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Additive functional equation and inequality are stable in Banach space and its applications

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Abstract

In this paper, the authors established the solution of the additive functional equation and inequality

$$f(x) + f(y + z) - f(x + y) = f(z)$$

and

$$||f(x) + f(y + z) - f(x + y)|| \leq ||f(z)||.$$

We also prove that the above functional equation and inequality are stable in Banach space in the sense of Ulam, Hyers, Rassias. An application of this functional equation is also studied.

Keywords: Additive functional equations, generalized Hyers - Ulam - Rassias stability.

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1 Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [21] concerning the stability of group homomorphisms. D.H. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [20] for linear mappings by considering an unbounded Cauchy difference.

The paper of Th.M. Rassias [20] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Th.M. Rassias theorem was obtained by P. Gavruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach.

In 1982, J.M. Rassias [14] followed the innovative approach of the Th.M. Rassias theorem [20] in which he replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in R$ with $p + q = 1$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [19] by considering the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 4, 5, 6, 8, 9, 11, 12, 16, 17]).

The solution and stability of the following additive functional equations

$$f(x + y) = f(x) + f(y), \tag{1.1}$$

$$f(2x - y) + f(x - 2y) = 3f(x) - 3f(y), \tag{1.2}$$

$$f(x + y - 2z) + f(2x + 2y - z) = 3f(x) + 3f(y) - 3f(z), \tag{1.3}$$

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$$f(2x \pm y \pm z) = f(x \pm y) + f(x \pm z), \tag{1.4}$$

were discussed in [1, 3, 13, 18].

One of the most famous functional equations is the additive functional equation (1.1). In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of A.L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function.

It is well known that if an additive function $f : R \to R$ satisfies one of the following conditions:

(a) f is continuous at a point;

(b) f is monotonic on an interval of positive length;

(c) f is bounded on an interval of positive length;

(d) f is integrable;

(e) f is measurable,

then f is of the form $f(x) = cx$ with a real constant c. That is to say f has the linearity. That is, if a solution of the additive equation (1.1) satisfies one of the very weak conditions (a) to (e), then it does have the linearity. But every additive functional which is not linear displays a very strange behavior. More precisely, the graph of every additive functional $f : R \to R$ which is not of the form $f(x) = cx$ is dense in R^2 .

In this paper, the authors established the solution and generalized Ulam-Hyers stability of the additive functional equation and inequality

$$f(x) + f(y + z) - f(x + y) = f(z) \tag{1.5}$$

$$||f(x) + f(y + z) - f(x + y)|| \leq ||f(z)||. \tag{1.6}$$

In Section 2, we proved the general solution of (1.5) and (1.6) is provided.
In Section 3, the generalized Ulam-Hyers stability of the functional equation (1.5) is investigated.
The generalized Ulam-Hyers stability of the functional inequality (1.6) is discussed in section 4.
In Section 5, the application of functional equation (1.5) is studied.

2 General Solution of (1.5) and (1.6)

In this section, the general solution of (1.5) and (1.6) are given. Through out this section let X and Y be real vector spaces.

Theorem 2.1. *The mapping $f : X \to Y$ satisfies the functional equation*

$$f(x + y) = f(x) + f(y) \tag{2.1}$$

if and only if $f : X \to Y$ satisfies the functional equation

$$f(x) + f(y + z) - f(x + y) = f(z) \tag{2.2}$$

for all $x, y, z \in X$ with $f(0) = 0$.

Proof. Let $f : X \to Y$ satisfies the functional equation (2.1). Setting $x = y = 0$ in (2.1), we get $f(0) = 0$. Set $x = -y$ in (2.1), we get $f(-y) = -f(y)$ for all $y \in X$. Therefore f is an odd function. Replacing y by x and y by $2x$ in (2.1), we obtain

$$f(2x) = 2f(x) \quad \text{and} \quad f(3x) = 3f(x) \tag{2.3}$$

for all $x \in X$. In general for any positive integer a, we have $f(ax) = af(x)$.

Replacing (x, y) by $(x, y + z)$ in (2.1), we get

$$f(x) + f(y + z) = f(x + y + z) \tag{2.4}$$

for all $x, y, z \in X$. Again replacing (x, y) by $(x+y, z)$ in (2.1), we obtain

$$f(x+y) + f(z) = f(x+y+z) \tag{2.5}$$

for all $x, y, z \in X$. From (2.4) and (2.5), we derive (1.5) for all $x, y, z \in X$.

