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Existence results for neutral functional fractional differential equations with state dependent-delay

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Abstract

In this paper, we provide sufficient conditions for the existence of mild solutions for a class of fractional differential equations with state-dependent delay. The results are obtained by using the nonlinear alternative of Leray-Schauder type [14] fixed point theorem. An example is provided to illustrate the main results.

Keywords: Functional differential equation, fractional derivative, fractional integral, state-dependent delay, fixed point.

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1 Introduction

In the last two decades, the theory of fractional calculus has gained importance and popularity, due to its wide range of applications in varied fields of sciences and engineering. In [1, 3, 6, 7, 13, 19, 25, 27, 31, 32, 33] applications are mentioned to fluid flow, rheology, dynamical processes in self-similar and porous structures, electrical networks, control theory of dynamical systems and so on.

In this work, we establish the existence of mild solutions for a class of fractional abstract differential equations with state-dependent delay described by

$$^{c}D^{q}x(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in J = [0, a],\ 0 < q < 1, \tag{1.1}$$

$$x(t) = \varphi(t) \in \mathcal{B},\ t \in (-\infty, 0], \tag{1.2}$$

where the unknown $x(\cdot)$ takes values in Banach space X with norm $\|\cdot\|$, $^{c}D^{q}$ is the Caputo fractional derivative of order $0 < q < 1$, A is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t), t \geq 0\}$ in X, $f : J \times \mathcal{B} \to X$ and $\rho : J \times \mathcal{B} \to (-\infty, a]$ are appropriate given functions, $\varphi \in \mathcal{B}$, $\varphi(0) = 0$ and $\mathcal{B}$ is called a phase space that will be defined in preliminaries.

An important point to note here it that when the delay is infinite the right notion is phase space. This concept was introduced by Hale and Kato [15] (see also Kappel and Schappacher [26] and Schumacher [34]) which enables to deduce important information about qualitative properties of differential equations with unbounded delay. For a detailed discussion on this topic, we refer the reader to the book by Hino et al. [24].

On the other hand, functional differential equations with state-dependent delay appears frequently in applications as models of equations. Investigations of these classes of delay equations essentially differ from once of equations with constant or time-dependent delay. For these reasons the theory of differential equations with state-dependent delay has drawn the attention of researchers in the recent years, see for instance [4, 5, 16, 17, 18, 20, 21, 22, 23, 28, 29] and the references therein. The investigation of the exitnece of mild

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solutions of fractional functional differential equations with state-dependent delay is very recent and limited, see for instance [2, 8, 9, 10].

The results in the present work are, on one side, an extension of results in [10] and [35] and, one the otherside, an interesting contribution to the study of qualitative properties for fractional differential equations with state-dependent delay. The topological method that we have choosen to study existence of mild solutions of the fractiona differential equations (1.1)-(1.2) is the theory of fixed points, which has been a very powerful and important tool to study the nonlilnear phenomena.

Our approach and techniques here are based on the nonlinear alternative of Leray-Schauder type [14] and probability density function given by EI-Borai [11] and was then developed by Zhou et al. [35, 36].

2 Preliminaries

In this section, we introduce notation, definitions and preliminary facts which are used throughtout this paper.

By $C(J,X)$ we denote the Banach space of continuous functions from J into X with the norm

$$\|x\|_\infty := \sup\{|x(t)| : t \in J\}.$$

Definition 2.1. *The fractional integral of order α with the lower limit 0 for the function $f : (0,a] \to X$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \ \alpha > 0,$$

provided the right hand side exists pointwise on $(0,a]$, where Γ is the gamma function.

For instance, $I^\alpha f$ exists for all $\alpha > 0$, where $f \in C((0,a],X) \cup L^1((0,a],X)$; note also that when $f \in C((0,a],X)$ then $I^\alpha f \in C((0,a],X)$ and moreover $I^\alpha f(0) = 0$.

Definition 2.2. *The Caputo derivative of order α with the lower limit zero for a function $f : (0,a] \to X$ can be written as*

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \ n-1 < \alpha < n.$$

If f is an abstract function with values in X, then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner's sence.

In this paper, we will employ an axiomatic definition, for the phase space $\mathcal{B}$ which is similar to those introduced in [24]. More precisely, $\mathcal{B}$ will be a linear space of all functions from $(-\infty,0]$ to X endowed with a seminorm $\|\cdot\|_\mathcal{B}$ satisfying the following axioms:

(A1) If $x : (-\infty,a] \to X, \ a > 0$ is continuous on J and $x_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:

  (i) x_t is in $\mathcal{B}$,

  (ii) $\| x(t) \| \leq H \| x_t \|_\mathcal{B}$,

  (iii) $\| x_t \|_\mathcal{B} \leq K(t) \sup\{\| x(s) \| : 0 \leq s \leq t\} + M(t) \| x_0 \|_\mathcal{B}$, where $H > 0$ is a constant, $K : [0,\infty) \to [1,\infty)$ is continuous, $M : [0,\infty) \to [1,\infty)$ is locally bounded and $H, \ K, \ M$ are independent of $x(\cdot)$.

(A2) For the function $x(\cdot)$ in (A1), x_t is a $\mathcal{B}$-valued continuous function on J.

