Malaya Journal of Matematik 1(1)(2013) 27–37

Malaya Journal of Matematik

www.malayajournal.org

MJM an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

On a nonlinear functional second order integrodifferential equation in Banach Spaces

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Abstract

In this paper, we study the existence, uniqueness and other properties of solutions of a nonlinear functional second order Volterra integrodifferential equation in a general Banach space. The techniques used in our analysis are the theory of the strongly continuous cosine family, Schauder fixed point theorem and Pachpatte's integral inequality.

Keywords: Integrodifferential equations, Cosine family, Schauder's fixed point theorem, Pachpatte's integral inequality.

2010 MSC: 45D05, 34G20, 45N05,45P05, 37C25.

1 Introduction

Let X denotes a Banach space with norm $\|\cdot\|$. Let $C = C([-r, 0], X),\ 0 < r < \infty$, be the Banach space of all continuous functions from $\psi : [-r, 0] \longrightarrow X$ with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(\theta)\| : -r \le \theta \le 0\}.$$

If x is a continuous function from $[-r,\, T]$, $T > 0$, to X and $t \in [0,\, T]$ then x_t stands for the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. Let $B = C\,([-r, T],\, X)$ denotes the Banach space of all continuous functions $x : [-r,\, T] \to X$ endowed with supremum norm $||\,x\,||_B \;=\; sup\ \{||\,x(t)\,||:\quad -r \le t \le T\}$. We investigate the abstract nonlinear functional second order Volterra integrodifferential equation of the form

$$x''(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t,s), g(s, x_s)ds\right), \quad t \in [0, T] \tag{1.1}$$

$$x_0(t) = \phi(t), \quad -r \le t \le 0, \tag{1.2}$$

$$x'(0) = \delta \tag{1.3}$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space X, $f : [0, T] \times C \times X \to X,\ k : [0, T] \times [0, T] \to \mathbb{R},\ g : [0, T] \times C \to X$ are continuous functions, ϕ and δ are given elements of $C = C([-r, 0], X)$ and X respectively.

Equations of these types (1.1)-(1.3) are their special forms commonly come across in almost all phases of physics and applied mathematics, see, for example [1-6] and the references cited therein. Many authors have been investigated the problems such as existence, uniqueness and other properties of solutions of equations (1.1)-(1.3) or their special forms by using various methods, see, for example [7, 8, 13, 17-22] and the references given therein. Our attempt is to generalize some results obtained by A. Pazy [15], and C. C. Travis and G. F. Webb [20]. It is advantageous to treat second order abstract differential equations directly rather than to convert into first order systems, see, for example Fitzgibbon [10]. In [10], Fitzgibbon used the second order

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abstract system for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. Our work in the present chapter is motivated by the interesting results obtained by Fattorini H. O. in [9] and is influenced by the work of Patcheu S. K. [14] and Travis C. C. and Webb G. F. [21].

The paper is organized as follows: In section 2, we present the preliminaries and statements of our results. Section 3 proves the Theorems 2.4 and 2.5 In section 4, we discuss the proofs of Theorems 2.6 - 2.8. Finally, section 5 presents an example to illustrate the application of our theorem.

2 Preliminaries

Before proceeding to the statements of our main results, we setforth some preliminaries from [11, 18, 20] and hypotheses used in our further discussion.

Definition 2.1. *A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space X is called a strongly continuous cosine family if and only if*

(a) *$C(0) = I$ (I is the identity operator);*

(b) *$C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in X$;*

(c) *$C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t,\ s \in \mathbb{R}$.*

The associated strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}. \tag{2.1}$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is the operator $A : X \longrightarrow X$ defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(.)x \in C^2(\mathbb{R}, X)\}$.

Definition 2.2. *Let $f \in L^1(0, T; X)$. The function $x \in B$ defined by*

$$x(t) = C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s) f\Big(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\Big) ds, \quad t \in [0, T] \tag{2.2}$$

$$x_0(t) = \phi(t), \quad -r \le t \le 0 \tag{2.3}$$

is called mild solution of the initial value problem (1.1)-(1.3).