Conversely, assume $f : X \to Y$ satisfies the functional equation (2.2) with $f(0) = 0$. Set (x, z) by $(-y, 0)$ in (2.2), we get $f(-y) = -f(y)$ for all $y \in X$. Therefore f is an odd function. Replacing (y, z) by $(x, 0)$ and $(2x, 0)$ respectively, in (2.2), we obtain

$$f(2x) = 2f(x) \quad \text{and} \quad f(3x) = 3f(x) \tag{2.6}$$

for all $x \in X$. In general for any positive integer a, we have $f(ax) = af(x)$.

Replacing z by 0 in (2.2), we derive (2.1) for all $x, y \in X$. □

Theorem 2.2. *The mapping $f : X \to Y$ satisfies the functional equation (2.1) if and only if $f : X \to Y$ satisfies the functional inequality*

$$||f(x) + f(y+z) - f(x+y)|| \leq ||f(z)|| \tag{2.7}$$

for all $x, y, z \in X$ with $||f(0)|| = 0$.

Proof. Let $f : X \to Y$ satisfies the functional equation (2.7). Setting $z = 0$ in (2.7), we get

$$||f(x) + f(y) - f(x+y)|| \leq ||f(0)|| \tag{2.8}$$

for all $x, y, z \in X$. It follows from (2.8) our result is desired.

Conversely, assume $f : X \to Y$ satisfies the functional equation (2.1). Adding $f(z)$ on both sides of (2.1) and using (2.1) and rewrite the equation, we have

$$f(x) + f(y+z) - f(x+y) = f(z) \tag{2.9}$$

for all $x, y, z \in X$. It follows from (2.9) our result is desired. □

Corollary 2.3. *For a mapping $f : X \to Y$ the following conditions are equivalent.*

(i) f *is additive*

(ii) $f(x) + f(y+z) - f(x+y) = f(z)$

(iii) $||f(x) + f(y+z) - f(x+y)|| \leq ||f(z)||$.

Hereafter through out this paper, let us consider X and Y to be a normed linear space and a Banach space, respectively.

3 Stability Results for Functional Equation (1.5)

In this section, the generalized Ulam-Hyers stability of the functional equation (1.5) is investigated.

Theorem 3.1. *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\alpha\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{nj}} \quad \text{converges in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{nj}} = 0 \tag{3.1}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a function satisfying the inequality

$$\|f(x) + f(y+z) - f(x+y) - f(z)\| \leq \alpha(x, y, z) \tag{3.2}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and satisfying the functional equation (1.5) such that

$$\|f(x) - A(x)\| \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x, 2^{kj}x, 0)}{2^{kj}} \tag{3.3}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{nj}} \tag{3.4}$$

for all $x \in X$.

Proof. Assume $j = 1$. Replacing (x, y, z) by $(x, x, 0)$ in (3.2), we get

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \leq \frac{\alpha\,(x, x, 0)}{2} \tag{3.5}$$

for all $x \in X$. Now replacing x by $2x$ and dividing by 2 in (3.5), we get

$$\left\| \frac{f(2x)}{2} - \frac{f(2^2x)}{2^2} \right\| \leq \frac{\alpha\,(2x, 2x, 0)}{2^2} \tag{3.6}$$

for all $x \in X$. From (3.5) and (3.6), we obtain

$$\begin{aligned} \left\| f(x) - \frac{f(2^2x)}{2^2} \right\| &\leq \left\| f(x) - \frac{f(2x)}{2} \right\| + \left\| \frac{f(2x)}{2} - \frac{f(2^2x)}{2^2} \right\| \\ &\leq \frac{1}{2}\left[\alpha(x, x, 0) + \frac{\alpha(2x, 2x, 0)}{2}\right] \end{aligned} \tag{3.7}$$

for all $x \in X$. In general for any positive integer n , we get

$$\begin{aligned} \left\| f(x) - \frac{f(2^nx)}{2^n} \right\| &\leq \frac{1}{2}\sum_{k=0}^{n-1} \frac{\alpha(2^kx, 2^kx, 0)}{2^k} \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty} \frac{\alpha(2^kx, 2^kx, 0)}{2^k} \end{aligned} \tag{3.8}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{ \frac{f(2^nx)}{2^n} \right\},$$