(A3) The space $\mathcal{B}$ is complete.

The next lemma is a consequence of the phase space axioms and is proved in [20].

Lemma 2.1. *Let $\varphi \in \mathcal{B}$ and $I = (\gamma,0]$ be such that $\varphi_t \in \mathcal{B}$ for every $t \in I$. Assume that there exists a locally bounded function $J^\varphi : I \to [0,\infty)$ such that $\|\varphi_t\|_\mathcal{B} \leq J^\varphi(t)\|\varphi\|_\mathcal{B}$ for every $t \in I$. If $x : (\infty,a] \to \mathbb{R}$ is continuous on J and $x_0 = \varphi$, then*

$$\|x_t\|_\mathcal{B} \leq (M_a + J^\varphi(\max\{\gamma,-|s|\})\|\varphi\|_\mathcal{B} + K_a \sup\{|x(\theta)| : \theta \in [0,\max\{0,s\}]\},$$

for $s \in (\gamma,a]$, where we denoted $K_a = \sup_{t\in J} K(t)$ and $M_a = \sup_{t\in J} M(t)$.

3 Existence results for functional fractional differential equations with state-dependent delay

In this section, we discuss the existence of mild solutions for the fractional differential equations with state-dependent delay of the form (1.1)-(1.2). Following [11, 12, 35], we will introduce now the definition of mild solution to (1.1)-(1.2).

Definition 3.1. *A function $x : (-\infty, a] \to X$ is said to be a mild solution of (1.1)-(1.2) if $x_0 = \varphi,\ x_{\rho(s,x_s)} \in \mathcal{B}$ for each $s \in J$ and*

$$x(t) = S_q(t)\varphi(0) + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_{\rho(s,x_s)}) ds, \quad t \in J,$$

where

$$\begin{aligned} S_q(t) &= \int_0^\infty \xi_q(\theta) T(t^q\theta) d\theta, \\ T_q(t) &= q\int_0^\infty \theta\xi_q(\theta) T(t^q\theta) d\theta, \\ \xi_q(\theta) &= \frac{1}{q}\theta^{-1-\frac{1}{q}}\overline{w}_q(\theta^{-\frac{1}{q}}) \geq 0, \\ \overline{w}_q(\theta) &= \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \theta \in (0,\infty), \end{aligned}$$

ξ_q is a probability density function on $(0, \infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0,\infty) \text{ and } \int_0^\infty \xi_q(\theta) d\theta = 1.$$

Remark 3.1. *It is not difficult to verify that for $v \in [0, 1]$*

$$\int_0^\infty \theta^v \xi_q(\theta) d\theta = \int_0^\infty \theta^{-qv}\overline{w}_q(\theta) d\theta = \frac{\Gamma(1+v)}{\Gamma(1+qv)}.$$

Lemma 3.1. *[35] For any $t \geq 0$, The operators $S_q(t)$ and $T_q(t)$ have the following properties:*

(a) For any fixed $t \geq 0$, S_q and T_q are linear and bounded operators, ie., for any $x \in X$,

$$\|S_q(t)x\| \leq M\|x\|, \quad \|T_q(t)x\| \leq \frac{qM}{\Gamma(1+q)}\|x\|.$$

(b) $\{S_q(t), t \geq 0\}$ and $\{T_q(t), t \geq 0\}$ are strongly continuous.

(c) For every $t > 0$, $S_q(t)$ and $T_q(t)$ are also compact operators.

To prove our results, we always assume that $\rho : J \times \mathcal{B} \to (-\infty, a]$ is continuous. In addition, we introduce the following conditions.

($\mathbf{H_1}$) The semigroup $T(t)$ is compact for $t > 0$.

($\mathbf{H_2}$) For each $t \in J$, the function $f(t, \cdot) : \mathcal{B} \to X$ is continuous and for each $\psi \in \mathcal{B}$, the function $f(\cdot, \psi) : J \to X$ is strongly measurable.

($\mathbf{H_3}$) There exist $p : J \to [0, \infty]$ and a continuous non-decreasing function $\Omega : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, \psi)\| \leq p(t)\Omega(\|\psi\|_{\mathcal{B}}) \text{ for } t \in J, \text{ and each } \psi \in \mathcal{B}.$$

($\mathbf{H_4}$) The function $t \to \varphi_t$ is well defined and continuous from the set

$$\mathcal{R}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in J \times \mathcal{B},\ \rho(s,\psi) \leq 0\}$$

into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

Remark 3.2. *We point out here that the condition* $(\mathbf{H_4})$ *is usually satisfied by functions that are continuous and bounded. For complementary details related this matter the reader can see [20].*

Theorem 3.1. *Let conditions* $(\mathbf{H_1})-(\mathbf{H_4})$ *hold with* $\rho(t,x)\le t$ *for every* $(t,x)\in J\times\mathcal{B}$ *and*

$$\frac{\|\xi\|_\infty}{(M_a+\overline{J}^{\varphi})\|\varphi\|_{\mathcal{B}}+MK_a\Omega(\|\xi\|_\infty)\|I^qp\|_\infty}>1$$

then there exists a mild solution of (1.1)-(1.2) on $(-\infty,a]$.

