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Second order Volterra-Fredholm functional integrodifferential equations

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Abstract

This paper deals with the study of global existence of solutions to initial value problem for second order nonlinear mixed Volterra-Fredholm functional integrodifferential equations in Banach spaces. The technique used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative and rely on a priori bounds of solution.

Keywords: Global solution; Volterra-Fredholm functional integrodifferential equation; Leray-Schauder alternative; Fixed point; priori bounds.

2010 MSC: 5G10, 47H10, 35R10.

1 Introduction

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = C([-r, 0], X)$, $0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r, 0] \to X$ endowed with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(\theta)\| : -r \le \theta \le 0\}.$$

Let $B = C([-r, T], X)$, $T > 0$, be the Banach space of all continuous functions $x : [-r, T] \to X$ with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : -r \le t \le T\}$. For any $x \in B$ and $t \in [0, T]$ we denote x_t the element of C given by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

In this paper we prove the global existence for second order abstract nonlinear mixed Volterra-Fredholm functional integrodifferential equation of the form

$$(\varrho(t)x'(t))' = f\left(t, x_t, \int_0^t a(t,s)g(s,x_s)ds, \int_0^T b(t,s)h(s,x_s)ds\right),\ t \in [0,T], \tag{1.1}$$

$$x(t) = \phi(t),\ -r \le t \le 0,\ \ x'(0) = \delta, \tag{1.2}$$

where $f : [0,T] \times C \times X \times X \to X$, $a,\ b : [0,T] \times [0,T] \to \mathbb{R}$, $g, h : [0,T] \times C \to X$ are continuous functions, $\varrho(t)$ is real valued positive sufficiently smooth function on $[0,T]$, $\phi \in C$ and $\delta \in X$ are given.

Equation of the form (1.1)-(1.2) and their special forms serve as an abstract formulation of many partial differential equations or partial integrodifferential equations which arising in heat flow in materials with memory, viscoelasticity and many other physical phenomena see [6, 8, 12] and the references given therein.

The problem of existence, uniqueness and other properties of solutions of the special forms of (1.1)-(1.2) have been studied by many authors by using different techniques, see [1–4, 7, 9–11, 14–17] and some of the references given therein. In an interesting paper [13], Ntouyas have investigated the global existence for Volterra functional integro-differential equations in $\mathbb{R}^n$ by using classical application of Leray-Schauder alternative. The

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present paper generalizes the result of [13]. The aim of this paper is to study the existence of global solutions of (1.1)-(1.2). The main tool used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence.

2 Preliminaries and Main Results

Firstly, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Definition 2.1. *A function* $x : [-r, T] \to X$ *is called solution of initial value problem (1.1)-(1.2) if* $x \in C([-r, T], X) \bigcap C^2([0, T], X)$ *and satisfies (1.1)-(1.2) on* $[-r, T]$.

Our results are based on the following lemma, which is a version of the topological transversality theorem given by Granas [5, p. 61].

Lemma 2.1. *Let* S *be a convex subset of a normed linear space* E *and assume* $0 \in S$. *Let* $F : S \to S$ *be a completely continuous operator, and let*

$$\varepsilon(F) = \{x \in S : x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\varepsilon(F)$ *is unbounded or* F *has a fixed point.*

We list the following hypotheses for our convenience.

($\mathbf{H_1}$) There exists a continuous function $p : [0, T] \to \mathbb{R}_+ = [0, \infty)$ such that

$$\|f(t, \psi, x, y)\| \leq p(t)(\|\psi\|_C + \|x\| + \|y\|),$$

for every $t \in [0, T], \psi \in C$ and $x, y \in X$.

($\mathbf{H_2}$) There exists a continuous function $m : [0, T] \to \mathbb{R}_+$ such that

$$\|g(t, \psi)\| \leq m(t)G(\|\psi\|_C),$$

for every $t \in [0, T]$ and $\psi \in C$, where $G : \mathbb{R}_+ \to (0, \infty)$ is continuous nondecreasing function.

($\mathbf{H_3}$) There exists a continuous function $n : [0, T] \to \mathbb{R}_+$ such that

$$\|h(t, \psi)\| \leq n(t)H(\|\psi\|_C),$$

for every $t \in [0, T]$ and $\psi \in C$, where $H : \mathbb{R}_+ \to (0, \infty)$ is continuous nondecreasing function.