Definition 2.3. *A set S in a Banach space X is said to be relatively compact set if its closure is compact.*

Definition 2.4. *An operator $T : X \to X$ is called compact if it maps bounded sets into relatively compact sets.*

Consider the following initial value problems

$$x''(t) = Ax(t) + h\Big(t, x_t, \int_{t_0}^t k(t,s), g(s, x_s) ds, \mu_1\Big), \quad t \in [0, T] \tag{2.4}$$

$$x_0(t) = \phi(t), \quad -r \le t \le 0, \tag{2.5}$$

$$x'(0) = \delta \tag{2.6}$$

and

$$x''(t) = Ax(t) + h\Big(t, x_t, \int_{t_0}^t k(t,s), g(s, x_s) ds, \mu_2\Big), \quad t \in [0, T] \tag{2.7}$$

$$x_0(t) = \phi(t), \quad -r \le t \le 0, \tag{2.8}$$

$$x'(0) = \delta \tag{2.9}$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space X, $h : [0, T] \times C \times X \times \mathbb{R} \to X$, $k : [0, T] \times [0, T] \to \mathbb{R}$, $g : [0, T] \times C \to X$ are continuous functions, μ_1, μ_2 are real parameters, $\phi \in C$ and $\delta \in X$ are given elements.

For our convenience, we list the following hypotheses.

(H_1) There are constants $K \geq 1$ and $K_1 > 0$ such that

$$\|C(t)\| \leq K \quad \text{and} \quad \|S(t)\| \leq K_1,$$

for all $t \in [0,T]$.

(H_2) For every $t \in [0,T],\ \psi \in C$ and $x \in X$, there exist a continuous function $p : [0,T] \to \mathbb{R}_+$ such that

$$\|f(t,\psi,x)\| \leq p(t)\big[\|\psi\|_C + \|x\|\big].$$

(H_3) There exist a continuous function $q : [0,T] \to \mathbb{R}_+$ such that

$$\|g(t,\psi)\| \leq q(t)\|\psi\|_C$$

for every $t \in [0,T]$ and $\psi \in C$.

(H_4) For every $t \in [0,T], \psi_1, \psi_2 \in C$ and $x_1, x_2 \in X$, there exists a constant M such that

$$\|f(t,\psi_1,x_1) - f(t,\psi_2,x_2)\| \leq M[\|\psi_1 - \psi_2\|_C + \|x_1 - x_2\|]$$

(H_5) There exists a constant N such that

$$\|g(t,\psi_1) - g(t,\psi_2)\| \leq N\|\psi_1 - \psi_2\|_C,$$

for all $t \in [0,T]$ and $\psi_1, \psi_2 \in C$.

(H_6) For each $t \in [0,T]$ the function $f(t,.,.) : [0,T] \times C \times X \to X$ is continuous and for each $\psi \in C$ and for each $x \in X$, the function $f(.,\psi,x) : [0,T] \times C \times X \to X$ is strongly measurable.

(H_7) For each $t \in [0,T]$ the function $g(t,.) : [0,T] \times C \to X$ is continuous and for each $\psi \in C$, the function $g(.,\psi) : [0,T] \times C \to X$ is strongly measurable.

(H_8) For every positive integer q there exists $\alpha_q \in L^1([0,T],[0,\infty))$ such that for a.e. $t \in [0,T]$ and $x \in B$

$$\sup_{\|x\|_B \leq q} \|f\left(t, x_t, \int_0^t k(t,s)g(s,x_s)ds\right)\| \leq \alpha_q(t)$$

and

$$\liminf_{q \to +\infty} \frac{1}{q}\int_0^T \alpha_q(s)ds = \zeta < \infty.$$

(H_9) There exist constants M_1 and M_2 such that

$$\|h(t,\psi_1,y_1,\rho) - h(t,\psi_2,y_2,\rho)\| \leq M_1[\|\psi_1 - \psi_2\|_C + \|y_1 - y_2\|]$$

and

$$\|h(t,\psi,y,\rho_1) - h(t,\psi,y,\rho_2)\| \leq M_2|\rho_1 - \rho_2|.$$

We use Schauder fixed point theorem to prove our results.

Lemma 2.1. *(Schauder fixed point theorem [16], p-37) Let S be a bounded, closed and convex subset of a Banach space X. If $f \in \mathcal{C}(S,S)$, where $\mathcal{C}(S,S)$ is the set of all compact maps from S into S, then f has at least one fixed point.*

The following Pachpatte's inequality is the key instrument in our subsequent discussion.

Lemma 2.2 ([12], p. 758). *Let $u(t), p(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on $\mathbb{R}_+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t p(s)\Big[u(s) + \int_0^s q(\tau)u(\tau)d\tau\Big]ds,$$

holds for all $t \in \mathbb{R}_+$, where u_0 is a nonnegative constant, then

$$u(t) \leq u_0\Big[1 + \int_0^t p(s)\exp\Big(\int_0^s \big(p(\tau) + q(\tau)\big)d\tau\Big)ds\Big],$$

holds for all $t \in \mathbb{R}_+$.