Proof. Let $Y=\{u\in C(J,X):u(0)=\varphi(0)=0\}$ endowed with the uniform operator topology and $\Phi:Y\to Y$ be the operator defined by

$$\Phi(x)(t)=\int_0^t(t-s)^{q-1}T_q(t-s)f(s,x_{\rho(s,x(s))})ds,\quad t\in J,$$

where $\overline{x}:(-\infty,a]\to X$ is such that $\overline{x}_0=\varphi$ and $\overline{x}=x$ on J. From axiom (A_1) and our assumption on φ, we infer that $\Phi(x)(\cdot)$ is well defined and continuous.

Let $\overline{\varphi}:(-\infty,a]\to X$ be the extension of φ to $(-\infty,a]$ such that $\overline{\varphi}(\theta)=\phi(0)=0$ on J and $\overline{J}^{\varphi}=\sup\{J^{\varphi}:s\in\mathcal{R}(\rho^-)\}$.

We will prove that $\Phi(\cdot)$ is completely continuous from $B_r(\overline{\varphi}|_J,Y)$ to $B_r(\overline{\varphi}|_J,Y)$.

We break the proof into several steps.

Step 1: Φ is continuous on $B_r(\overline{\varphi}|_J,Y)$.

Let $\{x^n\}\subset B_r(\overline{\varphi}|_J,Y)$ and $x\in B_r(\overline{\varphi}|_J,Y)$ with $x^n\to x$ $(n\to\infty)$. From axiom $A1$, it is easy to see that $(\overline{x^n})_s\to\overline{x}_s$ uniformly for $s\in(-\infty,a]$ as $n\to\infty$. By $(H3)$, we have

$$\begin{aligned}&\|f(s,\overline{x^n}_{\rho(s,(\overline{x^n})_s)})-f(s,\overline{x}_{\rho(s,(\overline{x})_s)})\|\\&\le\|f(s,\overline{x^n}_{\rho(s,(\overline{x^n})_s)})-f(s,\overline{x}_{\rho(s,(\overline{x^n})_s})\|+\|f(s,\overline{x}_{\rho(s,(\overline{x^n})_s)})-f(s,\overline{x}_{\rho(s,(\overline{x})_s})\|\end{aligned}$$

which implies that $f(s,\overline{x^n}_{\rho(s,(\overline{x^n})_s)})\to f(s,\overline{x}_{\rho(s,(\overline{x})_s})$ as $n\to\infty$ for each $s\in J$. By axiom A1(ii), Lemma(2.1) and the dominated convergence theorem, we obtain

$$\begin{aligned}\|\Phi(x^n)-\Phi(x)\|&\le\left\|\int_0^t(t-s)^{q-1}T_q(t-s)f(s,x_{\rho(s,x_s)})ds\right\|\\&\to 0\quad\text{as}\quad n\to\infty\end{aligned}$$

Therefore, Φ is continuous.

Step 2: Φ maps bounded sets into bounded sets. If $x\in B_r(\overline{\phi}|_J,Y)$, from Lemma(2.1), it follows that

$$\|\overline{x}_{\rho(t,\overline{x}_t)}\|_{\mathcal{B}}\le r^*:=(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+K_ar$$

and so

$$\begin{aligned}|\Phi(x)(t)|&\le\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}f(s,\overline{x}_{\rho(s,\overline{x}_s)})ds\\&\le\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega(\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}})ds\\&\le\frac{qM}{\Gamma(1+q)}\|p\|_\infty\Omega(r^*)\int_0^t(t-s)^{q-1}ds\\&\le\frac{Ma^q}{\Gamma(1+q)}\|p\|_\infty\Omega(r^*).\end{aligned}$$

Thus,

$$\|\Phi(x)\|_\infty\le\frac{Ma^q}{\Gamma(1+q)}\|p\|_\infty\Omega(r^*):=l.$$

Step 3: Φ maps bounded sets into equicontinuous sets.

Let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$ and B_r be a bounded set as in step 2. Let $\epsilon > 0$ be given. For each $t \in J$, we have

$$\begin{aligned}
&\|\Phi(x)(\tau_2) - \Phi(x)(\tau_1)\| \\
&\leq \int_0^{\tau_1-\epsilon} \Big\| \Big[(\tau_2 - s)^{q-1} T_q(\tau_2 - s) - (\tau_1 - s)^{q-1} T_q(\tau_1 - s)\Big] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\
&+ \int_{\tau_1-\epsilon}^{\tau_1} \Big\| \Big[(\tau_2 - s)^{q-1} T_q(\tau_2 - s) - (\tau_1 - s)^{q-1} T_q(\tau_1 - s)\Big] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\
&+ \int_{\tau_1}^{\tau_2} \Big\| (\tau_2 - s)^{q-1} T_q(\tau_2 - s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\
&\leq \|p\|_\infty \Omega(r^*) \Big[\int_0^{\tau_1-\epsilon} \Big[|(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}| \|T_q(\tau_2 - s)\| \\
&+ |(\tau_1 - s)^{q-1}| \|T_q(\tau_2 - s) - T_q(\tau_1 - s)\| \Big] ds \\
&+ \int_{\tau_1-\epsilon}^{\tau_1} \Big[|(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}| \|T_q(\tau_2 - s)\| \\
&+ |(\tau_1 - s)^{q-1}| \|T_q(\tau_2 - s) - T_q(\tau_1 - s)\| \Big] ds + \frac{M}{\Gamma(1+q)} (\tau_2 - \tau_1)^q \Big].
\end{aligned}$$