($\mathbf{H_4}$) There exists a constants K and L such that

$$|a(t, s)| \leq K, \text{ for } t \geq s \geq 0, \quad \text{and} \quad |b(t, s)| \leq L, \text{ for } s, t \in [0, T].$$

($\mathbf{H_5}$) For each $t \in [0, T]$ the function $f(t, ., ., .) : C \times X \times X \to X$ is continuous and for each $(\psi,\ x,\ y) \in C \times X \times X$ the function $f(., \psi,\ x,\ y) : [0, T] \to X$ is strongly measurable.

($\mathbf{H_6}$) For each $t \in [0, T]$ the functions $g(t, .),\quad h(t, .) : C \to X$ are continuous and for each $\psi \in C$ the functions $g(., \psi),\quad h(., \psi) : [0, T] \to X$ are strongly measurable.

With these preparations we state and prove our main results.

Theorem 2.1. *Suppose that the hypothesis* ($\mathbf{H_1}$)-($\mathbf{H_6}$) *holds. Then the initial-value problem (1.1)-(1.2) has a solution* x *on* $[-r, T]$ *if* T *satisfies*

$$\int_0^T M(s)ds < \int_\alpha^\infty \frac{ds}{s + G(s)}, \tag{2.1}$$

where

$$M(t) = max\left\{\frac{1}{R}\int_0^t p(s)ds, Km(t), Ln(t)\right\},\ t \in [0, T],$$

$$\alpha = \beta + \|\phi\|_C + \|\delta\|\varrho(0)\int_0^T \frac{ds}{\varrho(s)},$$

$R = min\{\varrho(t) : t \in [0,T]\}$ *and* β *is constant such that* $\int_0^T M(s)H(J(s))ds \leq \beta$ *for any continuous function* $J : [0,T] \to \mathbb{R}_+$.

Proof. First we establish the priori bounds on the solutions of the initial value problem

$$(\varrho(t)x'(t))' = \lambda f\left(t, x_t, \int_0^t a(t,s)g(s,x_s)ds, \int_0^T b(t,s)h(s,x_s)ds\right), \quad t \in [0,T], \tag{2.2}$$

with the initial condition (1.2) for $\lambda \in (0,1)$. Let $x(t)$ be a solution of the problem (2.2)-(1.2) then it satisfies the equivalent integral equation

$$\begin{aligned} x(t) &= \phi(0) + \delta\varrho(0)\int_0^t \frac{ds}{\varrho(s)} \\ &\quad + \lambda\int_0^t \frac{1}{\varrho(s)}\int_0^s f\left(\tau, x_\tau, \int_0^\tau a(\tau,\eta)g(\eta,x_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta,x_\eta)d\eta\right)d\tau ds, t \in [0,T] \quad (2.3) \\ x(t) &= \phi(t), \ -r \leq t \leq 0, \ x'(0) = \delta. \quad (2.4) \end{aligned}$$

Using (2.3), hypotheses $(\mathbf{H_1}) - (\mathbf{H_4})$ and the fact that $\lambda \in (0,1)$, for $t \in [0,T]$ we have

$$\begin{aligned} \|x(t)\| &\leq \|\phi\|_C + \|\delta\|\varrho(0)\int_0^t \frac{ds}{\varrho(s)} \\ &\quad + |\lambda|\int_0^t \frac{1}{\varrho(s)}\int_0^s \left\|f\left(\tau, x_\tau, \int_0^\tau a(\tau,\eta)g(\eta,x_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta,x_\eta)d\eta\right)\right\| d\tau ds \\ &\leq \|\phi\|_C + \|\delta\|\varrho(0)\int_0^t \frac{ds}{\varrho(s)} \\ &\quad + \int_0^t \frac{1}{\varrho(s)}\int_0^s p(\tau)\left[\|x_\tau\|_C + \int_0^\tau Km(\eta)G(\|x_\eta\|_C)d\eta + \int_0^T Ln(\eta)H(\|x_\eta\|_C)d\eta\right]d\tau ds. \quad (2.5) \end{aligned}$$

