Malaya Journal of Matematik 1(1)(2013) 18–26

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Certain subclasses of uniformly convex functions and corresponding class of starlike functions

N. Magesh,[a,*] and V. Prameela[b]

[a] P.G. and Research Department of Mathematics, Govt Arts College for Men , Krishnagiri - 635001, India.

[b] Department of Mathematics, Adhiyamaan College of Engineering (Autonomous), Hosur - 635109, Tamilnadu, India.

Abstract

In this paper, we defined a new subclass of uniformly convex functions and corresponding subclass of starlike functions with negative coefficients and obtain coefficient estimates. Further we investigate extreme points, growth and distortion bounds, radii of starlikeness and convexity and modified Hadamard products.

Keywords: Univalent functions, convex functions, starlike functions, uniformly convex functions, uniformly starlike functions.

2010 MSC: 30C45.

1 Introduction

Denoted by S the class of functions of the form

$$f(z) = z + \sum_{n=j+1}^{\infty} a_n z^n \tag{1.1}$$

that are analytic and univalent in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ and by ST and CV the subclasses of S that are respectively, starlike and convex. Goodman [5, 6] introduced and defined the following subclasses of CV and ST.

A function $f(z)$ is uniformly convex (uniformly starlike) in $\mathcal{U}$ if $f(z)$ is in CV (ST) and has the property that for every circular arc γ contained in $\mathcal{U}$, with center ξ also in $\mathcal{U}$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions denoted by UCV and the class of uniformly starlike functions by UST (for details see [5]). It is well known from [8, 11] that

$$f \in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| \leq \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}.$$

In [11], Rønning introduced a new class of starlike functions related to UCV and defined as

$$f \in S_p \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}.$$

Note that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$. Further Rønning generalized the class S_p by introducing a parameter $\alpha,\ -1 \leq \alpha < 1$,

$$f \in S_p(\alpha) \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\}.$$

*Corresponding author.

E-mail addresses: nmagi_2000@yahoo.co.in (N. Magesh) and vasuprameelak@gmail.com (V. Prameela).

Motivated by the works of Bharati et al [2], Frasin [3, 4], Murugusundaramoorthy and Magesh [10] and others [5, 6, 8, 11, 12, 17, 18], we define the following class:

For $\beta \geq 0$, $-1 \leq \alpha < 1$ and $0 \leq \lambda < 1$, we let $S(\lambda, \alpha, \beta, j)$ denote the subclass of S consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-\alpha\right\} \tag{1.2}$$

$$> \beta\left|\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right|, \quad z \in \mathcal{U}. \tag{1.3}$$

We also let $TS(\lambda, \alpha, \beta, j) = S(\lambda, \alpha, \beta, j) \bigcap T$ where T, the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=j+1}^{\infty} a_n z^n, a_n \geq 0, \forall\ n \geq j+1 \tag{1.4}$$

introduced and studied by Silverman [14].

We note that, by specializing the parameters j, λ, α, and β we obtain the following subclasses studied by various authors.

1. $TS(0, \alpha, 0, 1) = T^*(\alpha)$ and $TS(1, \alpha, 0, 1) = \mathcal{K}(\alpha)$ (Silverman [14])
2. $TS(0, \alpha, 0, j) = T^*(\alpha, j)$ and $TS(1, \alpha, 0, j) = \mathcal{K}(\alpha, j)$ (Srivastava et al. [15])
3. $TS(1/2, \alpha, 0, 1) = \mathcal{P}(\alpha)$ (Al-Amiri [1], Gupta and Jain [7] and Sarangi and Uralegaddi [13])
4. $TS(\lambda, \alpha, 0, j) = \mathcal{B}_{\mathcal{T}}(\lambda, \alpha, j)$ (Frasin [3, 4] and Magesh [9])
5. $TS(1/2, \alpha, \beta, 1) = TR(\alpha, \beta)$ (Rosy [12] and Stephen and Subramanian [16])
6. $TS(0, \alpha, \beta, 1) = TS(\alpha, \beta)$ and $TS(1, \alpha, \beta, 1) = UCV(\alpha, \beta)$ (Bharati et al. [2])
7. $TS(0, 0, \beta, 1) = TS_p(\beta)$ (Subramanian et al. [17])
8. $TS(1, 0, \beta, 1) = UCV(\beta)$ (Subramanian et al. [18])

The main object of this paper is to obtain a necessary and sufficient conditions for the functions $f(z)$ in the generalized class $TS(\lambda, \alpha, \beta, j)$. Further we investigate extreme points, growth and distortion bounds, radii of starlikeness and convexity and modified Hadamard products for class $TS(\lambda, \alpha, \beta, j)$.