The right hand side tends to zero as $\tau_2 - \tau_1 \to 0$, since $T_q(t),\ t \geq 0$ is a strongly continuous semigroup and $T_q(t)$ is compact for $t > 0$ (so $T_q(t)$ is continuous in the uniform operator topology for $t > 0$). The equicontinuous for the other cases $\tau_1 < \tau_2 \leq 0$ or $\tau_1 \leq 0 \leq \tau_2 \leq a$ are very simple.

Step 4: Φ is precompact.

Let $0 < t \leq s \leq a$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$, and $\delta > 0$. For $x \in B_r$, we define,

$$\begin{aligned}
\Phi_{\epsilon,\delta}(x)(t) &= q \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \\
&= T(\epsilon^q \delta) q \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta - \epsilon^q \delta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds,
\end{aligned}$$

Since $T(\epsilon^q \delta)$ is a compact operator for $\epsilon^q \delta > 0$, the set $Y_{\epsilon,\delta}(t) = \{\Phi_{\epsilon,\delta}(x)(t) : x \in B_r\}$ is precompact in X for every $\epsilon,\ 0 < \epsilon < t$. Moreover

$$\begin{aligned}
&\|\Phi(x)(t) - \Phi_{\epsilon,\delta}(x)(t)\| \\
&= q \Big[\Big\| \int_0^{t-\epsilon} \int_0^\delta \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \Big\| \\
&\quad + \Big\| \int_{t-\epsilon}^t \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \Big\| \Big] \\
&\leq \|p\|_\infty \Omega(r^*) \frac{qM}{\Gamma(1+q)} \Big[\int_0^{t-\epsilon} (t-s)^{q-1} ds \int_0^\delta \theta \xi_q(\theta) d\theta + \int_{t-\epsilon}^t (t-s)^{q-1} ds \int_\delta^\infty \theta \xi_q(\theta) d\theta \Big].
\end{aligned}$$

Therefore, there are precompact sets arbitrarly close to the set $Y(t) = \{\Phi_{\epsilon,\delta}(x)(t) : x \in B_r\}$ is precompact. Hence the set $Y(t) = \{\Phi_{\epsilon,\delta}(x)(t) : x \in B_r\}$ is precompact in X.

As a consequence of the Step 1 to Step 4 and the Arzela-Ascoli theorem, we can conclude that the operator Φ is completely continuous.

Step 5: We now show there exists an open set $U \subset Y$ with $x \neq \lambda \Phi(x)$ for $\lambda \in (0,1)$ and $x \in \partial U$. Let $x \in Y$ and $x = \lambda \Phi(x)$ for some $0 < \lambda < 1$. Then for each $t \in J$ we have,

$$x(t) = \lambda \int_0^t (t-s)^{q-1} T_q(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds.$$

This implies by (H3) and Lemma(2.1) that

$$\begin{aligned}
|x(t)| &\leq \int_0^t (t-s)^{q-1} \|T_q(t-s)\| \|f(s, \overline{x}_{\rho(s,\overline{x}(s))})\| ds \\
&\leq \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} p(s) \Omega\Big((M_a + \overline{J}^\phi) \|\phi\|_{\mathcal{B}} + K_a \sup\{|\overline{x}(s)| : s \in [0,t]\} \Big) ds,
\end{aligned}$$

since $\rho(s,\overline{x}_s) \leq s$ for every $s \in J$. Here $\overline{J}^{\phi} = \sup\{J^{\phi}(s) : s \in \mathcal{R}(\rho^-)\}$.

Set $\mu(t) = \sup\{|x(s)| : 0 \leq s \leq t\},\ t \in J$. Then we have

$$\mu(t) \leq \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} p(s)\Omega\Big((M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + K_a\mu(s)\Big)ds.$$

If $\xi(t) = (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + K_a\mu(t)$ then we have,

$$\begin{aligned} \xi(t) &\leq (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + \frac{qMK_a}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} p(s)\Omega(\xi(s))ds \\ &\leq (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + MK_a\Omega(\|\xi\|_\infty)\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} p(s)ds \\ &\leq (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + MK_a\Omega(\|\xi\|_\infty)\|I^q p\|_\infty. \end{aligned}$$

Then

$$\frac{\|\xi\|_\infty}{(M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + MK_a\Omega(\|\xi\|_\infty)\|I^q p\|_\infty} \leq 1.$$

Then there exists M^* such that $\|x\|_\infty \neq M^*$. Set $U = \{x \in Y : \|x\|_\infty < M^* + 1\}$.

