

Plancherel formula for the Shehu transform

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Abstract. We discuss some existence conditions of the Shehu transform, provide a Plancherel formula and also relate the Shehu equicontinuity to exponential L^2 -equivanishing.

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1. Introduction

Integral transforms have many applications in various fields of mathematical sciences and engineering such as physics, mechanics, chemistry, acoustic, etc. For example, integral transforms such as the Fourier transform and the Laplace transform are highly efficient in signal processing and solving differential equations.

In [9] the authors introduced a Laplace-type integral which they called the *Shehu transform*. The Shehu transform of a function $f : \mathfrak{R}_+ \rightarrow \mathbb{C}$ is defined by

$$\mathbb{S}\{f\}(s, u) = \int_0^\infty e^{-\frac{s}{u}t} f(t) dt, \quad s \geq 0, u > 0. \quad (1.1)$$

provided that this integral exists, the symbol \mathfrak{R}_+ stands for the set of nonnegative real numbers. This integral transform generalizes both the Laplace transform [5] and the Yang transform [10]. Many authors used the Shehu transform to solve partial or ordinary differential equations related to real life problems [4], [1], [2], [3], [7]. Authors in [6] extended the Shehu transform to distributions and measures.

This paper is mainly devoted to search a Plancherel formula for the Shehu transform. The remainder of the paper is structured as follows. In Section 2 we discuss some existence conditions after replacing the first variable of the Shehu transform of a function with a complex variable and in Section 3, a Plancherel formula is given and Shehu equicontinuity and exponential L^2 -equivanishing are related.

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2. Extension to complex variables and existence conditions

In order to obtain the Plancherel theorem in the next section for the Shehu transform, we want the first variable of $\mathbb{S}\{f\}(s, u)$ to be a complex variable. This is why we consider the Shehu transform of the function $f : \mathfrak{R}_+ \rightarrow \mathbb{C}$ in the form

$$\mathbb{S}\{f\}(z, u) = \int_0^\infty e^{-\frac{z}{u}t} f(t) dt, \quad z \in \mathbb{C}, u > 0. \quad (2.1)$$

In what follows, we discuss some existence conditions. The symbol $\mathcal{R}_e(z)$ denotes the real part of the complex number z . The complex vector spaces $L^1(\mathfrak{R}_+)$ and $L^2(\mathfrak{R}_+)$ are

$$L^1(\mathfrak{R}_+) = \left\{ f : \mathfrak{R}_+ \rightarrow \mathbb{C} : \int_0^\infty |f(t)| dt < \infty \right\} \quad (2.2)$$

and

$$L^2(\mathfrak{R}_+) = \left\{ f : \mathfrak{R}_+ \rightarrow \mathbb{C} : \int_0^\infty |f(t)|^2 dt < \infty \right\}. \quad (2.3)$$

Theorem 2.1. Consider the function $f : \mathfrak{R}_+ \rightarrow \mathbb{C}$. If $f \in L^1(\mathfrak{R}_+)$ and $\mathcal{R}_e(z) \geq 0$ then $\mathbb{S}\{f\}(z, u)$ exists.

Proof. Assume $f \in L^1(\mathfrak{R}_+)$. Set $z = x + iy$ with $\mathcal{R}_e(z) = x \geq 0$. Then

$$\begin{aligned} |e^{-\frac{z}{u}t} f(t)| &= |e^{-\frac{x}{u}t} e^{-i\frac{y}{u}t} f(t)| \\ &= e^{-\frac{x}{u}t} |f(t)| \\ &\leq |f(t)| \text{ because } e^{-\frac{x}{u}t} \leq 1. \end{aligned}$$

Finally $\int_0^\infty |f(t)| dt < \infty \Rightarrow \int_0^\infty |e^{-\frac{z}{u}t} f(t)| dt < \infty$. Thus $\mathbb{S}\{f\}(z, u)$ exists. ■

Theorem 2.2. Consider the function $f : \mathfrak{R}_+ \rightarrow \mathbb{C}$. If $f \in L^2(\mathfrak{R}_+)$ and $\mathcal{R}_e(z) > 0$ then $\mathbb{S}\{f\}(z, u)$ exists.