replace x by 2^mx and dividing by 2^m in (3.8), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| \frac{f(2^mx)}{2^m} - \frac{f(2^{n+m}x)}{2^{(n+m)}} \right\| &= \frac{1}{2^m} \left\| f(2^mx) - \frac{f(2^n \cdot 2^mx)}{2^n} \right\| \\ &\leq \frac{1}{2}\sum_{k=0}^{n-1} \frac{\alpha(2^{k+m}x, 2^{k+m}x, 0)}{2^{k+m}} \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty} \frac{\alpha(2^{k+m}x, 2^{k+m}x, 0)}{2^{k+m}} \\ &\to 0 \quad as \;\; m \to \infty \end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{f(2^nx)}{2^n} \right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{n\to\infty} \frac{f(2^nx)}{2^n} \quad \forall \; x \in X.$$

Letting $n \to \infty$ in (3.8) we see that (3.3) holds for all $x \in X$. To prove that A satisfies (1.5), replacing (x, y, z) by $(2^nx, 2^ny, 2^nz)$ and dividing by 2^n in (3.2), we obtain

$$\frac{1}{2^n}\left\| f(2^nx) + f(2^n(y+z)) - f(2^n(x+y)) - f(2^nz) \right\| \leq \frac{1}{2^n}\alpha(2^nx, 2^ny, 2^nz)$$

for all $x, y, z \in X$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$A(x) + A(y + z) = A(x + y) + A(z).$$

Hence A satisfies (1.5) for all $x, y, z \in X$. To prove A is unique, we let $B(x)$ be another mapping satisfying (1.5) and (3.3), then

$$\|A(x) - B(x)\| = \frac{1}{2^n}\, \|A(2^nx) - B(2^nx)\|$$

$$
\begin{aligned}
&\leq \frac{1}{2^n}\left\{\|A(2^n x)-f(2^n x)\|+\|f(2^n x)-B(2^n x)\|\right\}\\
&\leq \sum_{k=0}^{\infty}\frac{2\ \alpha(2^{k+n}x)}{2^{(k+n)}}\\
&\to 0 \quad as \ \ n\to\infty
\end{aligned}
$$

for all $x \in X$. Hence A is unique.

For $j=-1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers [10], Ulam-Hyers-Rassias [20] and Ulam-JRassias [19] stabilities of (1.5).

Corollary 3.2. *Let λ and s be nonnegative real numbers. Let a function $f : X \to Y$ satisfies the inequality*

$$
\begin{aligned}
&\|f(x)+f(y+z)-f(x+y)-f(z)\|\\
&\leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s+||y||^s+||z||^s\right), & s<1 \quad or \quad s>1;\\ \lambda\left\{||x||^s||y||^s||z||^s+\left(||x||^{3s}+||y||^{3s}+||z||^{3s}\right)\right\}, & s<\frac{1}{3} \quad or \quad s>\frac{1}{3}; \end{cases}
\end{aligned} \tag{3.9}
$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$
\|f(x)-A(x)\| \leq \begin{cases} \lambda, \\ \dfrac{2\ \lambda||x||^s}{|2-2^s|}, \\ \dfrac{2\ \lambda||x||^{3s}}{|2-2^{3s}|}, \end{cases} \tag{3.10}
$$

for all $x \in X$.

4 Stability Results for Functional Inequality (1.6)

In this section, we discussed the generalized Ulam-Hyers stability of the functional inequality (1.6).

Theorem 4.1. *Let $j \in \{-1, 1\}$ and $\beta : X^3 \to [0,\infty)$ be a function such that*

$$
\sum_{n=0}^{\infty}\frac{\beta\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{nj}} \ \ converges \ \ in \ \ \mathbb{R} \quad and \quad \lim_{n\to\infty}\frac{\beta\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{nj}}=0 \tag{4.1}
$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a function satisfying the functional inequality

$$
\|f(x)+f(y+z)-f(x+y)\| \leq ||f(z)||+\beta\left(x,y,z\right) \tag{4.2}
$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and satisfying the functional equation (1.6) such that

$$
\|f(x)-A(x)\| \leq \frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x, 2^{kj}x, 0)}{2^{kj}} \tag{4.3}
$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$
A(x)=\lim_{n\to\infty}\frac{f(2^{nj}x)}{2^{nj}} \tag{4.4}
$$

for all $x \in X$.