Then $\Phi : \overline{U} \to Y$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda\Phi(x)$ for $\lambda \in (0,1)$. As a consequence of the nonlinear aiternative of Leray- Schauder type [14], we deduce that Φ has a fixed point x in U, which is a solution of (1.1)-(1.2). □

4 Existence results for netural functional fractional differential equations with state-dependent delay

In this section, we study existence results for netural fractional differential equations with state-dependent delay of the form

$$^cD^q[x(t) - g(t,x_t)] = A[x(t) - g(t,x_t)] + f(t, x_{\rho(t,x_t)}), \quad t \in J = [0,a],\ 0 < q < 1, \tag{4.1}$$

$$x(t) = \phi(t) \in \mathcal{B},\ t \in (-\infty, 0], \tag{4.2}$$

where $A, f, \rho,$ and ϕ are same as defined in (1.1)-(1.2) and $g : J \times \mathcal{B} \to X$ is appropriate given function.

Definition 4.1. *A function $x : (-\infty, a] \to X$ is said to be a mild solution of (4.1)-(4.2) if $x_0 = \phi,\ x_{\rho(s,x_s)} \in \mathcal{B}$ for each $s \in J$ and*

$$x(t) = S_q(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_{\rho(s,x_s)})ds, \quad t \in J,$$

where

$$\begin{aligned} S_q(t) &= \int_0^\infty \xi_q(\theta) T(t^q\theta)d\theta, \\ T_q(t) &= q\int_0^\infty \theta\xi_q(\theta) T(t^q\theta)d\theta, \\ \xi_q(\theta) &= \frac{1}{q}\theta^{-1-\frac{1}{q}}\overline{w}_q(\theta^{-\frac{1}{q}}) \geq 0, \\ \overline{w}_q(\theta) &= \frac{1}{\pi}\sum_{k=1}^\infty (-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \theta \in (0,\infty), \end{aligned}$$

ξ_q is a probability density function on $(0,\infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0,\infty) \text{ and } \int_0^\infty \xi_q(\theta)d\theta = 1.$$

To prove the next theorems, in addition, we need the following hypotheses:

$(\mathbf{H_5})$ The function $g : J \times \mathcal{B} \to X$ is completely continuous and there exist positive constants c_1 and c_2 such that

$$\|g(t,\psi)\| \le c_1\|\psi\|_{\mathcal{B}} + c_2, \quad t \in J,\ \psi \in \mathcal{B}.$$

$(\mathbf{H_5})^*$ The function $g : J \times \mathcal{B} \to X$ is continuous and there exists $L_f > 0$ such that

$$\|g(t,\psi_1) - g(t,\psi_2)\| \le L_f\|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad t \in J,\ \psi_i \in \mathcal{B},\ i = 1,2.$$

Theorem 4.1. *Assume that the hypotheses $(\mathbf{H_1}) - (\mathbf{H_5})$ are fulfilled. If*

$$K_a\Big[L_f + \frac{qM}{\Gamma(1+q)}\liminf_{\xi\to\infty}\frac{\Omega(\xi)}{\xi}\int_0^a (t-s)^{q-1}p(s)ds\Big] < 1$$

then there exist a mild solution of (4.1)-(4.2) on J.

Proof. Let $\overline{\phi} : (-\infty, a] \to X$ be the extension of ϕ to $(-\infty, a]$ such that $\overline{\phi}(\theta) = \phi(0)$ on $J = [0, a]$. Consider the space $S(a) = \{u \in C(J;X) : u(0) = \phi(0)\}$ endowed with the uniform operator topology and define the operator $\Upsilon : S(a) \to S(a)$ by

$$\Upsilon x(t) = S_q(t)[\phi(0) - g(0,\phi(0)) + g(t,\overline{x}_t) + \int_0^t (t-s)^{q-1}T_q(t-s)f(s,\overline{x}_{\rho(s,x_s)})ds, \quad t \in J,$$

where $\overline{x} : (-\infty, a] \to X$ is such that $\overline{x}_0 = \phi$ and $\overline{x} = x$ on J. From our assumptions, it is easy to see that $\Upsilon S(a) \subset S(a)$.

We shall prove that there exists a $r > 0$ such that $\Upsilon(B_r(\overline{\phi}|_J, S(a))) \subset B_r(\overline{\phi}|_J, S(a))$. If this property is false, then for every $r > 0$ there exist $x^r \in B_r(\overline{\phi}|_J, S(a))$ and $t^r \in J$ such that $r < \|\Upsilon x^r(t^r) - \phi(0)\|$. Then from Lemma (2.1), we find,

$$\begin{aligned} r &\le \|\Upsilon x^r(t^r) - \phi(0)\| \\ &\le \|S_q(t^r)\phi(0) - \phi(0)\| + \|S_q(t^r)g(0,\phi) - g(0,\phi)\| + \|g(t,(\overline{x^r})_{t^r} - g(0,\phi)\| \\ &\quad + \int_0^{t^r} \big\|(t-s)^{q-1}T_q(t^r-s)f(s,\overline{x^r}_{\rho(s,(\overline{x^r})_s)})\big\|ds \\ &\le (M+1)H\|\phi\|_{\mathcal{B}} + \|S_q(t^r)g(0,\phi) - g(0,\phi)\| + L_f\Big(K_a r + (M_a + HK_a + 1)\|\phi\|_{\mathcal{B}}\Big) \\ &\quad + \frac{qM}{\Gamma(1+q)}\Omega\Big(\big(M_a + \overline{J}^{\phi}\big)\|\phi\|_{\mathcal{B}} + K_a\big(r + \|\phi(0)\|\big)\Big)\int_0^a (t-s)^{q-1}p(s)ds \end{aligned}$$

and hence

$$1 \le K_a\Big[L_f + \frac{qM}{\Gamma(1+q)}\liminf_{\xi\to\infty}\frac{\Omega(\xi)}{\xi}\int_0^a (t-s)^{q-1}p(s)ds\Big]$$

which contradicts our assumption.