We need the following result in the sequel.

Lemma 2.3. *([16], p.76) Let $C(t)$, (resp. $S(t)$), $t \in \mathbb{R}$ be a strongly continuous cosine (resp. sine) family on X. Then there exists constants $N \geq 1$ and $\omega \geq 0$ such that*

$$\|C(t)\| \leq Ne^{\omega|t|}, \text{ for } t \in \mathbb{R},$$

$$\|S(t_1) - S(t_2)\| \leq N \left| \int_{t_1}^{t_2} e^{\omega|s|} ds \right|, \text{ for } t_1, t_2 \in \mathbb{R}$$

For more details on strongly continuous cosine and sine families, we refer the reader to [19] and [21].

With these preperations, now, we are in position to state our main results.

Theorem 2.4. *Suppose that the hypotheses (H_1), $(H_6) - (H_8)$ hold. Then initial value problem (1.1)-(1.3) has at least one mild solution on $[-r, T]$ if $K_1\zeta < 1$.*

Theorem 2.5. *Suppose that the hypotheses $(H_1), (H_4)$ and (H_5) hold. Then initial value problem (1.1)-(1.3) has at most one mild solution on $[-r, T]$.*

Theorem 2.6. *Suppose that the hypotheses $(H_1) - (H_3)$ hold. Then, every solution of the initial value problem (1.1)-(1.3) is bounded on $[-r, T]$.*

Theorem 2.7. *Suppose that the hypotheses $(H_1), (H_4)$ and (H_5) hold. Let $x_1(t)$ and $x_2(t)$ be two solutions of the initial value problem (1.1) with initial conditions*

$$x_{1_0}(t) = \phi(t), \quad -r \leq t \leq 0, \ x_1'(0) = \delta$$

and

$$x_{2_0}(t) = \chi(t), \quad -r \leq t \leq 0, \ x_2'(0) = \sigma$$

respectively. Then

$$\|x_1 - x_2\|_B \leq \left[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\| \right] \left[1 + K_1 MT \exp\{(K_1 M + LN)T\} \right].$$

The following theorem investigates the continuous dependency of solutions of initial value problems (2.4) - (2.6) and (2.7) - (2.9) on parameters.

Theorem 2.8. *Suppose that the hypotheses (H_1), (H_5) and (H_9) hold. Let $x_1(t)$ and $x_2(t)$ be the solutions of initial value problem (2.4) - (2.6) and (2.7) - (2.9) respectively on $[-r, T]$. Then*

$$\|x_1 - x_2\|_B \leq K_1 M_2 T|\mu_1 - \mu_2| \left[1 + K_1 M_1 T \exp\{(K_1 M_1 + LN)T\} \right].$$

3 Proofs of the Theorems 2.4 and 2.5

Proof of Theorem 2.4. Define the operator $F : B \to B$ by

$$(Fx)(t) = \begin{cases} \phi(t), \quad t \in [-r, 0], \\ \\ C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s) f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right) ds, \quad t \in [0, T]. \end{cases}$$

Then the equivalent integral equation for the system (1.1) - (1.3) can be written as the fixed point problem $x = Fx$. We prove that F has a fixed point $x(\cdot)$ by applying the Schauder fixed point theorem. For each positive integer q, let

$$B_q = \{x \in B : x(t) = \phi(t), \, t \in [-r, 0] \text{ and } \|x\|_B \leq q\}.$$

Then for each q, B_q is clearly closed, convex and bounded subset in B. Obviously, F is well defined on B_q. We claim that there exists a positive integer q such that $FB_q \subseteq B_q$. If this were not true for some q, then for each positive integer q, there is a function $x_q \in B_q$ with $Fx_q \notin B_q$, that is $\|Fx_q\| > q$. Then $1 < \frac{1}{q}\|Fx_q\|$, and hence

$$1 \leq \liminf_{q\to+\infty} \frac{1}{q}\|Fx_q(t)\|, \ t \in [0,T] \tag{3.1}$$

However, on the other hand by using the hypotheses (H_1), (H_8) and condition in Theorem, we have

