

Semilinear functional differential equations with fractional order and finite delay

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Abstract

In this paper, we establish sufficient conditions for existence and uniqueness of solutions for semilinear functional differential equations with finite delay involving the Riemann-Liouville fractional derivative. Our approach is based on resolvent operators, the Banach contraction principle, and the nonlinear alternative of Leray-Schauder type.

Keywords: Semilinear functional differential equation, fractional derivative, fractional integral, fixed point, mild solutions, resolvent operator.

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1 Introduction

This paper is concerned with existence of solutions defined on a compact real interval for fractional order semilinear functional differential equations of the form

$$D^\alpha y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b], \quad 0 < \alpha < 1, \quad (1.1)$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad (1.2)$$

where D^α is the standard Riemann-Liouville fractional derivative, $f : J \times C([-r, 0], E) \rightarrow E$ is a continuous function, $A : D(A) \subset E \rightarrow E$ is a densely defined closed linear operator on E , $\phi : [-r, 0] \rightarrow E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Differential equations with fractional order have recently proved to be valuable tools for the description of hereditary properties of various materials and systems. For more details, see [9].

Fractional calculus appears in rheology, viscoelasticity, electrochemistry, electromagnetism, etc. For details, see the monographs of Kilbas et al. [8], Miller and Ross [10], Podlubny [13], Oldham et al. [12]. For some recent developments on the subject, see for instance [1, 2, 3, 4, 7, 11] and references cited therein.

The purpose of this paper is to study the existence and uniqueness of mild solutions for (1.1)-(1.2) by virtue of resolvent operator. In Section 2 we recall some definitions and preliminary facts which will be used in the sequel. In Section 3, we give our main existence and uniqueness results. An example will be presented in the last section illustrating the abstract theory.

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2 Preliminaries

In this section, we recall some definitions and propositions of fractional calculus and resolvent operators. Let E be a Banach space. By $C(J, E)$ we denote the Banach space of continuous functions from J into E with the norm

$$\|y\|_{\infty} = \sup\{|y(t)| : t \in J\}.$$

For $\phi \in C([-r, b], E)$ the norm of ϕ is defined by

$$\|\phi\|_{\mathfrak{D}} = \sup\{|\phi(\theta)| : \theta \in [-r, b]\}.$$

$C([-r, 0], E)$ is endowed with norm defined by

$$\|\psi\|_C = \sup\{|\psi(\theta)| : \theta \in [-r, 0]\}.$$

$\mathcal{L}(E)$ denotes the space of bounded linear operators from E into E , with norm

$$\|N\|_{\mathcal{L}(E)} = \sup\{|N(y)| : |y| = 1\}.$$

Definition 2.1. [8, 13] *The Riemann-Liouville fractional primitive of order $\alpha \in \mathbb{R}^+$ of a function $h : (0, b] \rightarrow E$ is defined by*

$$I_0^{\alpha} h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

provided the right hand side exists pointwise on $(0, b]$, where Γ is the gamma function.

Definition 2.2. [8, 13] *The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a continuous function $h : (0, b] \rightarrow E$ is defined by*

$$\begin{aligned} \frac{d^{\alpha} h(t)}{dt^{\alpha}} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds \\ &= \frac{d}{dt} I_0^{1-\alpha} h(t). \end{aligned}$$

Consider the fractional differential equation

$$D^{\alpha} y(t) = Ay(t) + f(t), \quad t \in J, \quad 0 < \alpha < 1, \quad y(0) = 0, \quad (2.1)$$

where A is a closed linear unbounded operator in E and $f \in C(J, E)$. Equation (2.1) is equivalent to the following integral equation [8]

$$y(t) = \frac{1}{\Gamma(\alpha)} A \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t \in J. \quad (2.2)$$

This equation can be written in the following form of integral equation

$$y(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ay(s) ds, \quad t \geq 0, \quad (2.3)$$

where

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \quad (2.4)$$

Examples where the exact solution of (2.1) and the integral equation (2.2) are the same, are given in [4]. Let us assume that the integral equation (2.3) has an associated resolvent operator $(S(t))_{t \geq 0}$ on E .

