

# Existence and uniqueness of solutions for random impulsive differential equation

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## Abstract

In this paper, we study the existence and uniqueness of the mild solutions for random impulsive differential equations through fixed point technique. An example is provided to illustrate the theory.

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## 1 Introduction

Many evolution processes from fields as diverse as physics, population dynamics, aeronautics, economics, telecommunications and engineering are characterized by the fact that they undergo abrupt change of state at certain moments of time between intervals of continuous evolution. The duration of these changes are often negligible compared to the total duration of process act instantaneously in the form of impulses. The impulses may be deterministic or random. There are lot of papers which investigate the qualitative properties of deterministic impulses see [1, 5, 7, 8] and the references therein.

When the impulses are exist at random points, the solutions of the differential systems are stochastic processes. It is very different from deterministic impulsive differential systems and also it is different from stochastic differential equations. Thus the random impulsive systems give more realistic than deterministic impulsive systems. The study of random impulsive differential equations is a new research area. There are few publications in this field, Iwankiewicz and Nielsen [6], investigated dynamic response of non-linear systems to poisson distributed random impulses. Sanz-Serna and Stuart [9] first brought dissipative differential equations with random impulses and used Markov chains to simulate such systems. Tatsuyuki et al [10] presented a mathematical model of random impulse to depict drift motion of granules in chara cells due to myosin-actin interaction. Shujin Wu and Meng [2004] first brought forward random impulsive ordinary differential equations and investigated boundedness of solutions to these models by Liapunov's direct function in [13]. Shujin Wu et al. [14, 15, 16, 17], studied some qualitative properties of random impulses. In [3], the author studied the existence and uniqueness of random impulsive differential system by relaxing the linear growth conditions, sufficient conditions for stability through continuous dependence on initial conditions and exponential stability via fixed point theory. In [2, 4, 11, 12] the author has studied some properties of random type impulsive differential systems.

Motivated by the above mentioned works, the main purpose of this paper is to study the random impulsive differential equations. We utilize the technique developed by [7, 8, 15].

This paper is organized as follows: Some preliminaries are presented in Section 2. In Section 3, we investigate the existence and uniqueness of solution of random impulsive differential equation by reducing the linear growth condition. Moreover, Lipschitz condition has to be relaxed on the impulsive terms in the deriving results. Finally in Section 4, we give an example to motivate our results.

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## 2 Preliminaries

Let  $X$  be a real separable Hilbert space and  $\Omega$  a nonempty set. Assume that  $\tau_k$  is a random variable defined from  $\Omega$  to  $D_k \stackrel{\text{def.}}{=} (0, d_k)$  for all  $k = 1, 2, \dots$ , where  $0 < d_k < +\infty$ . Furthermore, assume that  $\tau_i$  and  $\tau_j$  are independent with each other as  $i \neq j$  for  $i, j = 1, 2, \dots$ . Let  $\tau \in \mathfrak{R}$  be a constant. For the sake of simplicity, we denote  $\mathfrak{R}_\tau = [\tau, T]$ . We consider the differential equations with random impulses of the form

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \neq \xi_k, \quad t \geq \tau, \quad (2.1)$$

$$x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, \dots, \quad (2.2)$$

$$x_{t_0} = x_0, \quad (2.3)$$

where  $A$  is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators  $T(t)$  in  $X$ ;  $f : \mathfrak{R}_\tau \times X \rightarrow X$ ,  $b_k : D_k \rightarrow \mathfrak{R}$  for each  $k = 1, 2, \dots$ ;  $\xi_0 = t_0 \in [\tau, T]$  and  $\xi_k = \xi_{k-1} + \tau_k$  for  $k = 1, 2, \dots$ , here  $t_0 \in \mathfrak{R}_\tau$  is arbitrary real number. Obviously,  $t_0 = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_k < \dots$ ;  $x(\xi_k^-) = \lim_{t \uparrow \xi_k} x(t)$  according to their paths with the norm  $\|x\| = \sup_{\tau \leq s \leq t} |x(s)|$  for each  $t$  satisfying  $\tau \leq t \leq T$ .

Let us denote  $\{B_t, t \geq 0\}$  be the simple counting process generated by  $\{\xi_n\}$ , that is,  $\{B_t \geq n\} = \{\xi_n \leq t\}$ , and denote  $\mathcal{F}_t$  the  $\sigma$ -algebra generated by  $\{B_t, t \geq 0\}$ . Then  $(\Omega, P, \{\mathcal{F}_t\})$  is a probability space. Let  $L_2 = L_2(\Omega, \{\mathcal{F}_t\}, X)$  denote the Hilbert space of all  $\{\mathcal{F}_t\}$ -measurable square integrable random variables with values in  $X$ .

