappa = 0, \\ \rho_2\psi_{\mathbf{tt}} - b\psi_{\varkappa\varkappa} + k(\phi_\varkappa + \psi) + \mu_1\psi_{\mathbf{t}} + \int_{\iota_1}^{\iota_2} \eta_2(\tau)\psi_{\mathbf{t}}(\varkappa, \mathbf{t} - \tau)d\tau + f(\psi) = 0, \end{cases} \tag{1.1}$$

where $(\varkappa, \mathbf{t}) \in (0,1) \times \mathbb{R}^+$. The system (1.1) with $\mu_1 = \eta_2 = f = 0$, was first proposed by Timoshenko [24] as a model that describes the impact of vibrations on a thin elastic beam of length. The functions $\phi = \phi(\varkappa, \mathbf{t})$ and $\psi = \psi(\varkappa, \mathbf{t})$ describe the small transverse displacement of the beam and the rotation angle of the beam's filament. The parameters ρ_1,ρ_2, k and b are positive constants. The function $f(\psi)$ is a forcing term and $\mu_1\psi_{\mathbf{t}}$ designate a frictional damping. The distributed delay is given by $\int_{\iota_1}^{\iota_2} \eta_2(\tau)\psi_{\mathbf{t}}(\varkappa, \mathbf{t} - \tau)d\tau$, where, $\iota_1, \iota_2 > 0$.
We provide the system (1.1) with the initial data

$$\begin{cases} \phi(\varkappa, 0) = \phi_0,\ \phi_{\mathbf{t}}(\varkappa, 0) = \phi_1,\ \psi(\varkappa, 0) = \psi_0,\ \psi_{\mathbf{t}}(\varkappa, 0) = \psi_1 \\ \psi_{\mathbf{t}}(\varkappa, -\mathbf{t}) = f_0(\varkappa, \mathbf{t}),\ \varkappa \in (0,1), \end{cases} \tag{1.2}$$

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and the boundary conditions

$$\phi(0,\mathbf{t})=\phi(1,\mathbf{t})=\psi(0,\mathbf{t})=\psi(1,\mathbf{t})=0. \tag{1.3}$$

We remember that Timoshenko system without delay has been considered by many authors. Their goal was to achieve the asymptotic behaviour of the solutions of these systems by introducing different types of damping. See for instance [1–3, 9, 11, 12, 15, 19] and references therein.

In recent years, including the delay term makes the problems of EDPs more interesting. In fact, delays can cause destabilization of a system which is stable without the delays. Datko et al. [7] studied the the destabilizing effect of arbitrarily small delays in the boundary control of a wave equation. In [17], the authors proved an exponential decay result of the solution under suitable assumptions of the delayed wave equation where the delay is considered both in the boundary condition and in the internal feedback. Later [18] the same authors introduced a distributed delay on a part of the boundary, and they proved an exponential stability under some assumptions, they also studied the following problem with internal feedback

$$\begin{cases} u_{\mathbf{tt}}-\Delta u+\mu_0 u_{\mathbf{t}}+\int_{\iota_1}^{\iota_2} a(\varkappa)\mu(\tau)u_{\mathbf{t}}(\mathbf{t}-\tau)d\tau \\ u=0 \qquad on \quad \Gamma_0(0,\alpha) \\ \dfrac{\partial u}{\partial \nu}=0 \qquad on \quad \Gamma_1(0,\alpha) \\ u(\varkappa,0)=u_0(\varkappa) \;\; and \quad u_{\mathbf{t}}(\varkappa,0)=u_1(\varkappa) \qquad in\,\Omega \\ u_{\mathbf{t}}(\varkappa,-\mathbf{t})=f_0(\varkappa,-\mathbf{t}) \qquad in\;\; \Omega(0,\iota_2) \end{cases} \tag{1.4}$$

where $a\in L^2(\Omega)$ is a function satisfies

$$\mu_0>||a||_\alpha\int_{\iota_1}^{\iota_2}\mu(\tau)d\tau.$$

They obtained an exponential decay result for the energy.

In [22], the authors discussed the stability of a linear Timoshenko system with a constant delay

$$\begin{cases} \rho_1\phi_{\mathbf{tt}}-k(\phi_\varkappa+\psi)_\varkappa=0, \\ \rho_2\psi_{\mathbf{tt}}-b\psi_{\varkappa\varkappa}+k(\phi_\varkappa+\psi)+\mu_1\psi_{\mathbf{t}}+\eta_2\psi_{\mathbf{t}}(\varkappa,\mathbf{t}-\tau)=0. \end{cases} \tag{1.5}$$

a necessary condition which made the solutions of (1.5) exponentially stable is

$$\frac{k}{\rho_1}=\frac{b}{\rho_2}. \tag{1.6}$$

It is most important to report that most of results on Timoshenko types systems is based on the above condition, otherwise, only a polynomial stability was proved for the case of nonequal speeds (see [2, 9, 11, 12, 15]).

For a non linear Timoshenko system , Feng and Pelicer [8] added to (1.5) a forcing term $f(\psi)$ in the second equation and proved an exponential decay under an appropriate condition between the weights of the the delay term and frictional damping, their result was extended by Hao and Wei [12] to nonlinear heatTimoshenko system of based on the energy method. In the case where the speeds are nonequal, they established a polynomial decay estimate.

System (1.1) was recently investigated by Bouzettouta et al. [4] and they proved an exponential decay result of the energy when (1.6) holds, in this paper our goal is to complete their study for the case of non equal wave speeds.