Next we define the resolvent operator of the integral equation (2.3).

Definition 2.3. [14, Definition 1.1.3] *A one parameter family of bounded linear operators $(S(t))_{t \geq 0}$ on E is called a resolvent operator for (2.2) if the following conditions hold:*

- (a) $S(\cdot)x \in C([0, \infty), E)$ and $S(0)x = x$ for all $x \in E$;
- (b) $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and every $t \geq 0$;

(c) for every $x \in D(A)$ and $t \geq 0$,

$$S(t)x = x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} AS(s)x ds. \tag{2.5}$$

Here and hereafter we assume that the resolvent operator $(S(t))_{t \geq 0}$ is analytic [14, Chapter 2], and there exist a function $\phi_A \in L^1_{loc}([0, \infty), \mathbb{R}^+)$ such that $\|S'(t)x\| \leq \phi_A(t)\|x\|_{[D(A)]}$ for all $t > 0$ and each $x \in D(A)$.

We have the following concept of solution using Definition 1.1.1 in [14].

Definition 2.4. A function $u \in C(J, E)$ is called a mild solution of the integral equation (2.3) on J if $\int_0^t (t-s)^{\alpha-1} u(s) ds \in D(A)$ for all $t \in J$, $h(t) \in C(J, E)$ and

$$u(t) = \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + h(t), \quad \forall t \in J.$$

The next result follows from [14, Proposition I.1.2, Theorem II.2.4, Corollary II.2.6].

Lemma 2.1. Under the above conditions the following properties are valid.

(i) If $u(\cdot)$ is a mild solution of (2.3) on J , then the function $t \rightarrow \int_0^t S(t-s)h(s)ds$ is continuously differentiable on J , and

$$u(t) = \frac{d}{dt} \int_0^t S(t-s)h(s)ds, \quad \forall t \in J.$$

(ii) If $h \in C^\beta(J, E)$ for some $\beta \in (0, 1)$, then the function defined by

$$u(t) = S(t)(h(t) - h(0)) + \int_0^t S'(t-s)[h(s) - h(t)]ds + S(t)h(0), \quad t \in J,$$

is a mild solution of (2.3) on J .

(iii) If $h \in C(J, [D(A)])$ then the function $u : J \rightarrow E$ defined by

$$u(t) = \int_0^t S'(t-s)h(s)ds + h(t), \quad t \in J,$$

is a mild solution of (2.3) on J .

3 Main Results

In this section we give our main existence results for problem (1.1)-(1.2). This problem is equivalent to the following integral equation

$$y(t) = \begin{cases} \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds, & t \in J, \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

Motivated by Lemma 2.1 and the above representation, we introduce the concept of mild solution.

Definition 3.1. We say that a continuous function $y : [-r, b] \rightarrow E$ is a mild solution of problem (1.1)-(1.2) if:

1. $\int_0^t (t-s)^{\alpha-1} y(s) ds \in D(A)$ for $t \in J$,
2. $y(t) = \phi(t)$, $t \in [-r, 0]$, and
3. $y(t) = \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds$, $t \in J$.

Suppose that there exists a resolvent $(S(t))_{t \geq 0}$ which is differentiable and the function f is continuous. Then by Lemma 2.1 (iii), if $y : [-r, b] \rightarrow E$ is a mild solution of (1.1)-(1.2), then

$$y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau) d\tau \right) ds, & t \in J, \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

Our first existence result for problem (1.1)-(1.2) is based on the Banach's contraction principle.

Theorem 3.1. *Let $f : J \times C([-r, 0], E) \rightarrow E$ be continuous and there exists a constant $L > 0$ such that*

$$|f(t, u) - f(t, v)| \leq L \|u - v\|_C, \quad \text{for } t \in J \text{ and } u, v \in C([-r, 0], E).$$

If

$$\frac{Lb^\alpha}{\Gamma(\alpha+1)} (1 + \|\phi_A\|_{L^1}) < 1, \quad (3.1)$$

then the problem (1.1)-(1.2) has a unique mild solution on $[-r, b]$.