Let $r > 0$ be such that $\Upsilon(B_r(\overline{\phi}|_J, S(a))) \subset B_r(\overline{\phi}|_J, S(a))$, in what follows, r^* is the number defined by $r^* := (M_a + \overline{J}^{\phi})\|\phi\|_{\mathcal{B}} + K_a\big(r + \|\phi(0)\|\big)$. To prove that Υ is condensing operator, we introduce the decomposition $\Upsilon = \Upsilon_1 + \Upsilon_2$, where

$$\begin{aligned} \Upsilon_1 x(t) &= S_q(t)[\phi(0) - g(0,\phi) + g(t,\overline{x}_t), \\ \Upsilon_2 x(t) &= \int_0^t (t-s)^{q-1}T_q(t-s)f(s,\overline{x}_{\rho(s,x_s)})ds, \quad t \in J. \end{aligned}$$

Step 1: $\Upsilon_1(\cdot)$ is contraction on $B_r(\overline{\phi}|_J, S(a))$.

If $x, y \in B_r(\overline{\phi}|_J, S(a))$ and $t \in J$, then we have

$$\begin{aligned} \|\Upsilon_1 x(t) - \Upsilon_1 y(t)\| &\le \|g(t,\overline{x}_t) - g(t,\overline{y}_t)\| \\ &\le L_f K_a\|x - y\|_a, \end{aligned}$$

which proves that $\Upsilon_1(\cdot)$ is a contraction on $B_r(\overline{\phi}|_J, S(a))$.

Next we prove that $\Upsilon_2(\cdot)$ is completely continuous from $B_r(\overline{\phi}|_J, S(a))$ into $B_r(\overline{\phi}|_J, S(a))$.

Step 2: Υ_2 is continuous on $B_r(\overline{\phi}|_J, S(a))$.

Let $\{x^n\} \subset B_r(\overline{\phi}|_J, S(a))$ and $x \in B_r(\overline{\phi}|_J, S(a))$ with $x^n \to x$ $(n \to \infty)$. From axiom A_1, it is easy to see that $(\overline{x^n})_s \to \overline{x}_s$ uniformly for $s \in (-\infty, a]$ as $n \to \infty$. By $(H3)$, we have

$$\begin{aligned}
&\|f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) - f(s, \overline{x}_{\rho(s,(\overline{x})_s)})\| \\
&\leq \|f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) - f(s, \overline{x}_{\rho(s,(\overline{x^n})_s})\| + \|f(s, \overline{x}_{\rho(s,(\overline{x^n})_s)}) - f(s, \overline{x}_{\rho(s,(\overline{x})_s)})\|
\end{aligned}$$

which implies that $f(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}) \to f(s, \overline{x}_{\rho(s,(\overline{x})_s})$ as $n \to \infty$ for each $s \in J$. By axiom A1(ii), Lemma(2.1) and the dominated convergence theorem we obtain

$$\begin{aligned}
\|\Upsilon_2 x^n - \Upsilon_2 x\| \leq \Big\| \int_0^t (t-s)^{q-1} T_q(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds \Big\| \\
\to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

Therefore, Υ_2 is continuous.

Step 3: $\Upsilon_2(\cdot)$ is equicontinuous on J.

Let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$ and B_r be a bounded set as in step 2. Let $\epsilon > 0$ be given. For each $t \in J$, we have

$$\begin{aligned}
&\|\Upsilon_2(x)(\tau_2) - \Upsilon_2(x)(\tau_1)\| \\
&\leq \int_0^{\tau_1-\epsilon} \Big\| \Big[(\tau_2-s)^{q-1} T_q(\tau_2-s) - (\tau_1-s)^{q-1} T_q(\tau_1-s) \Big] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\
&+ \int_{\tau_1-\epsilon}^{\tau_1} \Big\| \Big[(\tau_2-s)^{q-1} T_q(\tau_2-s) - (\tau_1-s)^{q-1} T_q(\tau_1-s) \Big] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\
&+ \int_{\tau_1}^{\tau_2} \Big\| (\tau_2-s)^{q-1} T_q(\tau_2-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \Big\| ds \\
&\leq \Omega(r^*) \Big[\int_0^{\tau_1-\epsilon} \Big[|(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}| \|T_q(\tau_2-s)\| \\
&+ |(\tau_1-s)^{q-1}| \|T_q(\tau_2-s) - T_q(\tau_1-s)\| \Big] p(s) ds \\
&+ \int_{\tau_1-\epsilon}^{\tau_1} \Big[|(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}| \|T_q(\tau_2-s)\| \\
&+ |(\tau_1-s)^{q-1}| \|T_q(\tau_2-s) - T_q(\tau_1-s)\| \Big] p(s) ds + \frac{qM}{\Gamma(1+q)} \int_{\tau_1}^{\tau_2} |(\tau_2-s)^{q-1}| p(s) ds \Big].
\end{aligned}$$

The right hand side tends to zero as $\tau_2 - \tau_1 \to 0$, since $T_q(t),\ t \geq 0$ is a strongly continuous semigroup and $T_q(t)$ is compact for $t > 0$ (so $T_q(t)$ is continuous in the uniform operator topology for $t > 0$). The equicontinuous for the other cases $\tau_1 < \tau_2 \leq 0$ or $\tau_1 \leq 0 \leq \tau_2 \leq a$ are very simple.