$$\begin{aligned}
&\liminf_{q\to+\infty} \frac{1}{q}\|Fx_q(t)\| \\
=&\liminf_{q\to+\infty} \frac{1}{q}\|C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s)f\left(s, x_{q_s}, \int_0^s k(s,\tau)g(\tau, x_{q_\tau})d\tau\right)ds\| \\
\leq&\liminf_{q\to+\infty} \frac{1}{q}\left[\|C(t)\|\|\phi(0)\| + \|S(t)\|\|\delta\| + \int_0^t \|S(t-s)\|\|f\left(s, x_{q_s}, \int_0^s k(s,\tau)g(\tau, x_{q_\tau})d\tau\right)\|ds\right] \\
\leq&\liminf_{q\to+\infty} \frac{1}{q}\left[K\|\phi\|_C + K_1\|\delta\| + \int_0^t K_1\|f\left(s, x_{q_s}, \int_0^s k(s,\tau)g(\tau, x_{q_\tau})d\tau\right)\|ds\right] \\
\leq&\liminf_{q\to+\infty}\left[\frac{K\|\phi\|_C + K_1\|\delta\|}{q} + K_1\frac{1}{q}\int_0^t \alpha_q(s)ds\right] \\
=&K_1\zeta < 1,
\end{aligned}$$

which contradicts the condition (3.1). Therefore, for some positive integer q, we must have $FB_q \subseteq B_q$.

Next we prove that F is a compact operator on B_q . For this purpose, first we prove that F is continuous on B_q. Let $\{x_n\} \subseteq B_q$ with $x_n \to x$ in B_q. By using hypotheses (H_6) and (H_7), we have

$$f\left(t, x_{n_t}, \int_0^t k(t,s)g(s, x_{n_s})ds\right) \to f\left(t, x_t, \int_0^t k(t,s)g(s, x_s)ds\right) \text{ as } n \to \infty,$$

for each $t \in [0,T]$. Therefore by dominated convergence theorem,

$$\begin{aligned}
&\|(Fx_n)(t) - (Fx)(t)\| \\
&= \|\int_0^t S(t-s)\Big[f\left(s, x_{n_s}, \int_0^s k(s,\tau)g(\tau, x_{n_\tau})d\tau\right) \\
&\qquad - f\left(s, x_s, \int_0^s k(s,\tau)g(\tau, x_\tau)d\tau\right)\Big]ds\| \\
&= \int_0^t \|S(t-s)\| \ \|f\left(s, x_{n_s}, \int_0^s k(s,\tau)g(\tau, x_{n_\tau})d\tau\right) \\
&\qquad - f\left(s, x_s, \int_0^s k(s,\tau)g(\tau, x_\tau)d\tau\right)\|ds \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This implies that $\|Fx_n - Fx\|_B \to 0$ as $n \to \infty$. Therefore, F is continuous.

Next we prove that the family $\{Fx : x \in B_q\}$ is an equicontinuous family of functions. To do this, let $0 < t_1 < t_2 \leq T$; then

$$\begin{aligned}
&\|(Fx)(t_1) - (Fx)(t_2)\| \\
&\leq \|[C(t_1) - C(t_2)]\phi(0)\| + \|[S(t_1) - S(t_2)]\delta\| \\
&\quad + \|\int_0^{t_1} [S(t_1-s) - S(t_2-s)]f\left(s, x_s, \int_0^s k(s,\tau)g(\tau, x_\tau)d\tau\right)ds\| \\
&\quad + \|\int_{t_1}^{t_2} S(t_2-s)f\left(s, x_s, \int_0^s k(s,\tau)g(\tau, x_\tau)d\tau\right)ds\| \\
&\leq \|[C(t_1) - C(t_2)]\|\|\phi\|_C + \|S(t_1) - S(t_2)\|\|\delta\| \\
&\quad + \int_0^{t_1} \|S(t_1-s) - S(t_2-s)\|\alpha_q(s)ds + \int_{t_1}^{t_2} \|S(t_2-s)\|\alpha_q(s)ds
\end{aligned}$$

The right hand side of above inequality is independent of $x \in B_q$ and tends to zero as $(t_2 - t_1) \to 0$, since $C(t), S(t)$ are uniformly continuous for $t \in [0,T]$. The compactness of $C(t), S(t)$ for $t > 0$ imply the continuity in the uniform operator topology (see lemma 2.3). The compactness of $S(t)$ follows from that of $C(t)$. Thus F maps B_q into an equicontinuous family of functions. The equicontinuity for the cases $t_1 \leq t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ follows from the uniform continuity of ϕ on $[-r, 0]$ and from the relation

$$\|(Fy)(t_1) - (Fy)(t_2)\| \leq \|\phi(t_1) - \phi(0)\| + \|(Fy)(0) - (Fy)(t_2)\|$$

respectively.