Consider the function Z given by $Z(t) = \sup\{\|x(s)\| : -r \leq s \leq t\}, \ t \in [0,T]$. Let $t^* \in [-r,t]$ be such that $Z(t) = \|x(t^*)\|$. If $t^* \in [0,t]$ then from (2.5), we have

$$\begin{aligned} Z(t) &\leq \|\phi\|_C + \|\delta\|\varrho(0)\int_0^{t^*} \frac{ds}{\varrho(s)} \\ &\quad + \int_0^{t^*} \frac{1}{\varrho(s)}\int_0^s p(\tau)\left[\|x_\tau\|_C + \int_0^\tau Km(\eta)G(\|x_\eta\|_C)d\eta + \int_0^T Ln(\eta)H(\|x_\eta\|_C)d\eta\right]d\tau ds \\ &\leq \|\phi\|_C + \|\delta\|\varrho(0)\int_0^T \frac{ds}{\varrho(s)} \\ &\quad + \int_0^t \frac{1}{\varrho(s)}\int_0^s p(\tau)\left[Z(\tau) + \int_0^\tau M(\eta)G(Z(\eta))d\eta + \int_0^T M(\eta)H(Z(\eta))d\eta\right]d\tau ds. \quad (2.6) \end{aligned}$$

If $t^* \in [-r,0]$ then $Z(t) = \|\phi\|_C$ and the previous inequality (2.6) obviously holds. Denoting by $u(t)$ the right-hand side of the inequality (2.6), we have

$$u(0) = \|\phi\|_C + \|\delta\|\varrho(0)\int_0^T \frac{ds}{\varrho(s)}, \quad Z(t) \leq u(t), \quad t \in [0,T]$$

and

$$\begin{aligned} u'(t) &= \frac{1}{\varrho(t)}\int_0^t p(s)\left[Z(s) + \int_0^s M(\tau)G(Z(\tau))d\tau + \int_0^T M(\tau)H(Z(\tau))d\tau\right]ds \\ &\leq \frac{1}{R}\int_0^t p(s)\left[u(s) + \int_0^s M(\tau)G(u(\tau))d\tau + \int_0^T M(\tau)H(u(\tau))d\tau\right]ds \\ &\leq \frac{1}{R}\int_0^t p(s)\left[u(s) + \int_0^s M(\tau)G(u(\tau))d\tau + \beta\right]ds. \end{aligned}$$

Let $w(t)=u(t)+\int_0^t M(\tau)G(u(\tau))d\tau+\beta$. Then we have $u(t)\leq w(t),\ t\in[0,T]$. Since $u(t)$ is increasing, $w(t)$ is also increasing on [0,T] and $w(0)=\beta+\|\phi\|_C+\|\delta\|\varrho(0)\int_0^T\frac{ds}{\varrho(s)}=\alpha$. Now,

$$\begin{aligned} w'(t)= \quad & u'(t)+M(t)G(u(t)) \\ & \leq \frac{1}{R}\int_0^t p(s)w(s)ds+M(t)G(u(t)) \\ & \leq w(t)\frac{1}{R}\int_0^t p(s)ds+M(t)G(u(t)) \\ & \leq M(t)[w(t)+G(w(t))]. \end{aligned}$$

Therefore

$$\frac{w'(t)}{w(t)+G(w(t))}\leq M(t),\ t\in[0,T].$$

Integrating from 0 to t and using change of variables $t\rightarrow s=w(t)$ and the condition (2.1), we obtain

$$\int_\alpha^{w(t)}\frac{ds}{s+G(s)}\leq\int_0^t M(s)ds\leq\int_0^T M(s)ds<\int_\alpha^\infty\frac{ds}{s+G(s)},\ t\in[0,T]. \tag{2.7}$$

From the inequality (2.7) there exists a constant γ, independent of $\lambda\in(0,1)$ such that $w(t)\leq\gamma$ for $t\in[0,T]$. Hence $Z(t)\leq u(t)\leq w(t)\leq\gamma,\ t\in[0,T]$. Since for every $t\in[0,T]$, $\|x_t\|_C\leq Z(t)$, we have

$$\|x\|_B=\sup\{\|x(t)\|:t\in[-r,T]\}\leq\gamma.$$

Now, we rewrite initial value problem (1.1)-(1.2) as follows: For $\phi\in C$, define $\widehat{\phi}\in B, B=C([-r,T],X)$ by

$$\widehat{\phi}(t)=\begin{cases}\phi(t) & \text{if } -r\leq t\leq 0\\ \phi(0)+\delta\varrho(0)\int_0^t\frac{ds}{\varrho(s)} & \text{if } 0\leq t\leq T.\end{cases}$$