Proof. Assume $f \in L^2(\mathfrak{R}_+)$. Set $z = x + iy$ with $\mathcal{R}_e(z) = x > 0$. Then

$$\begin{aligned} |e^{-\frac{z}{u}t} f(t)| &= |e^{-\frac{x}{u}t} e^{-i\frac{y}{u}t} f(t)| \\ &= e^{-\frac{x}{u}t} |f(t)|. \end{aligned}$$

Now, $\int_0^\infty e^{-\frac{2x}{u}t} dt = \left[-\frac{u}{2x} e^{-\frac{2x}{u}t} \right]_0^\infty = \frac{u}{2x} < \infty$. Thus the function $t \mapsto e^{-\frac{x}{u}t}$ is in $L^2(\mathfrak{R}_+)$. Since f is assumed to be in $L^2(\mathfrak{R}_+)$ we see that the product $t \mapsto e^{-\frac{x}{u}t} f(t)$ is integrable (use the Hölder inequality). We have $\int_0^\infty e^{-\frac{x}{u}t} |f(t)| dt < \infty$. Therefore $\int_0^\infty |e^{-\frac{z}{u}t} f(t)| dt < \infty$. Thus $\mathbb{S}\{f\}(z, u)$ exists. ■

Let $\alpha \geq 0$. Consider the function $f : \mathfrak{R}_+ \rightarrow \mathbb{C}$ and set

$$f_\alpha(t) = f(t) e^{-\alpha t}, \quad t \in \mathfrak{R}_+. \quad (2.4)$$

Theorem 2.3. If $f_\alpha \in L^1(\mathfrak{R}_+)$ and $\mathcal{R}_e(z) \geq \alpha$ then $\mathbb{S}\{f\}(z, u)$ exists.

Proof. Assume $f_\alpha \in L^1(\mathfrak{R}_+)$. Set $z = x + iy \in \mathbb{C}$ with $\mathcal{R}_e(z) = x \geq \alpha$. We have

$$|e^{-\frac{z}{u}t} f(t)| = |e^{-\frac{x}{u}t} e^{-i\frac{y}{u}t} f(t)| = e^{-\frac{x}{u}t} |f(t)|.$$

$$\begin{aligned} \text{Now, } x \geq \alpha &\Rightarrow e^{-\frac{x}{u}t} |f(t)| \leq e^{-\frac{\alpha}{u}t} |f(t)| \\ &\Rightarrow \int_0^\infty e^{-\frac{x}{u}t} |f(t)| dt \leq \int_0^\infty e^{-\frac{\alpha}{u}t} |f(t)| dt < \infty. \end{aligned}$$

Therefore $\int_0^\infty |e^{-\frac{z}{u}t} f(t)| dt < \infty$. Thus $\mathbb{S}\{f\}(z, u)$ exists. ■

3. Plancherel formula and applications

In harmonic analysis, the Plancherel theorem holds for the Fourier transform. it states that the L^2 -norms of a function in the time domain and that of its Fourier transform in the Fourier domain are equal. In other words it expresses conservation of energy for signals in the time domain and the Fourier domain. We would like to obtain the analogue of this result for the Shehu transform. To achieve our goal we apply the Plancherel formula for the Laplace transform proved in [8]. We start with the following definition.

Definition 3.1. [8] *The function f is called a Laplace-Pego function of order α if $f_\alpha \in L^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$.*

Theorem 3.2. [8] *If f is a Laplace-Pego function of order $x \geq 0$, then*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{L}\{f\}(x + iy)|^2 dy = \int_0^{\infty} e^{-2xt} |f(t)|^2 dt. \quad (3.1)$$

Hereafter is the analogue of the Plancherel formula for the Shehu transform.