2 Coefficient Estimates

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $TS(\lambda, \alpha, \beta, j)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $S(\lambda, \alpha, \beta, j)$ if*

$$\sum_{n=j+1}^{\infty} [M_n(1+\beta) - (\alpha+\beta)F_n]|a_n| \leq 1-\alpha, \tag{2.1}$$

where

$$M_n = (2\lambda^2-\lambda)n^2 + (1+\lambda-2\lambda^2)n, \quad F_n = (2\lambda^2-\lambda)n + (1+2\lambda^2-3\lambda) \tag{2.2}$$

and $-1 \leq \alpha < 1$, $\frac{1}{2} \leq \lambda < 1$, $\beta \geq 0$.

Proof. It suffices to show that

$$\beta\left|\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right|$$
$$-\operatorname{Re}\left\{\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right\} \leq 1-\alpha.$$

We have

$$\begin{aligned}
&\beta\left|\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right| \\
-&\ \text{Re}\left\{\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right\} \\
\leq&\ (1+\beta)\left|\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}-1\right| \\
\leq&\ \frac{(1+\beta)\sum\limits_{n=j+1}^{\infty}(M_n-F_n)|a_n|}{1-\sum\limits_{n=j+1}^{\infty}|a_n|}.
\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]|a_n|\leq 1-\alpha,$$

and hence the proof is complete. □

Theorem 2.2. *A necessary and sufficient condition for $f(z)$ of the form (1.4) to be in the class $TS(\lambda,\alpha,\beta,j)$, is that*

$$\sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]\ a_n\leq 1-\alpha. \tag{2.3}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f(z)\in TS(\lambda,\alpha,\beta,j)$ and z is real then

$$\frac{1-\sum\limits_{n=j+1}^{\infty}M_n\ a_nz^{n-1}}{1-\sum\limits_{n=j+1}^{\infty}F_n\ a_nz^{n-1}}-\alpha\geq\beta\left|\frac{\sum\limits_{n=j+1}^{\infty}(M_n-F_n)\ a_nz^{n-1}}{1-\sum\limits_{n=j+1}^{\infty}F_n\ a_nz^{n-1}}\right|.$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]\ a_n\leq 1-\alpha,\ \ -1\leq\alpha<1,\ \ \beta\geq 0.$$

Finally, the function $f(z)$ given by

$$f(z)=z-\frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}z^{j+1}, \tag{2.4}$$

where M_{j+1} and F_{j+1} as written in (2.2), is extremal for the function. □

Corollary 2.3. *Let the function $f(z)$ defined by (1.4) be in the class $TS(\lambda,\alpha,\beta,j)$. Then*

$$a_n\leq\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]},\ \ n\geq j+1. \tag{2.5}$$

This equality in (2.5) is attained for the function $f(z)$ given by (2.4).

3 Growth and Distortion Theorem

Theorem 3.1. *Let the function $f(z)$ defined by (1.4) be in the class $TS(\lambda,\alpha,\beta,j)$. Then for $|z|<r=1$*

$$r-\frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^{j+1}\leq|f(z)|\leq r+\frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^{j+1}. \tag{3.1}$$

The result (3.1) is attained for the function $f(z)$ given by (2.4) for $z=\pm r$.