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $F : C([-r, b], E) \rightarrow C([-r, b], E)$ defined by:

$$F(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau) d\tau \right) ds, & t \in [0, b]. \end{cases}$$

We need to prove that F has a fixed point, which is a unique mild solution of (1.1)-(1.2) on $[-r, b]$. We shall show that F is a contraction. Let $y, z \in C([-r, b], E)$. For $t \in [0, b]$, we have

$$\begin{aligned} & |F(y)(t) - F(z)(t)| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, y_s) - f(s, z_s)] ds \right. \\ &\quad \left. + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} [f(\tau, y_\tau) - f(\tau, z_\tau)] d\tau \right) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y_s) - f(s, z_s)| ds \\ &\quad + \int_0^t \phi_A(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} |f(\tau, y_\tau) - f(\tau, z_\tau)| d\tau ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L \|y_\tau - z_\tau\|_C ds + \frac{1}{\Gamma(\alpha)} \int_0^t \phi_A(t-s) \int_0^s (s-\tau)^{\alpha-1} L \|y_\tau - z_\tau\|_C d\tau ds \\ &\leq \frac{L}{\Gamma(\alpha)} \|y - z\|_{\mathfrak{D}} \int_0^t (t-s)^{\alpha-1} ds + \frac{L}{\Gamma(\alpha)} \|y - z\|_{\mathfrak{D}} \int_0^t \phi_A(t-s) \int_0^s (s-\tau)^{\alpha-1} d\tau ds \\ &\leq \frac{Lb^\alpha}{\Gamma(\alpha+1)} \|y - z\|_{\mathfrak{D}} + \frac{\|\phi_A\|_{L^1} Lb^\alpha}{\Gamma(\alpha+1)} \|y - z\|_{\mathfrak{D}}. \end{aligned}$$

Taking the supremum over $t \in [-r, b]$, we get

$$\|F(y) - F(z)\|_{\mathfrak{D}} \leq \frac{Lb^\alpha}{\Gamma(\alpha+1)} (1 + \|\phi_A\|_{L^1}) \|y - z\|_{\mathfrak{D}}.$$

By (3.1) F is a contraction and thus, by the contraction mapping theorem, we deduce that F has a unique fixed point. This fixed point is the mild solution of (1.1)-(1.2). \square

Next, we give an existence result based upon the following nonlinear alternative of Leray-Schauder applied to completely continuous operators [5].

Theorem 3.2. *Let E a Banach space, and $U \subset E$ convex with $0 \in U$. Let $F : U \rightarrow U$ be a completely continuous operator. Then either*

- (a) F has a fixed point, or
- (b) The set $\mathcal{E} = \{x \in U : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded.

Our main result here reads:

Theorem 3.3. *Let $f : J \times C([-r, 0], E) \rightarrow E$ be continuous. Assume that:*

- (i). $S(t)$ is compact for all $t > 0$;
- (ii). there exist functions $p, q \in C(J, \mathbb{R}_+)$ such that

$$|f(t, u)| \leq p(t) + q(t)\|u\|_C, \quad t \in J \text{ and } u \in C([-r, 0], E).$$

Then, the problem (1.1)-(1.2) has at least one mild solution on $[-r, b]$, provided that

$$\frac{b^\alpha \|q\|_\infty}{\Gamma(\alpha + 1)} (1 + \|\phi_A\|_{L^1}) < 1.$$

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $F : C([-r, b], E) \rightarrow C([-r, b], E)$ defined in Theorem 3.1, namely,

$$F(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau) d\tau \right) ds, & t \in [0, b]. \end{cases}$$

In order to prove that F is completely continuous, we divide the operator F into two operators:

$$F_1(y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds,$$

and

$$F_2(y)(t) = \int_0^t S'(t-s) F_1(y)(s) ds.$$

We prove that F_1 and F_2 are completely continuous. We note that the condition (i) implies that $S'(t)$ is compact for all $t > 0$ (see [6, Lemma 2.2]).