2. Preliminaries

The necessary assumptions and transformations needed to obtain the desired results were presented in this section. As in [17], we use the following notation

$$\chi\left(\varkappa,\rho,\tau,\mathbf{t}\right)=\psi_{\mathbf{t}}\left(\varkappa,\mathbf{t}-\rho\tau\right),\;\varkappa\in(0,L)\,,\;\rho\in(0,L)\,,\;\mathbf{t},\tau\in(\iota_1,\iota_2).$$

The new variable χ satisfies the following differential equation

$$\tau \chi_{\mathbf{t}}(\varkappa, \rho, \tau, \mathbf{t}) + \chi_{\rho}(\varkappa, \rho, \tau, \mathbf{t}) = 0, \ (\varkappa, \rho, \tau, \mathbf{t}) \in (0, L) \times (0, L) \times (\iota_1, \iota_2) \times (0, +\infty).$$

Therefore, the problem (1.1) becomes

$$\begin{cases} \rho_1 \phi_{\mathbf{tt}} - k(\phi_\varkappa + \psi)_\varkappa = 0, \ \varkappa \in (0, L), \ \mathbf{t} > 0, \\ \rho_2 \psi_{\mathbf{tt}} - b\psi_{\varkappa\varkappa} + k(\phi_\varkappa + \psi) + \mu_1 \psi_{\mathbf{t}} \\ + \int_{\iota_1}^{\iota_2} \eta_2(\tau) \chi(\varkappa, 1, \tau, \mathbf{t}) d\tau + f(\psi) = 0, \ \varkappa \in (0, L), \ \mathbf{t} > 0, \\ \tau \chi_{\mathbf{t}}(\varkappa, \rho, \tau, \mathbf{t}) + \chi_\rho(\varkappa, \rho, \tau, \mathbf{t}) = 0, \ \rho \in (0, L), \ \tau \in (\iota_1, \iota_2), \ \mathbf{t} > 0, \end{cases} \tag{2.1}$$

with the initial data and boundary conditions

$$\begin{cases} \phi(\varkappa, 0) = \phi_0, \ \phi_{\mathbf{t}}(\varkappa, 0) = \phi_1, \ \varkappa \in (0, L), \\ \psi(\varkappa, 0) = \psi_0, \ \psi_{\mathbf{t}}(\varkappa, 0) = \psi_1, \ \varkappa \in (0, L), \\ \chi(\varkappa, \rho, \tau, 0) = f_0(\varkappa, \rho\tau), \ \varkappa \in (0, 1), \ \rho \in (0, L), \ \tau \in (0, \iota_2), \\ \phi(0, \mathbf{t}) = \phi(1, \mathbf{t}) = \psi(0, \mathbf{t}) = \psi(1, \mathbf{t}) = 0, \ \mathbf{t} > 0. \end{cases} \tag{2.2}$$

In what follows, we assume that

$$\int_{\iota_1}^{\iota_2} |\eta_2(\tau)| \, d\tau < \mu_1. \tag{2.3}$$

We assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\left| f(\psi^1) - f(\psi^2) \right| \leq k_0 \left(\left|\psi^1\right|^\theta - \left|\psi^2\right|^\theta \right) \left|\psi^1 - \psi^2\right| \tag{2.4}$$

for all $\psi^1, \psi^2 \in \mathbb{R}$, where $k_0 > 0$, $\theta > 0$. Also

$$0 \leq \tilde{f}(\psi) \leq f(\psi)\psi, \ \text{for all } \psi \in \mathbb{R}, \tag{2.5}$$

with

$$\tilde{f}(y) = \int_0^y f(\tau) d\tau.$$

Let $\mathbf{H}$ the Hilbert space,

$$\mathbf{H} = \mathbf{H}_0^1(0, L) \times L^2(0, L) \times \mathbf{H}_0^1(0, L) \times L^2(0, L) \times L^2((0, L) \times (0, L) \times (\iota_1, \iota_2)),$$

and for any $U = (\phi, u, \psi, v, \chi)^{\mathbf{t}} \in \mathbf{H}$, $\tilde{U} = \left(\widetilde{\phi}, \widetilde{u}, \widetilde{\psi}, \widetilde{v}, \widetilde{\chi}\right)^{\mathbf{t}} \in \mathbf{H}$, we equip the space $\mathbf{H}$ with the inner product

$$\begin{aligned} \left\langle U, \tilde{U} \right\rangle_{\mathbf{H}} &= \int_0^L \left[\rho_1 u \widetilde{u} + \rho_2 v \widetilde{v} + k(\phi_\varkappa + \psi)\left(\widetilde{\phi}_\varkappa + \widetilde{\psi}\right) + b\psi_\varkappa \widetilde{\psi}_\varkappa \right] d\varkappa \\ &+ \int_0^L \int_{\iota_1}^{\iota_2} \tau |\eta_2(\tau)| \int_0^L \chi(\varkappa, \rho, \tau, \mathbf{t}) \widetilde{\chi}(\varkappa, \rho, \tau, \mathbf{t}) \, d\rho d\tau d\varkappa. \end{aligned}$$

By introducing the variables $\phi_{\mathbf{t}} = u$ and $\psi_{\mathbf{t}} = v$, then the system (2.1)-(2.2) is equivalent to

$$\begin{cases} U_{\mathbf{t}} = AU + F, \ \mathbf{t} > 0 \\ U(\varkappa, 0) = U^0(\varkappa) = \left(\phi^0, \phi^1, \psi^0, \psi^1, f_0\right)^{\mathbf{t}}, \end{cases} \tag{2.6}$$

and

$$AU = \begin{pmatrix} u \\ \frac{k}{\rho_1}(\phi_{\varkappa\varkappa} + \psi_\varkappa) \\ v \\ \frac{b}{\rho_2}\psi_{\varkappa\varkappa} - \frac{k}{\rho_2}(\phi_\varkappa + \psi) - \frac{\mu_1}{\rho_2} v - \frac{\mu_1}{\rho_2}\int_{\iota_1}^{\iota_2} \eta_2(\tau) \chi(\varkappa, \rho, \tau, \mathbf{t}) \, d\tau \\ -\frac{1}{\tau}\chi_\rho(\varkappa, \rho, \tau, \mathbf{t}) \end{pmatrix}, \tag{2.7}$$

$$F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-1}{\rho^2} f(\psi) \\ 0 \end{pmatrix}$$

with the domain

$$D(A) = \left\{ (\phi, u, \psi, v, \chi)^{\mathbf{t}} \in \mathbf{H}_1 \right\},$$

with

$$\mathbf{H}_1 = \left(\mathbf{H}^2(0,L) \cap \mathbf{H}_0^1(0,L)\right) \times \mathbf{H}_0^1(0,L) \times \left(\mathbf{H}^2(0,L) \cap \mathbf{H}_0^1(0,L)\right) \times \mathbf{H}_0^1(0,L) \times L^2\left((0,L) \times (0,L) \times (\iota_1, \iota_2)\right).$$

We state the following well-posedness result (see [8]).