Step 4: Υ_2 is precompact.

Let $0 < t \leq s \leq a$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$, and $\delta > 0$. For $x \in B_r$, we define,

$$\begin{aligned}
\Upsilon_{2_{\epsilon,\delta}}(x)(t) &= q \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \\
&= T(\epsilon^q \delta) q \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta - \epsilon^q \delta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds,
\end{aligned}$$

Since $T(\epsilon^q \delta)$ is a compact operator for $\epsilon^q \delta > 0$, the set $V_{\epsilon,\delta}(t) = \{\Upsilon_{2_{\epsilon,\delta}}(x)(t) : x \in B_r\}$ is precompact in X for every $\epsilon,\ 0 < \epsilon < t$. Moreover

$$\begin{aligned}
&\|\Upsilon_2(x)(t) - \Upsilon_{2_{\epsilon,\delta}}(x)(t)\| \\
&= q \Big[\Big\| \int_0^{t-\epsilon} \int_0^\delta \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \Big\| \\
&\quad + \Big\| \int_{t-\epsilon}^t \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T((t-s)^q \theta) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) d\theta ds \Big\| \Big]
\end{aligned}$$

$$\leq \Omega(r^*)\frac{qM}{\Gamma(1+q)}\Big[\int_0^{t-\epsilon}(t-s)^{q-1}p(s)ds\int_0^{\delta}\theta\xi_q(\theta)d\theta+\int_{t-\epsilon}^{t}(t-s)^{q-1}p(s)ds\int_{\delta}^{\infty}\theta\xi_q(\theta)d\theta\Big].$$

Therefore, there are precompact sets arbitrarly close to the set $V(t)=\{\Upsilon_{2_{\epsilon,\delta}}(x)(t):x\in B_r\}$ is precompact. Hence the set $V(t)=\{\Upsilon_{2_{\epsilon,\delta}}(x)(t):x\in B_r\}$ is precompact in X.

As a consequence of the Step 2 to Step 4 and the Arzela-Ascoli theorem, we can conclude that the operator Υ_2 is completely continuous.

These arguments enable us to conclude that $\Upsilon=\Upsilon_1+\Upsilon_2$ is a condensing mapping on $B_r(\overline{\phi}|_J,S(a))$ and the existence of a mild solution for (4.1)-(4.2) is now a consequence of [[30], Theorem 4.3.2]. This completes the proof. □

Theorem 4.2. *Assume that the hypotheses* $(\mathbf{H_1})-(\mathbf{H_5})$ *and* $(\mathbf{H_5})^*$ *are fulfilled with* $\rho(t,\psi)\leq t$ *for every* $t\in J,\ \psi\in\mathcal{B}$. *If*

$$\frac{\|\xi\|_\infty}{(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+\frac{K_a}{1-c_1}\big[M(H+c_1)\|\phi\|_{\mathcal{B}}+c_2(1+M)+M\Omega(\|\xi\|_\infty)\|I^qp\|_\infty\big]}>1$$

then there exists a mild solution of (4.1)-(4.2) on J.

Proof. Let Υ be a function given in the proof of Theorem 4.1.

We show that there exists an open set $U_1\subset S(a)$ with $x\neq\lambda\Upsilon(x)$ for $\lambda\in(0,1)$ and $x\in\partial U_1$. Let $x\in S(a)$ and $x=\lambda\Upsilon(x)$ for some $0<\lambda<1$. Then

$$x(t)=\lambda\Big[S_q(t)[\phi(0)-g(0,\phi)+g(t,\overline{x}_t)+\int_0^t(t-s)^{q-1}T_q(t-s)f(s,\overline{x}_{\rho(s,x_s)})ds\Big],\quad t\in J,$$

and

$$\begin{aligned}|x(t)|&\leq MH\|\phi\|_{\mathcal{B}}+M[c_1\|\phi\|_{\mathcal{B}}+c_2]+c_1\|\overline{x}_t\|_{\mathcal{B}}+c_2\\&\quad+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega(\|\overline{x}_{\rho(s,(\overline{x}))}\|_{\mathcal{B}})ds\\&\leq M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]+c_1\|\overline{x}_t\|_{\mathcal{B}}\\&\quad+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega\Big((M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+K_a\|\overline{x}\|\Big)ds.\end{aligned}$$

If $\mu(t)=\sup\{|x(s)|:s\in[0,t]\}$ then

$$\begin{aligned}\mu(t)&\leq M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]+c_1\mu(t)\\&\quad+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega\Big((M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+K_a\mu(s)\Big)ds.\end{aligned}$$