Let  $\mathcal{B}$  denote Banach space  $\mathcal{B}([\tau, T], L_2)$ , the family of all  $\{\mathcal{F}_t\}$ -measurable random variables  $\psi$  with the norm

$$\|\psi\|^2 = \sup_{t \in [\tau, T]} E\|\psi(t)\|^2.$$

**Definition 2.1.** Consider the inhomogeneous initial value problem where  $f : [0, T] \rightarrow X$ .

$$\begin{aligned} x'(t) &= Ax(t) + f(t) \\ x(0) &= x_0. \end{aligned}$$

Let  $A$  be the infinitesimal generator of a  $C_0$  semigroup  $T(t)$ . Let  $x_0 \in X$  and  $f \in L^1(0, T; X)$ . Then the function  $x \in C([0, T]; X)$  is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq T$$

is the mild solution of the above initial value problem for  $t \in [0, T]$ .

**Definition 2.2.** A semigroup  $\{T(t), t \geq 0\}$  is said to be uniformly bounded if there exists a constant  $M \geq 1$  such that

$$\|T(t)\| \leq M, \quad \text{for } t \geq 0.$$

**Definition 2.3.** For a given  $T \in (\tau, +\infty)$ , a stochastic process  $\{x(t) \in \mathcal{B}, \tau \leq t \leq T\}$  is called a mild solution to equation (2.1)-(2.3) in  $(\Omega, P, \{\mathcal{F}_t\})$ , if

(i)  $x(t) \in X$  is  $\mathcal{F}_t$ -adapted;

(ii)

$$\begin{aligned} x(t) &= \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k b_i(\tau_i) T(t-t_0)x_0 + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t-s)f(s, x(s))ds \right. \\ &\quad \left. + \int_{\xi_k}^t T(t-s)f(s, x(s))ds \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [\tau, T], \end{aligned} \quad (2.4)$$

where  $\prod_{j=m}^n (\cdot) = 1$  as  $m > n$ ,  $\prod_{j=i}^k b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1}) \cdots b_i(\tau_i)$ , and  $I_A(\cdot)$  is the index function, i.e.,

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

### 3 Existence and Uniqueness

In this section, we discuss the existence and uniqueness of the mild solution for the system (2.1)-(2.3). Before proving the main results, we introduce the following hypotheses which are used in our results.

(**H<sub>1</sub>**) The function  $f$  satisfies the Lipschitz condition. That is., for  $\zeta, \varsigma \in X$  and  $\tau \leq t \leq T$  there exists a constant  $L > 0$  such that

$$\begin{aligned} E \|f(t, \zeta) - f(t, \varsigma)\|^2 &\leq L E \|\zeta - \varsigma\|^2, \\ E \|f(t, 0)\|^2 &\leq \kappa, \quad \text{where } \kappa \geq 0 \text{ is a constant.} \end{aligned}$$

(**H<sub>2</sub>**) The condition  $\max_{i,k} \left\{ \prod_{j=i}^k \|b_j(\tau_j)\| \right\}$  is uniformly bounded if, there is a constant  $C > 0$  such that

$$\max_{i,k} \left\{ \prod_{j=i}^k \|b_j(\tau_j)\| \right\} \leq C \quad \text{for all } \tau_j \in D_j, \quad j = 1, 2, \dots.$$

**Theorem 3.1.** *Let the hypotheses (**H<sub>1</sub>**) – (**H<sub>2</sub>**) be hold. If the following inequality*

$$\Lambda = M^2 \max\{1, C^2\}(T - \tau)^2 L < 1, \quad (3.1)$$

*is satisfied, then the system (2.1)-(2.3) has a unique mild solution in  $\mathcal{B}$ .*

*Proof.* Let  $T$  be an arbitrary number  $\tau < T < +\infty$ . First we define the nonlinear operator  $\mathcal{S} : \mathcal{B} \rightarrow \mathcal{B}$  as follows

$$\begin{aligned} (\mathcal{S}x)(t) &= \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k b_i(\tau_i) T(t - t_0) x_0 + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t - s) f(s, x(s)) ds \right. \\ &\quad \left. + \int_{\xi_k}^t T(t - s) f(s, x(s)) ds \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [\tau, T]. \end{aligned}$$

It is easy to prove the continuity of  $\mathcal{S}$ . Now, we have to show that  $\mathcal{S}$  maps  $\mathcal{B}$  into itself.