Theorem 2.1. *Let $U_0 \in$***H** *and suppose that*(2.3)-(2.5) *hold. Then, the problem* (2.1)-(2.2) *has a unique weak solution $U \in C(\mathbb{R}^+, \mathbf{H})$. If $U_0 \in D(A)$, then*

$$U \in C\left(\mathbb{R}^+, D(A)\right) \cap C\left(\mathbb{R}^+, \mathbf{H}\right).$$

3. Decay result

We exploit the multipliers technique, we show that the solution of (2.1)–(2.2) decays exponentially. First, we present the following lemmas.

Lemma 3.1. *The energy* $\mathbf{E}$ *of (2.1)–(2.2), defined by*

$$\mathbf{E}(\mathbf{t}) = \frac{1}{2}\int_0^L \left(\rho_1 \phi_{\mathbf{t}}^2 + \rho_2 \psi_{\mathbf{t}}^2\right) d\varkappa + \frac{1}{2}\int_0^L \left\{ K(\phi_\varkappa + \psi)^2 + b\psi_\varkappa^2 \right\} d\varkappa + \int_0^L \int_0^L \int_{\iota_1}^{\iota_2} \tau |\eta_2(\tau)| \chi(\varkappa, \rho, \tau, \mathbf{t})\, d\tau d\rho d\varkappa + \int_0^L \tilde{f}(\psi) d\varkappa \tag{3.1}$$

satisfies

$$\frac{d\mathbf{E}(\mathbf{t})}{d\mathbf{t}} \leq -m_1 \int_0^L \psi_{\mathbf{t}}^2 d\varkappa \leq 0, \tag{3.2}$$

where $m_1 = \mu_1 - \int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\, d\tau$.

Proof. Multiplying $(2.1)_1$ by $\phi_{\mathbf{t}}$, $(2.1)_2$ by $\psi_{\mathbf{t}}$, integrating and combining the results, we get

$$\frac{1}{2}\frac{d}{d\mathbf{t}}\int_0^L \left(\rho_1 \phi_{\mathbf{t}}^2 + \rho_2 \psi_{\mathbf{t}}^2\right) d\varkappa + \frac{1}{2}\frac{d}{d\mathbf{t}}\int_0^L \left\{ K(\phi_\varkappa + \psi)^2 + b\psi_\varkappa^2 \right\} d\varkappa = -\mu_1 \int_0^L \psi_{\mathbf{t}}^2 d\varkappa - \mu_1 \int_0^L f(\psi)\, \psi_{\mathbf{t}} d\varkappa - \int_0^L \int_{\iota_1}^{\iota_2} \psi_{\mathbf{t}} \eta_2(\tau) \chi(\varkappa, 1, \tau, \mathbf{t})\, d\tau d\varkappa. \tag{3.3}$$

Multiplying $(2.1)_3$ by $|\eta_2(\tau)|\, \chi(\varkappa, \rho, \tau, \mathbf{t})$, integrating over $(0,L) \times (0,L) \times (\iota_1, \iota_2)$, summing the result with (3.3) and applying Young's inequality, we have (3.1) and (3.2). ■

Lemma 3.2. *The functional*

$$I_1(\mathbf{t}) := -\int_0^L (\rho_1 \phi \phi_{\mathbf{t}} + \rho_2 \psi \psi_{\mathbf{t}}) d\varkappa - \frac{\mu_1}{2}\int_0^L \psi^2 d\varkappa. \tag{3.4}$$

satisfies

$$\frac{dI_1(\mathbf{t})}{d\mathbf{t}} \leq -\int_0^L (\rho_1\phi_{\mathbf{t}}^2 + \rho_2\psi_{\mathbf{t}}^2)d\varkappa + c_0\int_0^L \psi_\varkappa^2 d\varkappa + k\int_0^L (\phi_\varkappa + \psi)^2 d\varkappa$$
$$+ \frac{\mu_1}{4}\int_0^L \int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\,\chi^2(\varkappa, 1, \tau, \mathbf{t})d\tau d\varkappa, \tag{3.5}$$

Proof. Differentiating $I_1(\mathbf{t})$ with $(2.1)_1$, $(2.1)_2$ and Young's and Poincaré inequalities, we obtain (3.5). ■

Now, we introduce the following problem

$$-w_{\varkappa\varkappa} = \psi_\varkappa, \qquad w(0) = w(1) = 0, \tag{3.6}$$

where w the solution of the above problem is given by

$$w(\varkappa, \mathbf{t}) = -\int_0^\varkappa \psi(z, \mathbf{t})\, dz + \varkappa\left(\int_0^L \psi(z, \mathbf{t})\, dz\right).$$

Lemma 3.3. *The solution of* (3.6) *satisfies*

$$\int_0^L w_\varkappa^2 d\varkappa \leq \int_0^L \psi^2 d\varkappa \text{ and } \int_0^L w_{\mathbf{t}}^2 d\varkappa \leq \int_0^L \psi_{\mathbf{t}}^2 d\varkappa.$$

Proof. Multiplying (3.6) by w, integrating and introduce the Hölder inequality, we arrive at

$$\int_0^L w_\varkappa^2 d\varkappa \leq \int_0^L \psi^2 d\varkappa$$

Next, we differentiate (3.6) and using the same above technique, we get

$$\int_0^L w_{\mathbf{t}}^2 d\varkappa \leq \int_0^L \psi_{\mathbf{t}}^2 d\varkappa. \tag{3.7}$$