Proof. Note that

$$[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]\sum_{n=j+1}^{\infty}a_n \le \sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]\,a_n \le 1-\alpha,$$

this last inequality follows from Theorem 2.2. Thus

$$|f(z)| \ge |z| - \sum_{n=j+1}^{\infty}a_n|z|^n \ge r - r^{j+1}\sum_{n=j+1}^{\infty}a_n \ge r - \frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^{j+1}.$$

Similarly,

$$|f(z)| \le |z| + \sum_{n=j+1}^{\infty}a_n|z|^n \le r + r^{j+1}\sum_{n=j+1}^{\infty}a_n \le r + \frac{1-\alpha}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^{j+1}.$$

This completes the proof. □

Theorem 3.2. *Let the function $f(z)$ defined by (1.4) be in the class $TS(\lambda,\alpha,\beta,j)$. Then for $|z|<r=1$*

$$r - \frac{(j+1)(1-\alpha)}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^j \le |f^{'}(z)| \le r + \frac{(j+1)(1-\alpha)}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}r^j. \tag{3.2}$$

Proof. We have

$$|f'(z)| \ge 1 - \sum_{n=j+1}^{\infty}na_n|z|^{n-1} \ge 1 - r^j\sum_{n=j+1}^{\infty}na_n \tag{3.3}$$

and

$$|f'(z)| \le 1 + \sum_{n=j+1}^{\infty}na_n|z|^{n-1} \le 1 + r^j\sum_{n=j+1}^{\infty}na_n. \tag{3.4}$$

In view of Theorem 2.2,

$$\frac{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}{j+1}\sum_{n=j+1}^{\infty}na_n \le \sum_{n=j+1}^{\infty}[M_n(1+\beta)-(\alpha+\beta)F_n]a_n \le 1-\alpha, \tag{3.5}$$

or, equivalently

$$\sum_{n=j+1}^{\infty}na_n \le \frac{(j+1)(1-\alpha)}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]}. \tag{3.6}$$

A substitution of (3.6) into (3.3) and (3.4) yields the inequality (3.2). This completes the proof. □

Theorem 3.3. *Let $f_j(z)=z$, and*

$$f_n(z) = z - \frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}z^n, \quad n\ge j+1 \tag{3.7}$$

for $0\le\lambda\le1, \beta\ge0, -1\le\alpha<1$. Then $f(z)$ is in the class $TS(\lambda,\alpha,\beta,j)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=j}^{\infty}\mu_n f_n(z), \tag{3.8}$$

where $\mu_n\ge0 (n\ge j)$ and $\sum_{n=j}^{\infty}\mu_n=1$.

Proof. Assume that

$$\begin{aligned} f(z) &= \mu_j f_j(z) + \sum_{n=j+1}^{\infty}\mu_n\left[z - \frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}z^n\right] \\ &= \sum_{n=j}^{\infty}\mu_n z - \sum_{n=j+1}^{\infty}\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}\mu_n z^n. \end{aligned}$$

Then it follows that

$$\sum_{n=j+1}^{\infty}\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}\mu_n\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha}=\sum_{n=j+1}^{\infty}\mu_n\leq 1,$$

so by Theorem 2.2, $f(z)\in TS(\lambda,\alpha,\beta,j)$.

Conversely, assume that the function $f(z)$ defined by (1.4) belongs to the class $TS(\lambda,\alpha,\beta,j)$, then

$$a_n\leq\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]},\quad n\geq j+1.$$

Setting $\mu_n=\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha}\,a_n,(n\geq j+1)$ and $\mu_j=1-\sum_{n=j+1}^{\infty}\mu_n$, we have,

$$\begin{aligned} f(z) &= z-\sum_{n=j+1}^{\infty}a_nz^n\\ f(z) &= z-\sum_{n=j+1}^{\infty}\frac{1-\alpha}{[M_n(1+\beta)-(\alpha+\beta)F_n]}\mu_nz^n. \end{aligned}\tag{3.9}$$

Then (3.8) gives

$$\begin{aligned} f(z) &= z+\sum_{n=j+1}^{\infty}(f_n(z)-z)\mu_n\\ &= f_j(z)\mu_j+\sum_{n=j+1}^{\infty}f_n(z)\mu_n\\ &= \sum_{n=j}^{\infty}\mu_nf_n(z) \end{aligned}$$

and hence the proof is complete. □

4 Radii of close-to-convexity, Starlikeness and Convexity

In this subsection, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS(\lambda,\alpha,\beta,j)$.