Proof. Assume $j=1$. Replacing (x, y, z) by $(x, x, 0)$ in (4.2), we get

$$
\left\|f(x)-\frac{f(2x)}{2}\right\| \leq \frac{\beta\left(x,x,0\right)}{2} \tag{4.5}
$$

for all $x \in X$. Now replacing x by $2x$ and dividing by 2 in (4.5), we get

$$\left\| \frac{f(2x)}{2} - \frac{f(2^2x)}{2^2} \right\| \leq \frac{\beta\,(2x,2x,0)}{2^2} \tag{4.6}$$

for all $x \in X$. From (4.5) and (4.6), we obtain

$$\begin{aligned} \left\| f(x) - \frac{f(2^2x)}{2^2} \right\| &\leq \left\| f(x) - \frac{f(2x)}{2} \right\| + \left\| \frac{f(2x)}{2} - \frac{f(2^2x)}{2^2} \right\| \\ &\leq \frac{1}{2}\left[\beta(x,x,0) + \frac{\beta(2x,2x,0)}{2} \right] \end{aligned} \tag{4.7}$$

for all $x \in X$. In general for any positive integer n , we get

$$\begin{aligned} \left\| f(x) - \frac{f(2^nx)}{2^n} \right\| &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\beta(2^kx,2^kx,0)}{2^k} \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\beta(2^kx,2^kx,0)}{2^k} \end{aligned} \tag{4.8}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{ \frac{f(2^nx)}{2^n} \right\},$$

replace x by 2^mx and dividing by 2^m in (4.8), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| \frac{f(2^mx)}{2^m} - \frac{f(2^{n+m}x)}{2^{(n+m)}} \right\| &= \frac{1}{2^m} \left\| f(2^mx) - \frac{f(2^n \cdot 2^mx)}{2^n} \right\| \\ &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\beta(2^{k+m}x,2^{k+m}x,0)}{2^{k+m}} \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\beta(2^{k+m}x,2^{k+m}x,0)}{2^{k+m}} \\ &\to 0 \quad as \;\; m \to \infty \end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{f(2^nx)}{2^n} \right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{n\to\infty} \frac{f(2^nx)}{2^n} \quad \forall \; x \in X.$$

Letting $n \to \infty$ in (4.8) we see that (4.3) holds for all $x \in X$. In order to prove that A satisfies (1.6) and it is unique, the proof is similar to that of Theorem 3.1.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 4.1 concerning the Ulam-Hyers [10], Ulam-Hyers-Rassias [20] and Ulam-JRassias [19] stabilities of (1.6).

Corollary 4.2. *Let λ and r, s, t be nonnegative real numbers. Let a function $f : X \to Y$ satisfies the functional inequality*

$$\begin{aligned} &\|f(x) + f(y+z) - f(x+y)\| \\ &\leq ||f(z)|| + \begin{cases} \lambda, & \\ \lambda\left(||x||^r + ||y||^s + ||z||^t\right), & r,s < 1 \quad or \quad r,s > 1; \\ \lambda\left\{||x||^r||y||^s||z||^t + \left(||x||^{r+s+t} + ||y||^{r+s+t} + ||z||^{r+s+t}\right)\right\}, & \\ \qquad r+s+t < \frac{1}{3} \quad or \quad r+s+t > \frac{1}{3}; & \end{cases} \end{aligned} \tag{4.9}$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \begin{cases} \lambda, \\ \dfrac{\lambda||x||^r}{|2-2^r|} + \dfrac{\lambda||x||^s}{|2-2^s|}, \\ \dfrac{\lambda||x||^{r+s+t}}{|2-2^{r+s+t}|}, \end{cases} \tag{4.10}$$

for all $x \in X$.

5 Application of Functional Equation (1.5)

Consider the additive functional equation (1.5), that is

$$f(x) + f(y+z) - f(x+y) = f(z).$$

The above functional equation can be rewritten as

$$f(x) + f(y+z) = f(x+y) + f(z).$$

This functional equation is originating from an excellent definition of Group Theory which states the associative law for the binary operation "+".

Since $f(x) = x$ is the solution of the functional equation, the above equation is written as follows

$$x + (y+z) = (x+y) + z$$

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