It remains to prove that $V(t) = \{(Fx)(t) : x \in B_q\}$ is relatively compact in X for each $t \in [-r, T]$. This is trivial for $t \in [-r, 0]$, since $V(t) = \{\phi(t)\}$ which is singleton set. So let $0 < t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$; for $x \in B_q$, we define

$$(F_\epsilon x)(t) = C(t)\phi(0) + S(t)\delta + \int_0^{t-\epsilon} S(t-s) f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right) ds$$

Since $C(t)$ and $S(t)$ are compact operators the set $V\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_q\}$ is relative compact in X for every ϵ, $0 < \epsilon < t$. Moreover by making use of hypotheses (H_8), for every $x \in B_q$, we have

$$\begin{aligned}\|(Fx)(t) - (F_\epsilon x)(t)\| &= \int_{t-\epsilon}^t \|S(t-s) f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right) \| ds \\ &\leq \int_{t-\epsilon}^t \|S(t-s)\| \alpha_q(s) ds\end{aligned}$$

Therefore there are relative compact sets arbitrarily close to the set $V(t) = \{(Fx)(t) : x \in B_q\}$; hence the set $V(t)$ is also relative compact in X. Thus, by the Arzela-Ascoli theorem F is a compact operator and by Schauder's fixed point theorem there exists a fixed point $x(\cdot)$ for F, which is a solution of (1.1) - (1.3) satisfying $x(t) = \phi(t), -r \leq t \leq 0$. This completes proof of the Theorem 2.4. □

Proof of Theorem 2.5. Assume that x and y are two solutions of the initial value problem (1.1) - (1.3) on $[-r, T]$. The function $k : [0, T] \times [0, T] \longrightarrow \mathbb{R}$ being continuous on compact set, there exists a constant $L > 0$ such thawt

$$\|k(t, s)\| \leq L \quad \text{for} \quad 0 \leq s \leq t \leq T \tag{3.2}$$

From definition of mild solution given in (2.2) - (2.3) and using hypotheses $(H_1), (H_4), (H_5)$ and condition 3.2, we have

$$\begin{aligned}\|x(t) - y(t)\| &\leq \int_0^t \|S(t-s)\| \ \|f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right) - f\left(s, y_s, \int_0^s k(s,\tau) g(\tau, y_\tau) d\tau\right) \|ds \\ &\leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \end{aligned} \tag{3.3}$$

Case 1: Suppose $t \geq r$. Then, for every $\theta \in [-r, 0]$, we have $t + \theta \geq 0$. For such $\theta' s$, from (3.3) we have

$$\begin{aligned}\|x(t+\theta) - y(t+\theta)\| &\leq K_1 M \int_0^{t+\theta} \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \\ &\leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds,\end{aligned}$$

which implies

$$\|x_t - y_t\|_C \leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \tag{3.4}$$

Case 2: Suppose $0 \leq t < r$. Then for all $\theta \in [-r, -t)$, we have $t + \theta < 0$. For such $\theta' s$, we observe, from (2.2)-(2.3), that

$$\begin{aligned}\|x(t+\theta) - y(t+\theta)\| &= \|x_t(\theta) - y_t(\theta)\| \\ &= 0,\end{aligned}$$

which yields

$$\|x_t - y_t\|_C = 0. \tag{3.5}$$

For $\theta \in [-t, 0],\ t+\theta \geq 0$. Then, for such $\theta' s$ we obtain as in the case 1,

$$\|x_t - y_t\|_C \leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \tag{3.6}$$

Thus, for every $\theta \in [-r, 0]$, $(0 \leq t < r)$, from (3.5) and (3.6), we get

$$\|x_t - y_t\|_C \leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \tag{3.7}$$

For every $t \in [0, T]$, from inequalities (3.4) and (3.7), we have

$$\begin{aligned} \|x_t - y_t\|_C &\leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \\ &< \epsilon + K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \end{aligned} \tag{3.8}$$

for an arbitrary $\epsilon > 0$. Thanks to Pachpatte's integral inequality given in Lemma 2.2 and applying it to (3.8) with $u(t) = \|x_t - y_t\|_C$ we get

$$\begin{aligned} \|x_t - y_t\|_C &\leq \epsilon\left[1 + \int_0^t K_1 M \exp\left(\int_0^s (K_1 M + LN) d\tau\right) ds\right] \\ &< \epsilon\left[1 + K_1 M T \exp\left(\{K_1 M + LN\} T\right)\right] \end{aligned}$$