Step 1: F_1 is completely continuous.

At first, we prove that F_1 is continuous. Let $\{y_n\}$ be a sequence such that $y_n \rightarrow y$ as $n \rightarrow \infty$ in $C([-r, b], E)$. Then for $t \in [0, b]$ we have

$$\begin{aligned} |F_1(y_n)(t) - F_1(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, y_{ns}) - f(s, y_s) \right| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \|f(\cdot, y_n) - f(\cdot, y)\|_\infty \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{b^\alpha}{\Gamma(\alpha + 1)} \|f(\cdot, y_n) - f(\cdot, y)\|_\infty. \end{aligned}$$

Since f is a continuous function, we have

$$\|F_1(y_n) - F_1(y)\|_{\mathfrak{D}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus F_1 is continuous.

Next, we prove that F_1 maps bounded sets into bounded sets in $C([-r, b], E)$. Indeed, it is enough to show that for any $\rho > 0$, there exists a positive constant δ such that for each $y \in B_\rho = \{y \in C([-r, b], E) : \|y\|_{\mathfrak{D}} \leq \rho\}$ one has $F_1(y) \in B_\delta$. Let $y \in B_\rho$. Since f is a continuous function, we have for each $t \in [0, b]$

$$|F_1(y)(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds \right|$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y_s)| ds \\
&\leq \frac{b^\alpha}{\Gamma(\alpha+1)} (\|p\|_\infty + \rho \|q\|_\infty) = \delta^* < \infty.
\end{aligned}$$

Then, $\|F_1(y)\|_{\mathfrak{D}} = \max\{\|\phi\|_C, \delta^*\} = \delta$, and hence $F_1(y) \in B_\delta$.

Now, we prove that F_1 maps bounded sets into equicontinuous sets of $C([-r, b], E)$. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set. Let $y \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned}
&|F_1(y)(\tau_2) - F_1(y)(\tau_1)| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} f(s, y_s) ds - \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} f(s, y_s) ds \right| \\
&\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1 - \epsilon} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] f(s, y_s) ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \int_{\tau_1 - \epsilon}^{\tau_1} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] f(s, y_s) ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} f(s, y_s) ds \right| \\
&\leq \frac{\|p\|_\infty + \rho \|q\|_\infty}{\Gamma(\alpha)} \left(\int_0^{\tau_1 - \epsilon} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] ds \right. \\
&\quad \left. + \int_{\tau_1 - \epsilon}^{\tau_1} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] ds + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} ds \right).
\end{aligned}$$

As $\tau_1 \rightarrow \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_1 maps B_ρ into a precompact set in E .

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$ we define

$$F_{1\epsilon}(y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} f(s, y_s) ds.$$

Note that the set

$$\left\{ \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} f(s, y_s) ds : y \in B_\rho \right\}$$

is bounded since

$$\begin{aligned}
\left| \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} f(s, y_s) ds \right| &\leq (\|p\|_\infty + \rho \|q\|_\infty) \left| \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} ds \right| \\
&\leq \frac{\|p\|_\infty + \rho \|q\|_\infty}{\Gamma(\alpha+1)} (t-\epsilon)^\alpha.
\end{aligned}$$

Then for $t > 0$, the set

$$Y_\epsilon(t) = \{F_{1\epsilon}(y)(t) : y \in B_\rho\}$$

is precompact in E for every ϵ , $0 < \epsilon < t$. Moreover

$$\begin{aligned}
|F_1(y)(t) - F_{1\epsilon}(y)(t)| &\leq \frac{\|p\|_\infty + \rho \|q\|_\infty}{\Gamma(\alpha)} \left(\int_0^{t-\epsilon} [(t-s)^{\alpha-1} - (t-s-\epsilon)^{\alpha-1}] ds + \int_{t-\epsilon}^t (t-s)^{\alpha-1} ds \right) \\
&\leq \frac{\|p\|_\infty + \rho \|q\|_\infty}{\Gamma(\alpha+1)} (t^\alpha - (t-\epsilon)^\alpha).
\end{aligned}$$

Therefore, the set $Y(t) = \{F_1(y)(t) : y \in B_\rho\}$ is precompact in E . Hence the operator F_1 is completely continuous.