■

Lemma 3.4. *Let $\Phi = (\phi, \psi, \chi)$ be the solution of the system (2.1)–(2.2). Then, for any $\varepsilon_2 > 0$, the functional*

$$I_2(\mathbf{t}) := \int_0^L \left(\rho_2\psi_{\mathbf{t}}\psi + \rho_1\phi_{\mathbf{t}}w + \frac{\mu_1}{2}\psi^2\right) d\varkappa, \tag{3.8}$$

satisfies

$$\frac{dI_2(\mathbf{t})}{d\mathbf{t}} \leq -\frac{b}{2}\int_0^L \psi_\varkappa^2 d\varkappa + \left(\frac{\rho_1}{4\varepsilon_2} + \rho_2\right)\int_0^L \psi_{\mathbf{t}}^2 d\varkappa + \rho_1\varepsilon_2\int_0^L \phi_{\mathbf{t}}^2 d\varkappa \tag{3.9}$$
$$+ \frac{\mu_1}{4\varepsilon_2}\int_0^L \left(\int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\,\chi^2(\varkappa, 1, \tau, \mathbf{t})d\tau\right) d\varkappa - \int_0^L \tilde{f}(\psi)d\varkappa.$$

Proof. By differentiation $I_2(\mathbf{t})$ and using$(2.1)_1$, $(2.1)_2$, we obtain

$$\frac{dI_2(\mathbf{t})}{d\mathbf{t}} = \rho_2\int_0^L \psi_{\mathbf{t}}^2 d\varkappa - b\int_0^L \psi_\varkappa^2 d\varkappa + \rho_1\int_0^L \phi_{\mathbf{t}}w_{\mathbf{t}} d\varkappa - k\int_0^L \psi^2 d\varkappa + k\int_0^L w_\varkappa^2 d\varkappa$$
$$-\int_0^L f(\psi)\psi d\varkappa - \int_0^L \psi\left(\int_{\iota_1}^{\iota_2} \eta_2(\tau)\,\chi(\varkappa, 1, \tau, \mathbf{t})d\tau\right) d\varkappa. \tag{3.10}$$

Using (3.7), Young's, Cauchy-Schwarz, Poincaré inequalities, we have

$$\rho_1 \int_0^L \phi_{\mathbf{t}} w_{\mathbf{t}} d\varkappa \leq \rho_1 \varepsilon_2 \int_0^L \phi_{\mathbf{t}}^2 d\varkappa + \frac{\rho_1}{4\varepsilon_2} \int_0^L w_{\mathbf{t}}^2 d\varkappa$$
$$\leq \rho_1 \varepsilon_2 \int_0^L \phi_{\mathbf{t}}^2 d\varkappa + \frac{\rho_1}{4\varepsilon_2} \int_0^L \psi_{\mathbf{t}}^2 d\varkappa, \tag{3.11}$$

$$- \int_0^L \psi \left(\int_{\iota_1}^{\iota_2} \left| \eta_2 (\tau) \chi(\varkappa, 1, \tau, \mathbf{t}) \right| d\tau \right) d\varkappa$$
$$\leq \delta_1 \int_0^L \psi_{\varkappa}^2 d\varkappa + \frac{\mu_1}{4\delta_1} \int_0^L \left(\int_{\iota_1}^{\iota_2} \left| \eta_2 (\tau) \right| \chi^2(\varkappa, 1, \tau, \mathbf{t}) d\tau \right) d\varkappa, \tag{3.12}$$

$$\int_0^L \left| f(\psi)\psi \right| d\varkappa \leq \int_0^L |\psi|^\theta \, |\psi| \, |\psi| \, d\varkappa$$
$$\leq \|\psi\|_{2(\theta+1)}^\theta \, \|\psi\|_{2(\theta+1)} \, \|\psi\|$$
$$\leq c_1 \int_0^L \psi_{\varkappa}^2 d\varkappa. \tag{3.13}$$

By substituting (3.11), (3.12), (3.13) in (3.10), recalling (2.5) and letting $\delta_1 = \frac{b}{2}$, we obtain (3.9). ∎

Lemma 3.5. *The functional*

$$I_3(\mathbf{t}) := \rho_2 \int_0^L \psi_{\mathbf{t}} (\phi_{\varkappa} + \psi) + \rho_2 \int_0^L \psi_{\varkappa} \phi_{\mathbf{t}} d\varkappa,$$

satisfies

$$\frac{dI_3(\mathbf{t})}{d\mathbf{t}} \leq b \left[\psi_{\varkappa} \phi_{\varkappa} \right]_0^1 d\varkappa + \left(\rho_2 + \frac{\mu_1^2}{k} \right) \int_0^L \psi_{\mathbf{t}}^2 d\varkappa - \frac{k}{4} \int_0^L (\phi_{\varkappa} + \psi)^2 d\varkappa$$
$$+ c_1 \int_0^L \psi_{\varkappa}^2 d\varkappa + \frac{\mu_1}{k} \int_0^L \int_{\iota_1}^{\iota_2} \left| \eta_2 (\tau) \right| \chi^2(\varkappa, 1, \tau, \mathbf{t}) d\tau d\varkappa - \int_0^L \tilde{f}(\psi) d\varkappa$$
$$+ \left(\frac{\rho_2 k - \rho_1 b}{\rho_1} \right) \int_0^L \psi_{\varkappa} \left(\phi_{\varkappa} + \psi \right)_{\varkappa} d\varkappa, \tag{3.14}$$

where c_1 is a positive constant.