Since $\|x(t) - y(t)\| = 0\ \forall\ t \in [-r, 0]$, it follows, for $t \in [-r, T]$, that

$$\|x(t) - y(t)\| \leq \epsilon\left[1 + K_1 M T \exp\left\{(K_1 M + LN) T\right\}\right]$$

which yields

$$\|x - y\|_B \leq \epsilon\left[1 + K_1 M T \exp\left\{(K_1 M + LN) T\right\}\right]$$

Since $\epsilon > 0$ is an arbitrary, it follows that

$$\|x - y\|_B = 0$$

which implies $x(t) = y(t),\ \forall\ t \in [-r, T]$. This proves that the initial value problem (1.1)-(1.3) has at most one solution. □

4 Proofs of Theorems 2.6 and 2.8

Proof of Theorem 2.6. The solution of the initial value problem (1.1)-(1.3) is given by

$$x(t) = C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s) f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right) ds \qquad t \in [o, T] \tag{4.1}$$

$$x_0(t) = \phi(t), \quad -r \leq t \leq 0 \tag{4.2}$$

If $t \in [0, T]$ then from (4.1) and using the hypotheses $(H_1) - (H_3)$ and condition (3.2), we have

$$\begin{aligned} \|x(t)\| &\leq \|C(t)\|\ \|\phi(0)\| + \|S(t)\|\ \|\delta\| + \int_0^t \|S(t-s)\|\ \|f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right)\| ds \\ &\leq K\|\phi(0)\| + K_1\|\delta\| + \int_0^t K_1 p(s)\left[\|x_s\|_C + L\int_0^s q(\tau)\|x_\tau\|_C d\tau\right] ds \end{aligned}$$

Since $K \geq 1$, for $-r \leq t \leq T$, we get

$$\|x(t)\| \leq K\|\phi\|_C + K_1\|\delta\| + \int_0^t K_1 p(s)\left[\|x_s\|_C + L\int_0^s q(\tau)\|x_\tau\|_C d\tau\right] ds \tag{4.3}$$

From (4.3) and considering cases 1 and 2 as in the proof of the Theorem 2.5, we obtain

$$\|x_t\|_C \leq K\|\phi\|_C + K_1\|\delta\| + \int_0^t K_1 p(s)\|x_s\|_C ds + \int_0^t K_1 p(s) \int_0^s Lq(\tau)\|x_\tau\|_C d\tau ds \tag{4.4}$$

Thanks to Pachpatte's integral inequality given in Lemma 2.2 and applying it to (4.4) with $u(t) = \|x_t\|_C$, we get

$$\begin{aligned} \|x_t\|_C &\leq \Big[K\|\phi\|_C + K_1\|\delta\|\Big]\Big[1 + \int_0^t K_1 p(s) \exp\Big(\int_0^s (K_1 p(\tau) + Lq(\tau))d\tau\Big)ds\Big] \\ &\leq \Big[K\|\phi\|_C + K_1\|\delta\|\Big]\Big[1 + \big\{K_1 P \exp(K_1 P + LQ)T\big\}T\Big] \end{aligned} \tag{4.5}$$

where

$$P = \max_{t\in[0,T]} p(t), \ \ Q = \max_{t\in[0,T]} q(t).$$

It follows that solutions $x(t)$ of initial value problem (1.1) - (1.3) are bounded on closed interval $[-r, T]$ and proof of the Theorem 2.6 is complete. □

Remark 4.1. *We remark that our result in Theorem 2.6 also proves the stability of a solution $x(t)$ if $\|\phi\|_C$, $\|\delta\|$ are small enough.*

Remark 4.2. *We note that cosine family $C(t)$ and sine family $S(t)$ are not bounded in $\mathbb{R}$. $C(t)$ and $S(t)$ are bounded only in finite interval and may have exponential growth in $\mathbb{R}$. Consequently, all solutions of initial value problem (1.1)-(1.3) are not bounded on $\mathbb{R}_+$.*

Proof of Theorem 2.7. By making use of the definition of mild solution given in (2.2) - (2.3), the condition (3.2) and hypothesis $(H_1), (H_4)$ and (H_5), we get

$$\begin{aligned} \|x_1(t) - x_2(t)\| &\leq \|C(t)\| \ \|\phi(0) - \chi(0)\| + \|S(t)\| \ \|\delta - \sigma\| \\ &\quad + \int_0^t \|S(t-s)\| \ \Big\|f\Big(s, x_{1_s}, \int_0^s k(s,\tau), g(\tau, x_{1_\tau})d\tau\Big) \\ &\quad - f\Big(s, x_{2_s}, \int_0^s k(s,\tau), g(\tau, x_{2_\tau})d\tau\Big)\Big\| ds \\ &\leq K\|\phi(0) - \chi(0)\| + K_1\|\delta - \sigma\| + \int_0^t K_1 M\Big[\|x_{1_s} - x_{2_s}\|_C + LN\int_0^s \|x_{1_\tau} - x_{2_\tau}\|_C d\tau\Big]ds \end{aligned} \tag{4.6}$$