Step 2: F_2 is completely continuous.

The operator F_2 is continuous, since $S'(\cdot) \in C([0, b], \mathcal{L}(E))$ and F_1 is continuous as proved in Step 1.

Now, let B_ρ be a bounded set as in Step 1. For $y \in B_\rho$ we have

$$|F_2(y)(t)| \leq \int_0^t |S'(t-s)| |F_1(y)(s)| ds$$

$$\begin{aligned} &\leq \int_0^t \phi_A(t-s) \|F_1(y)(s)\|_{[D(A)]} ds \\ &\leq \frac{\|\phi\|_{L^1} b^\alpha (\|p\|_\infty + \rho \|q\|_\infty)}{\Gamma(\alpha+1)} = \delta'. \end{aligned}$$

Thus, there exists a positive number δ' such that $\|F_2(y)\|_{\mathfrak{D}} \leq \delta'$. This means that $F_2(y) \in B_{\delta'}$.

Next, we shall show that F_2 maps bounded sets into equicontinuous sets in $C([-r, b], E)$. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set as in Step 1. Let $y \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned} &|F_2(y)(\tau_2) - F_2(y)(\tau_1)| \\ &= \left| \int_0^{\tau_2} S'(\tau_2 - s) F_1(y)(\tau_2) ds - \int_0^{\tau_1} S'(\tau_1 - s) F_1(y)(\tau_1) ds \right| \\ &\leq \frac{b^\alpha (\|p\|_\infty + \rho \|q\|_\infty)}{\Gamma(\alpha+1)} \left(\int_0^{\tau_1 - \epsilon} |S'(\tau_2 - s) - S'(\tau_1 - s)| ds \right. \\ &\quad \left. + \int_{\tau_1 - \epsilon}^{\tau_1} |S'(\tau_2 - s) - S'(\tau_1 - s)| ds + \int_{\tau_1}^{\tau_2} |S'(\tau_2 - s)| ds \right). \end{aligned}$$

As $\tau_1 \rightarrow \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_2 maps B_ρ into a precompact set in E .

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$ we define

$$F_{2\epsilon}(y)(t) = S'(\epsilon) \int_0^{t-\epsilon} S'(t-s-\epsilon) F_1(y)(s) ds.$$

Since $S'(t)$ is a compact operator for $t > 0$, the set

$$Y_\epsilon(t) = \{F_{2\epsilon}(y)(t) : y \in B_\rho\}$$

is precompact in E for every ϵ , $0 < \epsilon < t$. Moreover

$$\left| F_2(y)(t) - F_{2\epsilon}(y)(t) \right| \leq \frac{\|\phi_A\|_{L^1} (\|p\|_\infty + \rho \|q\|_\infty)}{\Gamma(\alpha+1)} (t^\alpha - (t-\epsilon)^\alpha).$$

Then $Y(t) = \{F_2(y)(t) : y \in B_\rho\}$ is precompact in E . Hence the operator F_2 is completely continuous.

Step 3: A priori bound on solutions.