Proof. By differentiation $I_3(\mathbf{t})$ and exploiting $(2.1)_1$, $(2.1)_2$, we have

$$\frac{dI_3(\mathbf{t})}{d\mathbf{t}} = \rho_2 \int_0^L \psi_{\mathbf{tt}} (\phi_{\varkappa} + \psi) d\varkappa + \rho_2 \int_0^L \psi_{\mathbf{t}} (\phi_{\varkappa} + \psi)_{\mathbf{t}} d\varkappa + \rho_2 \int_0^L \psi_{\varkappa \mathbf{t}} \phi_{\mathbf{t}} d\varkappa$$
$$+ \rho_2 \int_0^L \psi_{\varkappa} \phi_{\mathbf{tt}} d\varkappa$$
$$= b \left[\psi_{\varkappa} \phi_{\varkappa} \right]_0^1 + \rho_2 \int_0^L \psi_{\mathbf{t}}^2 d\varkappa - k \int_0^L (\phi_{\varkappa} + \psi)^2 d\varkappa - \mu_1 \int_0^L \psi_{\mathbf{t}} (\phi_{\varkappa} + \psi) d\varkappa$$
$$- \int_0^L \int_{\iota_1}^{\iota_2} \eta_2 (\tau) (\phi_{\varkappa} + \psi) \chi(\varkappa, 1, \tau, \mathbf{t}) d\tau d\varkappa - \int_0^L f(\psi)(\phi_{\varkappa} + \psi) d\varkappa. \tag{3.15}$$

By functional inequalities, we arrive at

$$\mu_1 \int_0^L |\psi_{\mathbf{t}}(\phi_{\varkappa}+\psi)|\, d\varkappa \leq \frac{k}{4}\int_0^L (\phi_{\varkappa}+\psi)^2 d\varkappa + \frac{\mu_1^2}{k}\int_0^L \psi_{\mathbf{t}}^2 d\varkappa, \tag{3.16}$$

$$\begin{aligned}
&\int_0^L (\phi_{\varkappa}+\psi)\int_{\iota_1}^{\iota_2} |\eta_2(\tau)\,\chi(\varkappa,1,\tau,\mathbf{t})|\, d\tau d\varkappa \\
&\leq \frac{k}{4}\int_0^L (\phi_{\varkappa}+\psi)^2 d\varkappa + \frac{\mu_1^2}{k}\int_0^L\int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\,\chi^2(\varkappa,1,\tau,\mathbf{t}) d\tau d\varkappa,
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\int_0^L f(\psi)\phi_{\varkappa} d\varkappa &\leq \frac{\rho_0}{2b^2}\int_0^L \phi_{\varkappa}^2 d\varkappa + \frac{b^2}{2\rho_0\lambda_1}\int_0^L \psi_{\varkappa}^2 d\varkappa \\
&\leq \frac{\rho_0}{2b^2}\int_0^L (\phi_{\varkappa}+\psi)^2 d\varkappa + \frac{\rho_0}{2b^2}\int_0^L \psi^2 d\varkappa + \frac{b^2}{2\rho_0\lambda_1}\int_0^L \psi_{\varkappa}^2 d\varkappa \\
&\leq \frac{\rho_0}{2b^2}\int_0^L (\phi_{\varkappa}+\psi)^2 d\varkappa + \left(\frac{\rho_0}{2\lambda_1 b^2} + \frac{b^2}{2\rho_0\lambda_1}\right)\int_0^L \psi_{\varkappa}^2 d\varkappa.
\end{aligned} \tag{3.18}$$

Inserting (3.16)-(3.18) in (3.15) and letting $\rho_0 = \frac{1}{2}kb^2$, we obtain (3.14). ■

To manipulate the boundary terms appeared in (3.14), we introduce the function

$$q(\varkappa) = -4\varkappa + 2, \quad \varkappa \in (0,1).$$

So, we were able to find the following result.

Lemma 3.6. *For any $\varepsilon_1 > 0$, we have*

$$\begin{aligned}
b\left[\psi_{\varkappa}\phi_{\varkappa}\right]_0^1 \leq& -\frac{b\rho_2}{4\varepsilon_1}\frac{d}{d\mathbf{t}}\int_0^L q\psi_{\mathbf{t}}\psi_{\varkappa} d\varkappa - \frac{\rho_1\varepsilon_1}{k}\frac{d}{d\mathbf{t}}\int_0^L q\phi_{\mathbf{t}}\phi_{\varkappa} d\varkappa + 3\varepsilon_1\int_0^L \phi_{\varkappa}^2 d\varkappa \\
&+ \left(\frac{2\rho_1\varepsilon_1}{k} + \frac{b\rho_2}{2\varepsilon_1}\right)\int_0^L \psi_{\mathbf{t}}^2 d\varkappa + \left(\frac{k^2\varepsilon_1^2}{4} + \frac{\varepsilon_1}{4b^2}\right)\int_0^L (\phi_{\varkappa}+\psi)^2 d\varkappa \\
&+ \frac{b}{4\varepsilon_1}\int_0^L\int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\,\chi^2(\varkappa,1,\tau,\mathbf{t})\, d\varkappa \\
&+ \left(\frac{b^2}{2\varepsilon_1^2} + \frac{1}{4\lambda_1 b^2} + \frac{b^2}{8\varepsilon_1^2\lambda_1} + \frac{\mu_1 b}{4\varepsilon_1} + \frac{b^2}{4\varepsilon_1^3} + \varepsilon_1\right)\int_0^L \psi_{\varkappa}^2 d\varkappa
\end{aligned} \tag{3.19}$$

Proof. Young's inequality gives easily for $\varepsilon_1 > 0$,

$$b\left[\psi_{\varkappa}\phi_{\varkappa}\right]_0^1 \leq \varepsilon_1\left[\phi_{\varkappa}^2(1) + \phi_{\varkappa}^2(0)\right] + \frac{b^2}{4\varepsilon_1}\left[\psi_{\varkappa}^2(1) + \psi_{\varkappa}^2(0)\right], \tag{3.20}$$

we need the following fact

$$\frac{d}{d\mathbf{t}}\int_0^L b\rho_2 q\psi_{\mathbf{t}}\psi_{\varkappa} d\varkappa = b\rho_2\int_0^L q\psi_{\mathbf{tt}}\psi_{\varkappa} d\varkappa + b\rho_2\int_0^L q\psi_{\mathbf{t}}\psi_{\varkappa\mathbf{t}} d\varkappa.$$