Theorem 4.1. *Let $f\in TS(\lambda,\alpha,\beta,j)$. Then $f(z)$ is close-to-convex of order σ $(0\leq\sigma<1)$ in the disc $|z|<r_1$, where*

$$r_1:=\inf\left[\frac{(1-\sigma)[M_n(1+\beta)-(\alpha+\beta)F_n]}{n(1-\alpha)}\right]^{\frac{1}{n-1}},\quad n\geq j+1.\tag{4.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.4).

Proof. Given $f\in T$, and f is close-to-convex of order σ, we have

$$|f'(z)-1|<1-\sigma.\tag{4.2}$$

For the left hand side of (4.2) we have

$$|f'(z)-1|\leq\sum_{n=j+1}^{\infty}na_n|z|^{n-1}.$$

The last expression is less than $1-\sigma$ if

$$\sum_{n=j+1}^{\infty}\frac{n}{1-\sigma}a_n|z|^{n-1}<1.$$

Using the fact, that $f\in TS(\lambda,\alpha,\beta,j)$, if and only if

$$\sum_{n=j+1}^{\infty}\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}a_n\leq 1.$$

We can say (4.2) is true if

$$\frac{n}{1-\sigma}|z|^{n-1} \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\frac{(1-\sigma)[M_n(1+\beta)-(\alpha+\beta)F_n]}{n(1-\alpha)}\right],$$

which completes the proof. □

Theorem 4.2. *Let $f \in TS(\lambda, \alpha, \beta, j)$. Then*

(i) f is starlike of order $\sigma (0 \leq \sigma < 1)$ in the disc $|z| < r_2$; where

$$r_2 = \inf \left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}\right]^{\frac{1}{n-1}}, \ n \geq j+1, \tag{4.3}$$

(ii) f is convex of order σ $(0 \leq \sigma < 1)$ in the unit disc $|z| < r_3$, where

$$r_3 = \inf \left[\left(\frac{1-\sigma}{n(n-\sigma)}\right)\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}\right]^{\frac{1}{n-1}}, \ n \geq j+1. \tag{4.4}$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.4).

Proof. (i) Given $f \in T$, and f is starlike of order σ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\sigma. \tag{4.5}$$

For the left hand side of (4.5) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=j+1}^{\infty}(n-1)a_n\ |z|^{n-1}}{1-\sum\limits_{n=j+1}^{\infty}a_n\ |z|^{n-1}}.$$

The last expression is less than $1-\sigma$ if

$$\sum_{n=j+1}^{\infty}\frac{n-\sigma}{1-\sigma}a_n\ |z|^{n-1} < 1.$$

Using the fact, that $f \in TS(\lambda, \alpha, \beta, j)$ if and only if

$$\sum_{n=j+1}^{\infty}\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}a_n \leq 1.$$

We can say (4.5) is true if

$$\frac{n-\sigma}{1-\sigma}|z|^{n-1} < \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{(1-\alpha)}\right]$$

which yields the starlikeness of the family.
(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar to the proof of (i). □

5 Modified Hadamard Product

Let the functions $f_i(z)(i=1,2)$ be defined by

$$f_i(z) = z - \sum_{n=j+1}^{\infty} a_{n,i} z^n, \quad a_{n,i} \geq 0; j \in \mathbb{N}, \tag{5.1}$$

then we define the modified Hadamard product of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z - \sum_{n=j+1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{5.2}$$

Now, we prove the following.

Theorem 5.1. *Let each of the functions $f_i(z)(i=1,2)$ defined by (5.1) be in the class $TS(\lambda,\alpha,\beta,j)$. Then $(f_1 * f_2) \in TS(\lambda,\delta_1,\beta,j)$, for*

$$\delta_1 = \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]^2 - [M_n(1+\beta)-\beta F_n](1-\alpha)^2}{[M_n(1+\beta)-(\alpha+\beta)F_n]^2 - F_n(1-\alpha)^2}. \tag{5.3}$$

The result is sharp.