Since $0<c_1<1$, we have

$$\begin{aligned}\mu(t)\leq\frac{1}{1-c_1}\Big[&M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]\\&+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega\Big((M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+K_a\mu(s)\Big)ds\Big],\quad t\in J.\end{aligned}$$

If $\xi(t)=(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+K_a\mu(s)$ then we have

$$\begin{aligned}\xi(t)&=(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+\frac{K_a}{1-c_1}\Big[M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]\\&\quad+\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}p(s)\Omega(\xi(s))ds\Big]\\&\leq(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+\frac{K_a}{1-c_1}\Big[M[H+c_1]\|\phi\|_{\mathcal{B}}+c_2[1+M]+M\Omega(\|\xi\|_\infty)\|I^qp\|_\infty\Big].\end{aligned}$$

Consequently,

$$\frac{\|\xi\|_\infty}{(M_a+\overline{J}^{\phi})\|\phi\|_{\mathcal{B}}+\frac{K_a}{1-c_1}\big[M(H+c_1)\|\phi\|_{\mathcal{B}}+c_2(1+M)+M\Omega(\|\xi\|_\infty)\|I^qp\|_\infty\big]}\leq 1.$$

Now, there exist L^* such that $\|x\|_\infty \neq L^*$. set

$$U_1 = \{x \in Y : \|x\|_\infty < L^* + 1\}.$$

From the choice of U_1 there is no $x \in \partial U_1$ such that $x = \lambda \Upsilon(x)$ for $\lambda \in (0,1)$.

To prove that Υ is completely continuous on $S(a)$, we introduce the decomposition $\Upsilon = \Upsilon_1 + \Upsilon_2$ introduced in the proof of the Theorem 4.1. From the proof of Theorem 4.1, we obtain that Υ_2 is completely continuous on $S(a)$ and from the condition $(\mathbf{H_5})$ it follows that Υ_1 is completely continuous on $S(a)$. As a consequence of the nonlinear alternative of Leray-Schauder type [14], we deduce that Υ has a fixed point x in U_1. Then Υ has a fixed poinf, whixh is a solution of (4.1)-(4.2). $\square$

5 Example

In this section, we consider an applications of our abstract results. At first we introduce the required technical framework. In the rest of this secion, $X = L^2([0,\pi])$ and $A : D(A) \subset X \to X$ be the operator $Aw = w''$ with domain $D(A) := \{w \in X : w'' \in X,\ w(0) = w(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup on X.

Then

$$Aw = -\sum_{n=1}^{\infty} n^2 < w, e_n > e_n, \quad w \in D(A),$$

where $e_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi),\ 0 \leq \xi \leq \pi,\ n = 1, 2, \ldots$. Clearly A generates a compact semigroup $T(t), t > 0$ in X and it is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} < w, e_n > e_n, \text{ for every } w \in X.$$

Clearly the assumption $(\mathbf{H_1})$ is satisfied. Consider the fractional differential system

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t,\xi) = \frac{\partial^2}{\partial \xi^2} u(t,\xi) + \int_{-\infty}^{t} a_2(s-t) u(s - \rho_1(t)\rho_2(\|u(t)\|), \xi) ds,\ t \in J,\ \xi \in [0,\pi], \tag{5.1}$$

submitted to the conditions

$$u(t,0) = u(t,\pi) = 0, \quad t \geq 0, \tag{5.2}$$
$$u(\theta,\xi) = \varphi(\theta,\xi), \quad \theta \leq 0,\ 0 \leq \xi \leq \pi, \tag{5.3}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is a Caputo fractional partial derivative of order $0 < \alpha < 1$. In the sequel, $\mathcal{B} = C_0 \times L^2(g, X)$ is the space introduced in [20]; $\varphi \in \mathcal{B}$ with the identification $\varphi(s)(\theta) = \varphi(s,\theta)$.

To treat this system, we assume that $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$, are continuous functions and the following condition.

(a) The functions $a_1 : \mathbb{R} \to \mathbb{R}$, are continuous and $L_1 = \left(\int_{-\infty}^{0} \frac{(a_1(s))^2}{g(s)} ds\right)^{1/2} < \infty$.

Under these conditions, we can define the operators $f : J \times \mathcal{B} \to X$, and $\rho : J \times \mathcal{B} \to \mathbb{R}$ by

$$f(t,\varphi)(\xi) = \int_{-\infty}^{0} a_1(s)\varphi(s,\xi) ds, \tag{5.4}$$
$$\rho(s,\varphi) = s - \rho_1(s)\rho_2(\| \varphi(0) \|), \tag{5.5}$$

which permit to transform system (5.1)-(5.3) into the abstract Cauchy problem (1.1)-(1.2). Moreover, the maps f is bounded linear operators with $\| f \|_{\mathcal{L}(\mathcal{B},X)} \leq L_1$. The following result is a direct consequence of Theorem 3.1.

Proposition 5.1. *Let $\varphi \in \mathcal{B}$ be such that condition (H4) holds. Then there exists a mild solution of (5.1)-(5.3).*

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