On the other hand

$$
\begin{aligned}
b\rho_2 \int_0^L q\psi_{\mathbf{tt}}\psi_{\varkappa} d\varkappa &= b^2 \int_0^L q\psi_{\varkappa\varkappa}\psi_{\varkappa} d\varkappa - kb \int_0^L q(\phi_{\mathbf{t}} + \psi)\psi_{\varkappa} d\varkappa \\
&- b \int_0^L \int_{\iota_1}^{\iota_2} q\psi_{\varkappa}\eta_2(\tau)\chi(\varkappa, 1, \tau, \mathbf{t})\, d\tau d\varkappa - b \int_0^L qf(\psi)\phi_{\varkappa} d\varkappa \\
&\leq -b^2 \left[\psi_{\varkappa}^2(1) + \psi_{\varkappa}^2(0)\right] + 2b^2 \int_0^L \psi_{\varkappa}^2 d\varkappa \\
&+ (k^2\varepsilon_1^2 + \frac{\varepsilon_1}{b^2}) \int_0^L (\phi_{\varkappa} + \psi)^2 d\varkappa \\
&+ (\frac{b^2}{\varepsilon_1^2} + \frac{\varepsilon_1}{2\lambda_1 b^2} + \frac{b^2}{2\varepsilon_1\lambda_1} + \mu_1 b) \int_0^L \psi_{\varkappa}^2 d\varkappa \\
&+ b \int_0^L \int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\, \chi^2(\varkappa, 1, \tau, \mathbf{t})\, d\tau d\varkappa. \qquad (3.21)
\end{aligned}
$$

Therefore

$$
b\rho_2 \int_0^L q\psi_{\mathbf{t}}\psi_{\varkappa\mathbf{t}} d\varkappa = 2\rho_2 b \int_0^L \psi_{\mathbf{t}}^2 d\varkappa
$$

Similarly

$$
\begin{aligned}
\frac{d}{d\mathbf{t}} \int_0^L \rho_1 q\phi_{\mathbf{t}}\phi_{\varkappa} d\varkappa &= \int_0^L q(\phi_{\mathbf{t}} + \psi)\phi_{\varkappa} d\varkappa + \int_0^L \rho_1 q\phi_{\mathbf{t}}\phi_{\varkappa\mathbf{t}} d\varkappa \\
&\leq -k\left[\phi_{\varkappa}^2(1) + \phi_{\varkappa}^2(0)\right] + 3k \int_0^L \phi_{\varkappa}^2 d\varkappa \\
&+ k \int_0^L \psi_{\varkappa}^2 d\varkappa + 2\rho_1 \int_0^L \psi_{\mathbf{t}}^2 d\varkappa
\end{aligned}
$$

which gives us (3.19) by exploiting (3.20)-(3.21). ■

Lemma 3.7. *([13]) For $\eta_1 > 0$, the functional*

$$
I_4(\mathbf{t}) = \int_0^L \int_0^L \int_{\iota_1}^{\iota_2} \tau \mathbf{e}^{-\tau\rho} |\eta_2(\tau)|\, \chi^2(\varkappa, \rho, \tau, \mathbf{t})\, d\tau d\rho d\varkappa, \qquad (3.22)
$$

satisfies

$$
\begin{aligned}
\frac{dI_4(\mathbf{t})}{d\mathbf{t}} &\leq -\eta_1 \int_0^L \int_0^L \int_{\iota_1}^{\iota_2} \tau\, |\eta_2(\tau)|\, \chi^2(\varkappa, \rho, \tau, \mathbf{t}) d\tau d\rho d\varkappa \\
&- \eta_1 \int_0^L \int_{\iota_1}^{\iota_2} |\eta_2(\tau)|\, \chi^2(\varkappa, 1, \tau, \mathbf{t})\, d\tau d\varkappa + \beta \int_0^L \phi_{\mathbf{t}}^2 d\varkappa. \qquad (3.23)
\end{aligned}
$$

where β is a positive constant.

Let $\mathcal{L}(\mathbf{t})$ the Lyapunov functional given by

$$
\mathcal{L}(\mathbf{t}) = N\mathbf{E}(\mathbf{t}) + \frac{1}{8} I_1(\mathbf{t}) + N_1 I_2(\mathbf{t}) + I_3(\mathbf{t}) + N_2 I_4(\mathbf{t}), \qquad (3.24)
$$

where $N_1, N_2, N > 0$.

Lemma 3.8. *There exist $\beta_1, \beta_2 > 0$, such that $\mathcal{L}(\mathbf{t})$ verifies*

$$\beta_1 \mathbf{E}(\mathbf{t}) \leq \mathcal{L}(\mathbf{t}) \leq \beta_2 \mathbf{E}(\mathbf{t}), \qquad \forall \mathbf{t} \geq 0, \tag{3.25}$$

and

$$\mathcal{L}'(\mathbf{t}) \leq -\lambda_1 \mathbf{E}(\mathbf{t}) + \left(\frac{\rho_2 k - \rho_1 b}{\rho_1}\right) \int_0^L \psi_\varkappa (\phi_\varkappa + \psi)_\varkappa \, d\varkappa. \tag{3.26}$$

Proof. Let

$$\mathcal{L}(\mathbf{t}) := N\mathbf{E}(\mathbf{t}) + \frac{1}{8} I_1(\mathbf{t}) + N_1 I_2(\mathbf{t}) + I_3(\mathbf{t}) + N_2 I_4(\mathbf{t}),$$

then

$$\begin{aligned}
|\mathcal{L}(\mathbf{t}) - N\mathbf{E}(\mathbf{t})| \leq{}& \frac{\rho_1}{8} \int_0^L |\phi\phi_\mathbf{t}| \, d\varkappa + \frac{\rho_2}{8} \int_0^L |\psi\psi_\mathbf{t}| \, d\varkappa + \frac{\mu_1}{16} \int_0^L \psi^2 d\varkappa \\
&+ N_1 \rho_2 \int_0^L |\psi_\mathbf{t} \psi| \, d\varkappa + N_1 \rho_1 \int_0^L |\phi_\mathbf{t} w| \, d\varkappa + N_1 \frac{\mu_1}{2} \int_0^L \psi^2 d\varkappa \\
&+ \rho_2 \int_0^L |\psi_\mathbf{t}(\phi_\varkappa + \psi)| \, d\varkappa + \rho_2 \int_0^L |\psi_\varkappa \phi_\mathbf{t}| \, d\varkappa \\
&+ N_2 \int_0^L \int_0^L \int_{\iota_1}^{\iota_2} \tau \mathbf{e}^{-\tau\rho} |\eta_2(\tau)| \chi^2(\varkappa, \rho, \tau, \mathbf{t}) \, d\tau d\rho d\varkappa.
\end{aligned}$$