Proof. We need to prove the largest δ_1 such that

$$\sum_{n=j+1}^{\infty} \frac{[M_n(1+\beta)-(\delta_1+\beta)F_n]}{1-\delta_1} a_{n,1} a_{n,2} \leq 1. \tag{5.4}$$

From Theorem 2.2, we have

$$\sum_{n=j+1}^{\infty} \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha} a_{n,1} \leq 1$$

and

$$\sum_{n=j+1}^{\infty} \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha} a_{n,2} \leq 1,$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{n=j+1}^{\infty} \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha} \sqrt{a_{n,1} a_{n,2}} \leq 1. \tag{5.5}$$

Thus it is sufficient to show that

$$\frac{[M_n(1+\beta)-(\delta_1+\beta)F_n]}{1-\delta_1} a_{n,1} a_{n,2} \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]}{1-\alpha} \sqrt{a_{n,1} a_{n,2}}, \ n \geq j+1 \tag{5.6}$$

that is

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n](1-\delta_1)}{[M_n(1+\beta)-(\delta_1+\beta)F_n](1-\alpha)}, \ n \geq j+1. \tag{5.7}$$

Note that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(1-\alpha)}{[M_n(1+\beta)-(\alpha+\beta)F_n]}, \ n \geq j+1. \tag{5.8}$$

Consequently, we need only to prove that

$$\frac{(1-\alpha)}{[M_n(1+\beta)-(\alpha+\beta)F_n]} \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n](1-\delta_1)}{[M_n(1+\beta)-(\delta_1+\beta)F_n](1-\alpha)}, \tag{5.9}$$

or equivalently

$$\delta_1 \leq \frac{[M_n(1+\beta)-(\alpha+\beta)F_n]^2 - [M_n(1+\beta)-\beta F_n](1-\alpha)^2}{[M_n(1+\beta)-(\alpha+\beta)F_n]^2 - F_n(1-\alpha)^2} = \Delta(n). \tag{5.10}$$

Since $\Delta(n)$ is an increasing function of $n(n \geq j+1)$, letting $n = j+1$ in (5.10) we obtain

$$\delta_1 \leq \Delta(j+1) = \frac{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]^2 - [M_{j+1}(1+\beta)-\beta F_{j+1}](1-\alpha)^2}{[M_{j+1}(1+\beta)-(\alpha+\beta)F_{j+1}]^2 - F_{j+1}(1-\alpha)^2} \tag{5.11}$$

which proves the main assertion of Theorem 5.1. The result is sharp for the functions defined by (2.4). □

Theorem 5.2. *Let the function $f_i(z)(i = 1, 2)$ defined by (5.1) be in the class $TS(\lambda, \alpha, \beta, j)$. If the sequence $\{[M_n(1+\beta) - (\alpha+\beta)F_n]\}$ is non-decreasing. Then the function*

$$h(z) = z - \sum_{n=j+1}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n \tag{5.12}$$

belongs to the class $TS(\lambda, \delta_2, \beta, j)$ where

$$\delta_2 = \frac{[M_n(1+\beta) - (\alpha+\beta)F_n]^2 - 2[M_n(1+\beta) - \beta F_n](1-\alpha)^2}{[M_n(1+\beta) - (\alpha+\beta)F_n]^2 - 2F_n(1-\alpha)^2}.$$

Proof. By virtue of Theorem 2.2, we have for $f_j(z)(j = 1, 2) \in TS(\lambda, \alpha, \beta, j)$ we have

$$\sum_{n=j+1}^{\infty} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha}\right]^2 a_{n,1}^2 \leq \sum_{n=j+1}^{\infty} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha} a_{n,1}\right]^2 \leq 1 \tag{5.13}$$

and

$$\sum_{n=j+1}^{\infty} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha}\right]^2 a_{n,2}^2 \leq \sum_{n=j+1}^{\infty} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha} a_{n,2}\right]^2 \leq 1. \tag{5.14}$$

It follows from (5.13) and (5.14) that

$$\sum_{n=j+1}^{\infty} \frac{1}{2} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha}\right]^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \tag{5.15}$$