Theorem 3.3. *If f is a Laplace-Pego function of order $\frac{x}{u}$ then*

$$\frac{1}{2\pi u} \int_{-\infty}^{\infty} |\mathbb{S}\{f\}(x + iy, u)|^2 dy = \int_0^{\infty} e^{-\frac{2x}{u}t} |f(t)|^2 dt. \quad (3.2)$$

Proof. Assume that f is a Laplace-Pego function of order $\frac{x}{u}$. From Theorem 3.2, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{L}\{f\}(x + iy)|^2 dy = \int_0^{\infty} e^{-2xt} |f(t)|^2 dt.$$

Replacing x and y by $\frac{x}{u}$ and $\frac{y}{u}$ respectively we obtain

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{L}\{f\}(\frac{x}{u} + i\frac{y}{u})|^2 d\frac{y}{u} = \int_0^{\infty} e^{-2\frac{x}{u}t} |f(t)|^2 dt; \\ & \Rightarrow \frac{1}{2\pi u} \int_{-\infty}^{\infty} |\mathcal{L}(f)(\frac{x + iy}{u})|^2 dy = \int_0^{\infty} e^{-2\frac{x}{u}t} |f(t)|^2 dt \\ & \Rightarrow \frac{1}{2\pi u} \int_{-\infty}^{\infty} |\mathbb{S}(f)(x + iy, u)|^2 dy = \int_0^{\infty} e^{-\frac{2x}{u}t} |f(t)|^2 dt. \end{aligned}$$

■

Definition 3.4. [8] *A family \mathcal{A} of Laplace-Pego functions of common order x is said to be exponentially L^2 -equivanishing at x if for any given strictly positive number ε , we can find a strictly positive number T that depends only on ε and the order x , such that for any function f in \mathcal{A} , we have*

$$\int_T^{\infty} e^{-2xt} |f(t)|^2 dt < \varepsilon. \quad (3.3)$$

The author in [8] related the concept of exponential L^2 -equivanishing to the notion of Laplace equicontinuity. We obtain the analogue result for Shehu equicontinuity as an application of the Plancherel formula.

Definition 3.5. *A family \mathcal{A} of functions is said to be Shehu equicontinuous at x if for any given strictly positive number ε , we can find a strictly positive number η that depends only on ε , such that for any strictly positive number u and any function f in \mathcal{A} , we have*

$$\frac{1}{2\pi u} \int_{-\infty}^{\infty} |\mathbb{S}\{f\}(x + iy + \eta, u) - \mathbb{S}\{f\}(x + iy, u)|^2 dy < \varepsilon. \quad (3.4)$$

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If \mathcal{A} is a family of Laplace-Pego functions of common order α , we set

$$\mathcal{A}_\alpha = \{f_\alpha : f \in \mathcal{A}\}.$$

We recall that $f_\alpha(t) = f(t)e^{-\alpha t}$, $t \geq 0$.

Theorem 3.6. *Let \mathcal{A} be a family of Laplace-Pego functions with common order $\frac{x}{u} \geq 0$. If \mathcal{A} is Shehu equicontinuous at x then it is exponentially L^2 -equivanishing at $\frac{x}{u}$. Moreover, if $\mathcal{A}_{\frac{x}{u}}$ is L^2 -bounded, then the inverse is also true.*

Proof. We follow the great lines of the proof of [8, Theorem 6]. Let $\varepsilon > 0$. Let η be such that (3.4) holds. There exists $T > 0$ such that $|e^{-\frac{\eta}{u}T} - 1| \geq \frac{1}{2}$. Then

$$\begin{aligned} & \frac{1}{2\pi u} \int_{-\infty}^{\infty} |\mathbb{S}\{f\}(x + iy + \eta, u) - \mathbb{S}\{f\}(x + iy, u)|^2 dy < \varepsilon \\ \Rightarrow & \frac{1}{2\pi u} \int_{-\infty}^{\infty} \left| \int_0^{\infty} \left[e^{-\frac{x+iy}{u}t} e^{-\frac{\eta}{u}t} f(t) - e^{-\frac{x+iy}{u}t} f(t) \right] dt \right|^2 dy < \varepsilon \\ & \Rightarrow \frac{1}{2\pi u} \int_{-\infty}^{\infty} \left| \int_0^{\infty} e^{-\frac{x+iy}{u}t} (e^{-\frac{\eta}{u}t} - 1) f(t) dt \right|^2 dy < \varepsilon \end{aligned}$$