If $y\in B$ and $x(t)=y(t)+\widehat{\phi}(t)$, $t\in[-r,T]$ then it is easy to see that $y(t)$ satisfies

$$\begin{aligned} & y(t)=y_0=0; \quad -r\leq t\leq 0 \quad \text{and} \\ & y(t)=\int_0^t\frac{1}{\varrho(s)}\int_0^s f\left(\tau,y_\tau+\widehat{\phi}_\tau,\int_0^\tau a(\tau,\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta,\int_0^T b(\tau,\eta)h(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\right)d\tau ds,\ t\in[0,T], \end{aligned}$$

if and only if $x(t)$ satisfies

$$\begin{aligned} x(t)=\ & \phi(0)+\delta\varrho(0)\int_0^t\frac{ds}{\varrho(s)} \\ & +\int_0^t\frac{1}{\varrho(s)}\int_0^s f\left(\tau,x_\tau,\int_0^\tau a(\tau,\eta)g(\eta,x_\eta)d\eta,\int_0^T b(\tau,\eta)h(\eta,x_\eta)d\eta\right)d\tau ds, t\in[0,T], \end{aligned} \tag{2.8}$$

$$x(t)=\phi(t),\ -r\leq t\leq 0,\ x'(0)=\delta. \tag{2.9}$$

We define the operator $F:B_0\rightarrow B_0$, $B_0=\{y\in B:y_0=0\}$ by

$$(Fy)(t)=\begin{cases}0 \quad \text{if } -r\leq t\leq 0\\ \int_0^t\frac{1}{\varrho(s)}\int_0^s f\left(\tau,y_\tau+\widehat{\phi}_\tau,\int_0^\tau a(\tau,\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta,\int_0^T b(\tau,\eta)h(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\right)d\tau ds, \text{ if } t\in[0,T].\end{cases} \tag{2.10}$$

From the definition of operator F equations (2.8)-(2.9) can be written as $y=Fy$, and the equations (2.3)-(2.4) can be written as $y=\lambda Fy$.

Now, we prove that F is completely continuous. First, we prove that $F:B_0\rightarrow B_0$ is continuous. Let $\{u_m\}$ be a sequence of elements of B_0 converging to u in B_0. Then by using hypothesis ($\mathbf{H_5}$) and ($\mathbf{H_6}$) we have

$$f\left(t,u_{m_t}+\widehat{\phi}_t,\int_0^t a(t,s)g(s,u_{m_s}+\widehat{\phi}_s)ds,\int_0^T b(t,s)h(s,u_{m_s}+\widehat{\phi}_s)ds\right)$$

$$\to f\left(t, u_t + \widehat{\phi}_t, \int_0^t a(t,s)g(s, u_s + \widehat{\phi}_s)ds, \int_0^T b(t,s)h(s, u_s + \widehat{\phi}_s)ds\right)$$

for each $t \in [0,T]$. Then by dominated convergence theorem, we have

$$\begin{aligned}
&\|(Fu_m)(t) - (Fu)(t)\| \\
&\le \int_0^t \frac{1}{\varrho(s)} \int_0^s \left\| f\left(\tau, u_{m_\tau} + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, u_{m_\eta} + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, u_{m_\eta} + \widehat{\phi}_\eta)d\eta\right)\right. \\
&\qquad \left. -f\left(\tau, u_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, u_\eta + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, u_\eta + \widehat{\phi}_\eta)d\eta\right)\right\| d\tau ds \\
&\to 0 \quad \text{as} \quad n \to \infty, \forall\ t \in [0,T].
\end{aligned}$$

Since, $\|Fu_m - Fu\|_B = \sup_{t\in[-r,T]} \|(Fu_m)(t) - (Fu)(t)\|$, it follows that $\|Fu_m - Fu\|_B \to 0$ as $n \to \infty$ which implies $Fu_m \to Fu$ in B_0 as $u_m \to u$ in B_0. Therefore, F is continuous.