Exploiting some functional inequalities, we arrive at

$$\begin{aligned}
|\mathcal{L}(\mathbf{t}) - N\mathbf{E}(\mathbf{t})| &\leq C \int_0^L \left(\psi_\varkappa^2 + \psi_\mathbf{t}^2 + \phi_\mathbf{t}^2 + (\phi_\varkappa + \psi)^2\right) d\varkappa \\
&\quad + \int_0^L \int_{\iota_1}^{\iota_2} \tau |\eta_2(\tau)| \chi^2(\varkappa, 1, \tau, \mathbf{t}) \, d\tau d\varkappa + \int_0^L \tilde{f}(\psi) d\varkappa \\
&\leq C\mathbf{E}(\mathbf{t}),
\end{aligned}$$

By (3.2), (3.5), (3.9), (3.14), (3.23) and (3.19), we get

$$\begin{aligned}
\frac{d\mathcal{L}(\mathbf{t})}{d\mathbf{t}} ={}& -\left(Nm_1 - N_1\left(\frac{\rho_1}{4\varepsilon_2} + \rho_2\right) - N_2\mu_1 - \left(\rho_2 + \frac{\mu_1^2}{k}\right) - (\frac{2\rho_1\varepsilon_1}{k} + \frac{b\rho_2}{2\varepsilon_1}) + \rho_2\right) \int_0^L \psi_\mathbf{t}^2 d\varkappa \\
&- \left(\frac{b}{2} N_1 - \frac{c_0}{8} - (\frac{b^2}{2\varepsilon_1^2} + \frac{1}{4b^2} + \frac{b^2}{8\varepsilon_1^2} + \frac{\mu_1 b}{4\varepsilon_1} + \frac{b^2}{4\varepsilon_1^3} + \varepsilon_1)\right) \int_0^L \psi_\varkappa^2 d\varkappa \\
&- \left(\frac{k}{8} - \varepsilon_1\left(k^2\varepsilon_1 + \frac{1}{b^2}\right)\right) \int_0^L (\phi_\varkappa + \psi)^2 d\varkappa \\
&- N_2\beta \int_0^L \int_0^L \int_{\iota_1}^{\iota_2} \tau |\eta_2(\tau)| \chi^2(\varkappa, \rho, \tau, \mathbf{t}) \, d\tau d\rho d\varkappa \\
&- \left(N_2\beta - N_1 \frac{\mu_1}{4\varepsilon_2} - \frac{\mu_1}{32} - \frac{\mu_1}{k} - \frac{b}{4\varepsilon_1}\right) \int_0^L \int_{\iota_1}^{\iota_2} |\eta_2(\tau)| \chi^2(\varkappa, 1, \tau, \mathbf{t}) d\tau d\varkappa \\
&- \left(\frac{\rho_1}{8} - \rho_1 \varepsilon_2 N_1\right) \int_0^L \phi_\mathbf{t}^2 d\varkappa \\
&- (N_1 + 1) \int_0^L \tilde{f}(\psi) d\varkappa, \\
&+ \left(\frac{\rho_2 k - \rho_1 b}{\rho_1}\right) \int_0^L \psi_\varkappa (\phi_\varkappa + \psi)_\varkappa \, d\varkappa.
\end{aligned}$$

By setting $\varepsilon_2 = \dfrac{\rho_1}{16N_1}$, we getFirst, we choose ε_1 small to hold

$$\frac{k}{8} - \varepsilon_1 \left(k^2\varepsilon_1 + \frac{1}{b^2} \right) > 0.$$

Choosing N_1 large to verify

$$\frac{b}{2}N_1 - \frac{c_0}{8} - \left(\frac{b^2}{2\varepsilon_1^2} + \frac{1}{4b^2} + \frac{b^2}{8\varepsilon_1^2} + \frac{\mu_1 b}{4\varepsilon_1} + \frac{b^2}{4\varepsilon_1^3} + \varepsilon_1\right) > 0.$$

Then, we select N_2 large to satisfies

$$N_2\beta - N_1\frac{\mu_1}{4\varepsilon_2} - \frac{\mu_1}{32} - \frac{\mu_1}{k} - \frac{b}{4\varepsilon_1} > 0.$$

Choosing N large such that

$$Nm_1 - N_1\left(\frac{\rho_1}{4\varepsilon_2} + \rho_2\right) - N_2\mu_1 - \left(\rho_2 + \frac{\mu_1^2}{k}\right) - \left(\frac{2\rho_1\varepsilon_1}{k} + \frac{b\rho_2}{2\varepsilon_1}\right) + \rho_2 > 0,$$

and so that (3.25) remains valid. We obtain (3.26). ■

Here is the following polynomial stability result.