From (4.6) and considering cases 1 and 2 as in the proof of Theorem 2.5, for every $t \in [0, T]$, we get

$$\|x_{1_t} - x_{2_t}\|_C \leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big] + \int_0^t K_1 M\Big[\|x_{1_s} - x_{2_s}\|_C + LN\int_0^s \|x_{1_\tau} - x_{2_\tau}\|_C d\tau\Big]ds \tag{4.7}$$

Applying Pachpatte's inequality given in Lemma 2.2, to the inequality (4.7) with $u(t) = \|x_{1_t} - x_{2_t}\|_C$, we obtain

$$\begin{aligned} \|x_{1_t} - x_{2_t}\|_C &\leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big]\Big[1 + \int_0^t K_1 M \exp\Big(\int_0^s (K_1 M + LN)d\tau\Big)ds\Big] \\ &\leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big]\Big[1 + K_1 M T \exp\Big\{(K_1 M + LN)T\Big\}\Big] \end{aligned}$$

which yields, for every $t \in [-r, T]$,

$$\|x_1(t) - x_2(t)\| \leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big]\Big[1 + K_1 M T \exp\Big\{(K_1 M + LN)T\Big\}\Big]$$

and therefore, we have

$$\|x_1 - x_2\|_B \leq \big[K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|\big]\Big[1 + K_1 M T \exp\Big\{(K_1 M + LN)T\Big\}\Big]$$

This completes the proof of the Theorem 2.7. □

Proof of Theorem 2.8. Using the hypotheses $(H_1), (H_5), (H_9)$ and condition (3.2) we have

$$\begin{aligned}
&\|x_1(t) - x_2(t)\| \qquad (4.8)\\
&= \int_0^t \|S(t-s)\| \, \Big\|h\Big(s, x_{1_s}, \int_0^s k(s,\tau)g(\tau, x_{1_\tau})d\tau, \mu_1\Big) - h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_1\Big)\\
&\quad + h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_1\Big) - h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_2\Big)\Big\| ds\\
&\le \int_0^t \|S(t-s)\| \, \Big\|h\Big(s, x_{1_s}, \int_0^s k(s,\tau)g(\tau, x_{1_\tau})d\tau, \mu_1\Big) - h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_1\Big)\Big\| ds\\
&\quad + \int_0^t \|S(t-s)\| \, \Big\|h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_1\Big) - h\Big(s, x_{2_s}, \int_0^s k(s,\tau)g(\tau, x_{2_\tau})d\tau, \mu_2\Big)\Big\| ds\\
&\le \int_0^t K_1 M_1 \Big[\|x_{1_s} - x_{2_s}\|_C + \int_0^s LN\|x_{1_\tau} - x_{1_\tau}\|_C d\tau\Big] ds + \int_0^t K_1 M_2 |\mu_1 - \mu_2| ds\\
&\le K_1 M_2 T|\mu_1 - \mu_2| + \int_0^t K_1 M_1 \Big[\|x_{1_s} - x_{2_s}\|_C + \int_0^s LN\|x_{1_\tau} - x_{1_\tau}\|_C d\tau\Big] ds \qquad (4.9)
\end{aligned}$$

From (4.9) and considering cases 1 and 2 as in the proof of the Theorem 2.5, we get

$$\|x_{1_t} - x_{2_t}\|_C \le K_1 M_2 T|\mu_1 - \mu_2| + \int_0^t K_1 M_1 \Big[\|x_{1_s} - x_{2_s}\|_C + \int_0^s LN\|x_{1_\tau} - x_{1_\tau}\|_C d\tau\Big] ds \tag{4.10}$$

Once again, thanks to Pachpatte's integral inequality given in Lemma 2.2 and applying it to (4.10) with $u(t) = \|x_{1_t} - x_{2_t}\|_C$, we obtain

$$\begin{aligned}
\|x_{1_t} - x_{2_t}\|_C &\le K_1 M_2 T|\mu_1 - \mu_2| \Big[1 + \int_0^t K_1 M_1 \exp\Big(\int_0^s (K_1 M_1 + LN) d\tau\Big) ds\Big]\\
&\le |\mu_1 - \mu_2| K_1 M_2 T \Big[1 + K_1 M_1 T \exp(\{K_1 M_1 + LN\}T)\Big]
\end{aligned} \tag{4.11}$$