We prove that F maps a bounded set of B_0 into a precompact set of B_0. Let $B_k = \{y \in B_0 : \|y\|_B \le k\}$ for $k \ge 1$. We show that FB_k is uniformly bounded. Let $M^* = \sup\{M(t) : t \in [0,T]\}$ and $\|\phi\|_C = c$. Then from the definition of F in (2.10) and using hypotheses $(\mathbf{H_1}) - (\mathbf{H_4})$ and the fact that $\|y\|_B \le k, y \in B_k$ implies $\|y_t\|_C \le k, t \in [0,T]$ we obtain

$$\begin{aligned}
&\|(Fy)(t)\| \\
&\le \int_0^t \frac{1}{\varrho(s)} \int_0^s p(\tau)\left[\|y_\tau + \widehat{\phi}_\tau\|_C + \int_0^\tau Km(\eta)G(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta + \int_0^T Ln(\eta)H(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta\right] d\tau ds \\
&\le \int_0^t \frac{1}{\varrho(s)} \int_0^s p(\tau)\left[\|k + c + \int_0^\tau M(\eta)G(k+c)d\eta + \int_0^T M(\eta)H(k+c)d\eta\right] d\tau ds \\
&\le [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_0^t \frac{1}{R} \int_0^s p(\tau)d\tau ds \\
&\le [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_0^T M(s)ds.
\end{aligned}$$

This implies that the set $\{(Fy)(t) : \|y\|_B \le k,\ -r \le t \le T\}$ is uniformly bounded in X and hence FB_k is uniformly bounded.

Next we show that F maps B_k into an equicontinuous family of functions with values in X. Let $y \in B_k$ and $t_1, t_2 \in [-r,T]$. Then from the equation (2.10) and using the hypotheses $(\mathbf{H_1}) - (\mathbf{H_4})$ we have three cases:

Case 1 : Suppose $0 \le t_1 \le t_2 \le T$

$$\begin{aligned}
&\|(Fy)(t_2) - (Fy)(t_1)\| \\
&\le \int_{t_1}^{t_2} \frac{1}{\varrho(s)} \int_0^s \left\| f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta)g(\eta, y_\eta + \widehat{\phi}_\eta)d\eta, \int_0^T b(\tau,\eta)h(\eta, y_\eta + \widehat{\phi}_\eta\right) d\eta\right\| d\tau ds \\
&\le \int_{t_1}^{t_2} \frac{1}{\varrho(s)} \int_0^s p(\tau)\left[\|y_\tau + \widehat{\phi}_\tau\|_C + \int_0^\tau Km(\eta)G(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta + \int_0^T Ln(\eta)H(\|y_\eta + \widehat{\phi}_\eta\|_C)d\eta\right] d\tau ds \\
&\le \int_{t_1}^{t_2} \frac{1}{\varrho(s)} \int_0^s p(\tau)\left[k + c + \int_0^\tau M(\eta)G(k+c)d\eta + \int_0^T M(\eta)H(k+c)d\eta\right] d\tau ds \\
&\le [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_{t_1}^{t_2} \frac{1}{R} \int_0^s p(\tau)d\tau ds \\
&\le [k + c + M^*TG(k+c) + M^*TH(k+c)] \int_{t_1}^{t_2} M(s)ds.
\end{aligned}$$

Case 2 : Suppose $-r \leq t_1 \leq 0 \leq t_2 \leq T$. Proceeding as in Case 1, we get

$$\begin{aligned}
&\|(Fy)(t_2) - (Fy)(t_1)\| \\
&\leq \int_0^{t_2} \frac{1}{\varrho(s)} \int_0^s \left\| f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta) g(\eta, y_\eta + \widehat{\phi}_\eta) d\eta, \int_0^T b(\tau,\eta) h(\eta, y_\eta + \widehat{\phi}_\eta\right) d\eta \right\| d\tau ds \\
&\leq [k + c + M^* T G(k+c) + M^* T H(k+c)] \int_0^{t_2} M(s) ds.
\end{aligned}$$

Case 3 : Suppose$-r \leq t_1 \leq t_2 \leq 0$. Then$\|(Fy)(t_2) - (Fy)(t_1)\| = 0$.

From Cases 1-3, we see that $\|(Fy)(t_2) - (Fy)(t_1)\| \to 0$ as $(t_2 - t_1) \to 0$ and we conclude that FB_k is an equicontinuous family of functions with values in X.