Lemma 3.9. *Let $\Phi = (\phi, \psi, \chi)$ be the solution of the system (2.1)–(2.2) and suppose that $\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$ holds. Therefore, the solution (ϕ, ψ, χ) decays in polynomial manner, i.e. there exists $a > 0$ such that*

$$\mathbf{E}(\mathbf{t}) \leq \frac{a}{\mathbf{t}}, \ \mathbf{t} > 0.$$

Proof. Using (3.26) and $(1)_1$, we obtain

$$\mathcal{L}'(\mathbf{t}) \leq -\lambda_1\mathbf{E}(\mathbf{t}) + \varsigma\int_0^L \psi_\varkappa\phi_{\mathbf{tt}}d\varkappa, \tag{3.27}$$

where $\varsigma = \dfrac{k}{\rho_1}\left(\dfrac{\rho_2 k - \rho_1 b}{\rho_1}\right)$, and by applying Young's inequality, we get

$$\varsigma\int_0^L \psi_\varkappa\phi_{\mathbf{tt}}d\varkappa \leq -\varsigma\frac{d}{d\mathbf{t}}\int_0^L (\phi_{\varkappa\mathbf{t}}\psi - \phi_\varkappa\psi_\mathbf{t})\,d\varkappa + |\varsigma|\,\rho\int_0^L \phi_\varkappa^2 d\varkappa + \frac{|\varsigma|}{4\rho}\int_0^L \psi_{\mathbf{tt}}^2 d\varkappa, \ \rho > 0. \tag{3.28}$$

Because

$$\int_0^L \phi_\varkappa^2 d\varkappa \leq 2\int_0^L (\phi_\varkappa + \psi)^2\,d\varkappa + 2\int_0^L \psi^2 d\varkappa, \tag{3.29}$$

we arrive at

$$\begin{aligned}\varsigma\int_0^L \psi_\varkappa\phi_{\mathbf{tt}}d\varkappa \leq& -\varsigma\frac{d}{d\mathbf{t}}\int_0^L (\phi_{\varkappa\mathbf{t}}\psi - \phi_\varkappa\psi_\mathbf{t})\,d\varkappa + C_0\,|\varsigma|\,\rho\mathbf{E}(\mathbf{t}) \\ &+ \frac{|\varsigma|}{4\rho}\int_0^L \psi_{\mathbf{tt}}^2 d\varkappa.\end{aligned} \tag{3.30}$$

By substituting (3.30) in (3.27) and letting $\rho = \dfrac{\lambda_1}{2C_0\,|\varsigma|}$, the inequality becomes as

$$\mathcal{L}'(\mathbf{t}) + \varsigma\frac{d}{d\mathbf{t}}\int_0^L (\phi_{\varkappa\mathbf{t}}\psi - \phi_\varkappa\psi_\mathbf{t})\,d\varkappa \leq -\iota_1\mathbf{E}(\mathbf{t}) + \iota_2\int_0^L \psi_{\mathbf{tt}}^2 d\varkappa, \ \iota_1, \iota_2 > 0. \tag{3.31}$$

Now, let the functional

$$\mathcal{L}_1(\mathbf{t}) = \mathcal{L}(\mathbf{t}) + \varsigma \int_0^L (\phi_{\varkappa\mathbf{t}}\psi - \phi_\varkappa\psi_\mathbf{t})\, d\varkappa + N_3(\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t})),$$

where $\mathbf{E}_2(\mathbf{t})$ is the second order energy of the problem (2.1)–(2.2) and we choice

$$N_3 > \max\left\{\frac{2|\varsigma|}{K}, \frac{|\varsigma|}{\rho_2}, \frac{3|\varsigma|}{b}, \frac{\iota_2}{m_1}\right\}, \tag{3.32}$$

for that $\mathcal{L}_1(\mathbf{t})$ be equivalent to $\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t})$. Indeed,

$$|\mathcal{L}_1(\mathbf{t}) - N_3(\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t}))| \le \beta_2\mathbf{E}(\mathbf{t}) + |\varsigma| \int_0^L |\phi_{\varkappa\mathbf{t}}\psi|\, d\varkappa + |\varsigma| \int_0^L |\phi_\varkappa\psi_\mathbf{t}|\, d\varkappa,$$

and by using some functional inequalities, we have

$$|\mathcal{L}_1(\mathbf{t}) - N_3(\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t}))| \le \beta_2\mathbf{E}(\mathbf{t}) + \frac{|\varsigma|}{2}\int_0^L \phi_{\varkappa\mathbf{t}}^2 d\varkappa + \frac{|\varsigma|}{2}\int_0^L \psi^2 d\varkappa$$
$$+ \frac{|\varsigma|}{2}\int_0^L \phi_\varkappa^2 d\varkappa + \frac{|\varsigma|}{2}\int_0^L \psi_\mathbf{t}^2 d\varkappa.$$

It's easily to show that

$$|\mathcal{L}_1(\mathbf{t}) - N_3(\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t}))| \le \max\left\{\frac{2|\varsigma|}{K}, \frac{|\varsigma|}{\rho_2}, \frac{3|\varsigma|}{b}\right\}(\mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t})),$$

and by recalling (3.32), we deduce that

$$\mathcal{L}_1(\mathbf{t}) \sim \mathbf{E}(\mathbf{t}) + \mathbf{E}_2(\mathbf{t}).$$

By using (3.31) and because $\mathbf{E}'(\mathbf{t}) \le 0$, we can conclude that

$$\mathcal{L}_1'(\mathbf{t}) \le -\iota_1\mathbf{E}(\mathbf{t}) - (N_3 m_1 - \iota_2)\int_0^L \psi_{\mathbf{tt}}^2 d\varkappa,$$

the choice of N_3 given in (3.31) leads to

$$\mathcal{L}_1'(\mathbf{t}) \le -\iota_1\mathbf{E}(\mathbf{t}). \tag{3.33}$$

Integrating the inequation (3.33), we obtain

$$\iota_1 \int_0^\mathbf{t} \mathbf{E}(\mathbf{t})\, d\mathbf{t} \le -\int_0^\mathbf{t} \mathcal{L}_1(\mathbf{t})\, d\mathbf{t},$$

and because $\mathbf{E}(\mathbf{t})$ is decreasing, we have

$$\iota_1\mathbf{E}(\mathbf{t})\,\mathbf{t} \le \mathcal{L}_1(0),$$

which gives (3.33) by taking $a = \dfrac{\mathcal{L}_1(0)}{\iota_1}$. Which complete the proof. ■

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Conflict of interest

There is no conflict of interest.

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