$$\begin{aligned} \|(\mathcal{S}x)(t)\|^2 &\leq \left[ \sum_{k=0}^{+\infty} \left[ \left\| \prod_{i=1}^k b_i(\tau_i) \right\| \|T(t - t_0)\| \|x_0\| \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^k \left\| \prod_{j=i}^k b_j(\tau_j) \right\| \left\{ \int_{\xi_{i-1}}^{\xi_i} \|T(t - s) f(s, x(s))\| ds \right\} \right. \right. \\ &\quad \left. \left. + \int_{\xi_k}^t \|T(t - s) f(s, x(s))\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \\ &\leq 2 \left[ \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k \|b_i(\tau_i)\|^2 \|T(t - t_0)\|^2 \|x_0\|^2 I_{[\xi_k, \xi_{k+1})}(t) \right] \right. \\ &\quad \left. + \left[ \sum_{k=0}^{+\infty} \left[ \sum_{i=1}^k \left\| \prod_{j=i}^k b_j(\tau_j) \right\| \left\{ \int_{\xi_{i-1}}^{\xi_i} \|T(t - s)\| \|f(s, x(s))\| ds \right\} \right. \right. \right. \\ &\quad \left. \left. + \int_{\xi_k}^t \|T(t - s)\| \|f(s, x(s))\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \right] \\ &\leq 2M^2 \max_k \left\{ \prod_{i=1}^k \|b_i(\tau_i)\|^2 \right\} \|x_0\|^2 \\ &\quad + 2M^2 \left[ \max_{i,k} \left\{ 1, \prod_{j=i}^k \|b_j(\tau_j)\| \right\} \right]^2 \\ &\quad \times \left( \int_{t_0}^t \|f(s, x(s))\| ds I_{[\xi_k, \xi_{k+1})}(t) \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq 2M^2C^2\|x_0\|^2 + 2M^2 \max\{1, C^2\} \left( \int_{t_0}^t \|f(s, x(s))\| ds \right)^2 \\
&\leq 2M^2C^2\|x_0\|^2 + 2M^2 \max\{1, C^2\} (t - t_0) \int_{t_0}^t \|f(s, x(s))\|^2 ds \\
E\|(\mathcal{S}x)(t)\|^2 &\leq 2M^2C^2\|x_0\|^2 + 2M^2 \max\{1, C^2\} (T - \tau) \int_{t_0}^t E \|f(s, x(s))\|^2 ds \\
&\leq 2M^2C^2\|x_0\|^2 + 4M^2 \max\{1, C^2\} (T - \tau)^2 \kappa \\
&\quad + 4M^2 \max\{1, C^2\} (T - \tau) L \int_{t_0}^t E \|x(s)\|^2 ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{t \in [\tau, T]} E\|(\mathcal{S}x)(t)\|^2 &\leq 2M^2C^2\|x_0\|^2 + 4M^2 \max\{1, C^2\} (T - \tau)^2 \kappa \\
&\quad + 4M^2 \max\{1, C^2\} (T - \tau) L \int_{t_0}^t \sup_{s \in [\tau, t]} E \|x(s)\|^2 ds \\
&\leq 2M^2C^2\|x_0\|^2 + 4M^2 \max\{1, C^2\} (T - \tau)^2 \kappa \\
&\quad + 4M^2 \max\{1, C^2\} (T - \tau)^2 L \sup_{t \in [\tau, T]} E \|x(t)\|^2
\end{aligned}$$

for all  $t \in [\tau, T]$ , therefore  $\mathcal{S}$  maps  $\mathcal{B}$  into itself.

Now, we have to show  $\mathcal{S}$  is a contraction mapping

$$\begin{aligned}
\|(\mathcal{S}x)(t) - (\mathcal{S}y)(t)\|^2 &\leq \left[ \sum_{k=0}^{+\infty} \left[ \sum_{i=1}^k \prod_{j=i}^k \|b_j(\tau_j)\| \right. \right. \\
&\quad \times \int_{\xi_{i-1}}^{\xi_i} \|T(t-s)\| \|f(s, x(s)) - f(s, y(s))\| ds \\
&\quad \left. \left. + \int_{\xi_k}^t \|T(t-s)\| \|f(s, x(s)) - f(s, y(s))\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \\
&\leq M^2 \left[ \max_{i,k} \left\{ 1, \prod_{j=i}^k \|b_j(\tau_j)\| \right\} \right]^2 \\
&\quad \times \left( \int_{t_0}^t \|f(s, x(s)) - f(s, y(s))\| ds I_{[\xi_k, \xi_{k+1})}(t) \right)^2 \\
&\leq M^2 \max\{1, C^2\} (t - t_0) \int_{t_0}^t \|f(s, x(s)) - f(s, y(s))\|^2 ds \\
E\|(\mathcal{S}x)(t) - (\mathcal{S}y)(t)\|^2 &\leq M^2 \max\{1, C^2\} (t - t_0) \int_{t_0}^t E \|f(s, x(s)) - f(s, y(s))\|^2 ds \\
&\leq M^2 \max\{1, C^2\} (T - \tau) L \int_{t_0}^t E \|x(s) - y(s)\|^2 ds.
\end{aligned}$$