Now using Theorem 3.3, we obtain

$$\begin{aligned} & \int_0^{\infty} e^{-\frac{2x}{u}t} |f(t)(e^{-\frac{\eta}{u}t} - 1)|^2 dt < \varepsilon, \\ \Rightarrow & \int_T^{\infty} e^{-\frac{2x}{u}t} |f(t)(e^{-\frac{\eta}{u}t} - 1)|^2 dt < \varepsilon, \\ & \Rightarrow \frac{1}{2} \int_T^{\infty} e^{-\frac{2x}{u}t} |f(t)|^2 dt < \varepsilon, \\ & \Rightarrow \int_T^{\infty} e^{-\frac{2x}{u}t} |f(t)|^2 dt < 2\varepsilon. \end{aligned}$$

Thus \mathcal{A} is exponentially L^2 -equivanishing at $\frac{x}{u}$.

Now assume that $\mathcal{A}_{\frac{x}{u}}$ is L^2 -bounded and that \mathcal{A} is exponentially L^2 -equivanishing at $\frac{x}{u}$. Let $\varepsilon > 0$ and choose T such that (3.3) holds for $\frac{x}{u}$ instead of x . As $\mathcal{A}_{\frac{x}{u}}$ is L^2 -bounded, there exists a constant $M > 0$ such that for any $f \in \mathcal{A}$ we have $\int_0^{\infty} e^{-\frac{2x}{u}t} |f(t)|^2 dt < M$.

Let $\eta > 0$ be such that $|e^{-\frac{\eta}{u}T} - 1|^2 M < \varepsilon$. Set

$$B = \frac{1}{2\pi u} \int_{-\infty}^{\infty} |\mathbb{S}\{f\}(x + iy + \eta, u) - \mathbb{S}\{f\}(x + iy, u)|^2 dy.$$

We have

$$B = \int_{-\infty}^{\infty} |\mathbb{S}\{g\}(x + iy, u)|^2 dy$$

where $g(t) = (e^{-\frac{\eta}{u}t} - 1)f(t)$. Now, using Theorem 3.3 we have

$$B = \int_0^T e^{-\frac{2x}{u}t} |f(t)|^2 \left| e^{-\frac{\eta}{u}t} - 1 \right|^2 dt + \int_T^\infty e^{-\frac{2x}{u}t} |f(t)|^2 \left| e^{-\frac{\eta}{u}t} - 1 \right|^2 dt.$$

For $t \leq T$, $|e^{-\frac{\eta}{u}t} - 1|^2 \leq |e^{-\frac{\eta}{u}T} - 1|^2$. Therefore,

$$\int_0^T e^{-\frac{2x}{u}t} |f(t)|^2 \left| e^{-\frac{\eta}{u}t} - 1 \right|^2 dt < \frac{\varepsilon}{M} \int_0^T e^{-\frac{2x}{u}t} |f(t)|^2 dt < \frac{\varepsilon}{M} \int_0^\infty e^{-\frac{2x}{u}t} |f(t)|^2 dt < \varepsilon.$$

On the other hand, note that $|e^{-\frac{\eta}{u}t} - 1|^2 < 1$. It follows that

$$\int_T^\infty e^{-\frac{2x}{u}t} |f(t)|^2 \left| e^{-\frac{\eta}{u}t} - 1 \right|^2 dt < \int_T^\infty e^{-\frac{2x}{u}t} |f(t)|^2 dt < \varepsilon.$$

Therefore, $B < 2\varepsilon$, and hence \mathcal{A} is Shehu equicontinuous at x . ■

4. Conclusion

In this paper, some existence conditions of the Shehu integral transform of a function have been discussed, a Plancherel formula provided and Shehu equicontinuity and exponential L^2 -equivanishing are related.

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