We have already shown that FB_k is an equicontinuous and uniformly bounded collection. To prove the set FB_k is precompact in B, it is sufficient, by Arzela-Ascoli's argument, to show that the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X for each $t \in [-r, T]$. Since $(Fy)(t) = 0$ for $t \in [-r, 0]$ and $y \in B_k$, it is sufficient to show this for $0 < t \leq T$. Let $0 < t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_k$, we define

$$(F_\epsilon y)(t) = \int_0^{t-\epsilon} \frac{1}{\varrho(s)} \int_0^s f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta) g(\eta, y_\eta + \widehat{\phi}_\eta) d\eta, \int_0^T b(\tau,\eta) h(\eta, y_\eta + \widehat{\phi}_\eta) d\eta\right) d\tau ds.$$

Since the set FB_k is bounded in B, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $y \in B_k$, we have

$$(Fy)(t) - (F_\epsilon y)(t) = \int_{t-\epsilon}^t \frac{1}{\varrho(s)} \int_0^s f\left(\tau, y_\tau + \widehat{\phi}_\tau, \int_0^\tau a(\tau,\eta) g(\eta, y_\eta + \widehat{\phi}_\eta) d\eta, \int_0^T b(\tau,\eta) h(\eta, y_\eta + \widehat{\phi}_\eta) d\eta\right) d\tau ds.$$

By making use of hypotheses $(\mathbf{H_1}) - (\mathbf{H_4})$ and the fact that $\|y\|_B \leq k,\ y \in B_k$ implies $\|y_t\|_C \leq k,\ t \in [0, T]$, we have

$$\begin{aligned}
&\|(Fy)(t) - (F_\epsilon y)(t))\| \\
&\leq \int_{t-\epsilon}^t \frac{1}{\varrho(s)} \int_0^s p(\tau) \left[\|y_\tau + \widehat{\phi}_\tau\|_C + \int_0^\tau K m(\eta) G(\|y_\eta + \widehat{\phi}_\eta\|_C) d\eta + \int_0^T L n(\eta) H(\|y_\eta + \widehat{\phi}_\eta\|_C) d\eta \right] d\tau ds \\
&\leq \int_{t-\epsilon}^t \frac{1}{\varrho(s)} \int_0^s p(\tau) \left[k + c + \int_0^\tau M(\eta) G(k+c) d\eta + \int_0^T M(\eta) H(k+c) d\eta \right] d\tau ds \\
&\leq [k + c + M^* T G(k+c) + M^* T H(k+c)] \int_{t-\epsilon}^t \frac{1}{\varrho(s)} \int_0^s p(\tau) d\tau ds \\
&\leq [k + c + M^* T G(k+c) + M^* T H(k+c)] \int_{t-\epsilon}^t M(s) ds.
\end{aligned}$$

This shows that there exists precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_k\}$ hence the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X. Thus we have shown that F is completely continuous operator. Moreover, the set

$$\varepsilon(F) = \{y \in B_0 : y = \lambda F y,\ 0 < \lambda < 1\},$$

is bounded in B, since for every y in $\varepsilon(F)$, the function $x(t) = y(t) + \widehat{\phi}(t)$ is a solution of initial value problem (2.2)-(1.2) for which we have proved that $\|x\|_B \leq \gamma$ and hence $\|y\|_B \leq \gamma + c$. Now, by virtue of Lemma 2.1, the operator F has a fixed point $\tilde{y}$ in B_0. Then $\tilde{x} = \tilde{y} + \widehat{\phi}$ is a solution of the initial value problem (1.1)-(1.2) . This completes the proof of the Theorem 2.1. □

In concluding this paper, we remark that one can easily extend the ideas of this paper to study the global existence of solutions to second order nonlinear mixed Volterra-Fredholm functional integrodifferential equation of the form

$$(\varrho(t) x'(t))' = f\left(t, x_t, x'(t), \int_0^t a(t,s) g(s, x_s, x'(s)) ds, \int_0^T b(t,s) h(s, x_s, x'(s)) ds\right),\ t \in [0, T],$$

$$x(t) = \phi(t),\ -r \leq t \leq 0,\ x'(0) = \delta.$$

with conditions given in $(\mathbf{H_1}) - (\mathbf{H_6})$ and suitable condition similar to that given in (2.1). The precise formulation of this result is very close to that of the result given in our Theorem 2.1 with suitable modification and hence we omit details.

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