Therefore we need to find the largest δ_2, such that

$$\frac{[M_n(1+\beta) - (\delta_2+\beta)F_n]}{1-\delta_2} \leq \frac{1}{2} \left[\frac{[M_n(1+\beta) - (\alpha+\beta)F_n]}{1-\alpha}\right]^2, \quad n \geq j+1$$

that is

$$\delta_2 \leq \frac{[M_n(1+\beta) - (\alpha+\beta)F_n]^2 - 2[M_n(1+\beta) - \beta F_n](1-\alpha)^2}{[M_n(1+\beta) - (\alpha+\beta)F_n]^2 - 2F_n(1-\alpha)^2} = \Psi(n).$$

Since $\Psi(n)$ is an increasing function of $n, (n \geq j+1)$, we readily have

$$\delta_2 \leq \Psi(j+1) = \frac{[M_{j+1}(1+\beta) - (\alpha+\beta)F_{j+1}]^2 - 2[M_{j+1}(1+\beta) - \beta F_{j+1}](1-\alpha)^2}{[M_{j+1}(1+\beta) - (\alpha+\beta)F_{j+1}]^2 - 2F_{j+1}(1-\alpha)^2}$$

which completes the proof. □

6 Acknowledgement

The authors are very much thankful to the referee for his/her useful suggestions and comments.

References

[1] H.S. Al-Amiri, On a subckass of close-to-convex functions with negative coefficients, Mathematics, *Tome*, 31(1)(54)(1989), 1–7.

[2] R. Bharati, R. Parvatham, and A.Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, *Tamkang J. Math.*, 28(1)(1997), 17–32.

[3] B.A. Frasin, On the analytic functions with negative coefficients, *Soochow J. Math.*, 31(3)(2005), 349–359.

[4] B.A. Frasin, Some applications of fractional calculus operators to certain subclass of analytic functions with negative coefficients, *Acta Universitatis Apulensis*, 23(2010), 123-132.

[5] A.W. Goodman, On uniformly convex functions, *Ann. polon. Math.*, 56(1991), 87-92.

[6] A.W. Goodman, On uniformly starlike functions, *J. Math. Anal. Appl.*, 155(1991), 364-370.

[7] V.P. Gupta and P.K. Jain, Certain classes univalent analytic functions with negative coefficients II, *Bull. Austral. Math. Soc.*, 15(1976), 467-473.

[8] W. Ma and D. Minda, Uniformly convex functions, *Ann. Polon. Math.*, 57(1992), 165-175.

[9] N. Magesh, On certain subclasses of analytic and bi-univalent functions, *Preprint*.

[10] G. Murugusundaramoorthy and N. Magesh, On certain subclasses of analytic functions associated with hypergeometric functions, *Applied Mathematics Letters*, 24(2011), 494-500.

[11] F. Running, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, 118(1993), 189-196.

[12] T. Rosy, *Study on subclasses of univalent functions*, Thesis, University of Madras, 2002.

[13] S. M. Sarangi and B.A. Uralegaddi, The radius of convexity and starlikeness of certain classes of analytic functions with negative coefficients I, *Atti. Accad. Naz. Lincei rend. Cl. Sci. Fis. Mat. Natur.*, 65(8)(1978), 34-42.

[14] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, 51(1975), 109–116.

[15] H. M. Srivastava, S. Owa and S. K. Chatterjea, A note on certain classes of starlike func- tions, *Rend. Sem. Mat. Univ. Padova*, 77(1987), 115–124.

[16] B.A. Stephand and K.G. Subramanian, On a subclass of Noshiro-Type analytic functions with negative coefficients, *Conference Proc. Ramanujan Mathematical Society*, 1998.

[17] K.G. Subramanian, T.V. Sudharsan, P. Balasubrahmanyam and H. Silverman, Class of uniformly starlike functions, *Publ. Math. Debercen.*, 53(4)(1998), 309 - 315.

[18] K.G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahmanyam and H. Silverman, Subclasses of uniformly convex and uniformly starlike functions, *Math. Japonica*, 42(3)(1995), 517–522.

Received: October 17, 2012; *Accepted*: November 30, 2012

UNIVERSITY PRESS