Thus, for $t \in [-r, T]$, we have

$$\|x_1(t) - x_2(t)\| \le K_1 M_2 T|\mu_1 - \mu_2| \Big[1 + K_1 M_1 T \exp(\{K_1 M_1 + LN\}T)\Big]$$

and hence

$$\|x_1 - x_2\|_B \le K_1 M_2 T|\mu_1 - \mu_2| \Big[1 + K_1 M_1 T \exp(\{K_1 M_1 + LN\}T)\Big]$$

This follows that the solutions of initial value problem (2.4) - (2.6) and (2.7) - (2.9) depend continuously on the parameters. This completes the proof of the Theorem 2.8 □

5 Application

To illustrate the application of our main result, consider the following nonlinear partial integrodifferential equation of the form

$$z_{tt}(w,t) = z_{ww}(w,t) + Q\Big(t, z(w, t-r), \int_0^t k_1(t,s) g_1(s, z(w, s-r)) ds\Big) ds, \quad t \in [0,T], \quad 0 \le w \le \pi \tag{5.1}$$

$$z(0,t) = z(\pi, t) = 0, \ t \in [0,T], \tag{5.2}$$

$$z(w,t) = \phi(w,t), \ 0 \le w \le \pi, -r \le t \le 0, \tag{5.3}$$

$$z_t(w,0) = z_0(w), \ 0 \le w \le \pi, \tag{5.4}$$

where ϕ is continuous, $Q : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g_1 : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous and strongly measurable and $k_1 : [0,T] \times [0,T] \to \mathbb{R}$ is continuous. We assume that the following condition is satisfied.

(1) For every positive integer q_1 there exists $\alpha'_{q_1} \in L^1([0,T],[0,\infty))$ such that for a.e. $t \in [0,T]$ and $z \in \mathbb{R}$

$$\sup_{|z|\le k_1} \left| Q\left(t, z(w,t-r), \int_0^t k_1(t,s) g_1(s, z(w,s-r))ds\right)\right| \le \alpha'_{q_1}(t),$$

and

$$\liminf_{q_1\to+\infty} \frac{1}{q_1}\int_0^b \alpha'_{q_1}(s)ds = \zeta' < \infty.$$

Let $X = L^2[0,\pi]$ be endowed with usual norm $\|\cdot\|_{L^2}$. Define the operator $A : X \to X$ by $Ay = y''$ with domain $D(A) = \{y \in X : y, y'$ are absolutely continuous, $y'' \in X$ and $y(0) = y(\pi) = 0\}$. Then

$$Ay = \sum_{n=1}^{\infty} -n^2(y,y_n)y_n, \quad y \in D(A),$$

where $y_n(s) = (\sqrt{2/\pi})\sin ns,\ n = 1,2,3,...$ is the orthogonal set of eigenvectors of A and it can be easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in \mathbb{R}$, in X and is given by (see[18])

$$C(t)y = \sum_{n=1}^{\infty} \cos nt(y,y_n)y_n, \quad y \in X.$$

The associated sine family is given by

$$S(t)y = \sum_{n=1}^{\infty} \frac{1}{n}\sin nt(y,y_n)y_n, \quad y \in X.$$

Further assume that $K_1\zeta' < 1$, where $K_1 = \sup\{\|S(t)\| : t \in [0,T]\}$.

Define the functions $f : [0,T]\times C\times X \to X$, $g : [0,T]\times C \to X$, $k : [0,T]\times[0,T] \to \mathbb{R}$, as follows

$$\begin{aligned} &f(t,\psi,x)(v) = Q(t,\psi(-r)(v), x(v)),\\ &g(t,\psi)(v) = g_1(t\psi(-r)(v)),\\ &k(t,s) = k_1(t,s), \end{aligned}$$

for $t \in [0,T], x \in X, \psi \in C$ and $v \in \mathbb{R}$. Then the above partial differential system (5.1)-(5.4) can be formulated abstractly as

$$x''(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t,s)g(s,x_s)ds\right), \quad t \in [0,T] \tag{5.5}$$

$$x_0(t) = \phi(t), \quad -r \le t \le 0 \tag{5.6}$$

$$x'(0) = \delta \tag{5.7}$$

Since all the hypotheses of the Theorem 2.4 are satisfied, and hence, by an application of the Theorem 2.4, the partial differential equations (5.1) - (5.4) have at least one solution on $[-r,T]$.

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Received: October 16, 2012; *Accepted*: December 17, 2012