Now, it remains to show that the set

$$\mathcal{E} = \{y \in \mathcal{C}([-r, b], E) : y = \lambda F(y), \quad 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}$ be any element. Then, for each $t \in [0, b]$,

$$y(t) = \lambda F(y)(t) = \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds + \lambda \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau) d\tau \right) ds.$$

Then

$$\begin{aligned} |y(t)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau) d\tau \right) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y_s)| ds + \int_0^t \phi_A(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} |f(\tau, y_\tau)| d\tau ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\|p\|_\infty + \|q\|_\infty \|y_s\|_C] ds \\ &\quad + \int_0^t \phi_A(t-s) \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} [\|p\|_\infty + \|q\|_\infty \|y_s\|_C] d\tau ds \\ &\leq \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)} \|y_s\|_C + \frac{\|\phi_A\|_{L^1} b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{\|\phi_A\|_{L^1} b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)} \|y_s\|_C \\ &\leq \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} (1 + \|\phi_A\|_{L^1}) + \frac{b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)} (1 + \|\phi_A\|_{L^1}) \|y\|_{\mathfrak{D}}, \end{aligned}$$

and consequently

$$\|y\|_{\mathfrak{D}} \leq \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha + 1)} (1 + \|\phi_A\|_{L^1}) \left\{ 1 - \frac{b^\alpha \|q\|_\infty}{\Gamma(\alpha + 1)} (1 + \|\phi_A\|_{L^1}) \right\}^{-1}.$$

Hence the set \mathcal{E} is bounded. As a consequence of Theorem 3.2 we deduce that F has at least a fixed point which gives rise to a mild solution of problem (1.1)-(1.2) on $[-r, b]$. \square

4 Example

As an application of our results we consider the following fractional time partial functional differential equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + Q(t, u(t-r, x)), \quad x \in [0, \pi], t \in [0, b], \alpha \in (0, 1), \quad (4.1)$$

$$u(t, 0) = u(t, \pi) = 0, \quad t \in [0, b], \quad (4.2)$$

$$u(t, x) = \phi(t, x), \quad x \in [0, \pi], \quad t \in [-r, 0], \quad (4.3)$$

where $r > 0$, $\phi : [-r, 0] \times [0, \pi] \rightarrow \mathbb{R}$ is continuous and $Q : [0, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function.

To study this system, we take $E = L^2[0, \pi]$ and let A be the operator given by $Aw = w''$ with domain $D(A) = \{w \in E, w, w' \text{ are absolutely continuous, } w'' \in E, w(0) = w(\pi) = 0\}$.

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A),$$

where (\cdot, \cdot) is the inner product in L^2 and $w_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(nx)$, $n = 1, 2, \dots$ is the orthogonal set of eigenvectors of A . It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on E and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} (w, w_n) w_n, \quad w \in E.$$

From these expressions it follows that $(T(t))_{t \geq 0}$ is uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is compact operator for all $\lambda \in \rho(A)$.

From [14, Example 2.2.1] we know that the integral equation

$$u(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s) ds, \quad s \geq 0,$$

has an associated analytic resolvent operator $(S(t))_{t \geq 0}$ on E given by

$$S(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda, & t > 0, \\ I, & t = 0, \end{cases}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in (\pi, \frac{\pi}{2})$. $S(t)$ is differentiable (Proposition 2.15 in [3]) and there exists a constant $M > 0$ such that $\|S'(t)x\| \leq M\|x\|$, for $x \in D(A)$, $t > 0$.

To represent the differential system (4.1) – (4.3) in the abstract form (1.1)-(1.2), let

$$\begin{aligned} y(t)(x) &= u(t, x), \quad t \in [0, b], \quad x \in [0, \pi] \\ \phi(\theta)(x) &= \phi(\theta, x), \quad \theta \in [-r, 0], \quad x \in [0, \pi] \\ f(t, \phi)(x) &= Q(t, \phi(\theta, x)), \quad \theta \in [-r, 0], \quad x \in [0, \pi] \end{aligned}$$

Choose b such that

$$\frac{Lb^\alpha}{\Gamma(\alpha + 1)} (1 + M) < 1.$$

Since the conditions of Theorem 3.1 are satisfied, there is a function $u \in C([-r, b], L^2[0, \pi])$ which is a mild solution of (4.1)-(4.3).

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