Taking supremum over  $t$ , we get,

$$\|(\mathcal{S}x) - (\mathcal{S}y)\|^2 \leq M^2 \max\{1, C^2\} (T - \tau)^2 L \|x - y\|^2.$$

Thus,

$$\|(\mathcal{S}x) - (\mathcal{S}y)\|^2 \leq \Lambda \|x - y\|^2,$$

since  $0 < \Lambda < 1$ . This shows that the operator  $\mathcal{S}$  satisfies the contraction mapping principle and therefore,  $\mathcal{S}$  has a unique fixed point which is the mild solution of the system (2.1)-(2.3). This completes the proof.  $\square$

## 4 Example

As an application for the problem (2.1)-(2.3), consider a one dimensional rod of length  $\pi$  whose ends are maintained at  $0^0$  and whose sides are insulated. Suppose there is an exothermic reaction taking place inside

the rod with heat being produced proportionally to the temperature at a previous time  $t - r$  (for the sake of simplicity, we assume the delay  $r \geq 0$  is constant). Consequently, the temperature in the rod may be modeled to satisfy

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + \rho u(x, t - r), & 0 < x < \pi, & t > 0, \\ u(0, t) &= u(\pi, t) = 0, \\ u(x, t) &= \varphi(x, t), & -r \leq t \leq 0, & 0 \leq x \leq \pi. \end{cases} \quad (4.1)$$

where  $\rho$  depends on the rate of reaction and  $\varphi : [-r, 0] \times [0, \pi] \rightarrow \mathfrak{R}$  is a given function. We observe that, when there is no heat production (i.e.,  $\rho = 0$ ), the problem (4.1) has solution given by

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx,$$

where  $r = 0$  and  $\varphi(x, 0) = \sum_{n=1}^{\infty} a_n \sin nx$ .

However, it often occurs that the exothermic reaction can be related with random impulses. In some cases, the equation (4.1) may be written in the generalized form with  $r = 0$ ,

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + \rho u(x, t), & 0 < x < \pi, & t > 0, & t \neq \xi_k, \\ u(x, \xi_k) &= q(k) \tau_k u(x, \xi_k^-), & t = \xi_k, \\ u(0, t) &= u(\pi, t) = 0, \\ u(x, t) &= \varphi(x, t), & 0 \leq x \leq \pi, \end{cases} \quad (4.2)$$

and setting  $X = L^2[0, \pi]$  and the operator  $A = \frac{\partial^2}{\partial x^2}$  with the domain

$$D(A) = \left\{ u \in X \mid u \text{ and } \frac{\partial u}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 u}{\partial x^2} \in X, u(0) = u(\pi) = 0 \right\}.$$

It is well known that  $A$  generates a strongly continuous semigroup  $T(t)$  which is compact, analytic and self adjoint and

$$\|T(t)\| \leq M, \quad \text{for } t \geq 0, \text{ where } M > 0.$$

Thus  $T(t)$  is bounded.

Furthermore, we may assume that the impulsive nature satisfy the following condition

$$E \left[ \max_{i,k} \left\{ \prod_{j=i}^k \|q(j)(\tau_j)\|^2 \right\} \right] < \infty.$$

Under these condition, we can define the functions  $f$  and  $b_k$  as

$$f(t, x(t)) = \rho u(x, t) \quad \text{and} \quad b_k(\tau_k) = q(k) \tau_k.$$

Then the problem (4.2) can be modeled as the abstract random impulsive differential equations of the form (2.1)-(2.3) .

The next result is consequence of Theorem 3.1

**Proposition 4.1.** *Let the hypotheses  $(\mathbf{H}_1) - (\mathbf{H}_2)$  be hold. Then there exist a unique mild solution  $u$  of the system (4.2) provided,*

$$M^2 \max\{1, C^2\} (T - \tau)^2 L < 1, \quad (4.3)$$

is satisfied.

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