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Discontinuous dynamical system represents the Logistic retarded functional equation with two different delays

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Abstract

In this work we are concerned with the discontinuous dynamical system representing the problem of the logistic retarded functional equation with two different delays,

$$x(t) = \rho x(t - r_1)[1 - x(t - r_2)], \quad t \in (0, T],$$
$$x(t) = x_0, \quad t \le 0.$$

The existence of a unique solution $x \in L^1[0, T]$ which is continuously dependence on the initial data, will be proved. The local stability at the equilibrium points will be studied. The bifurcation analysis and chaos will be discussed.

Keywords: Logistic functional equation, existence, uniqueness, equilibrium points, local stability, Chaos and Bifurcation.

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1 Introduction

Let R_+ be the set of positive real numbers and let $r \in R_+$. Consider the problem of retarded functional equation

$$x(t) = f(t, x(t - r)), \quad t \in (0, T] \tag{1.1}$$

$$x(t) = x_o, \quad t \le 0. \tag{1.2}$$

Now, if T be positive integer, $r = 1$, and $t = n = 1, 2, 3, \cdots T$, then the problem (1.1)-(1.2) will be the discrete dynamical system

$$x_n = f(n, x_{n-1}), \quad n = 1, 2, 3, \cdots T \tag{1.3}$$

$$x_0 = x_o, \quad t \le 0. \tag{1.4}$$

This shows that the discrete dynamical system (1.3)-(1.4) is a special case of the problem of retarded functional equation (1.1)-(1.2).

2 Discontinuous dynamical systems

The discontinuous dynamical systems have been studied, recently, in [3]-[5]. The results in [4] and [5] shows the richness of the models of discontinuous dynamical systems.
Consider the problem of retarded functional equation

$$x(t) = f(x(t - r)), \quad t \in (0, T] \tag{2.5}$$

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$$x(t) \; = \; x_o, \quad t \leq \; 0.$$

Let $t \in (0, r]$, then $t - r \in (-r, 0]$ and the solution of (1.1) − (1.2) is given by

$$x(t) \; = \; x_r(t) \; = \; f(x_o), \quad t \in (0, r].$$

For $t \in (r, 2r]$, we find that $t - r \in (0, r]$ and the solution of (1.1)-(1.2) is given by

$$x(t) \; = \; x_{2r}(t) \; = \; f(x_r(t)) \; = \; f(f(x_o)) \; = \; f^2(x_o), \quad t \in (r, 2r].$$

Repeating the process we can deduce that the solution of the problem (1.1)-(1.2) is given by

$$x(t) \; = \; x_{nr}(t) \; = \; f^n(x_o), \quad t \in ((n-1)r, nr],$$

which is continuous on each subinterval $((k-1)r, kr)$, $k \; = \; 1, 2, \cdots, n$, but

$$\lim_{t \to kr^+} \; x_{(k+1)r}(t) \; = \; f^{k+1}(x_o) \; \neq x_{kr}(t),$$

which implies that the solution of the problem (1.1)-(1.2) is discontinuous (sectionally continuous) on $(0, T]$ and we have proved the following theorem

Theorem 2.1. *The solution of the problem of retarded functional equation (1.1)-(1.2) is discontinuous (sectionally continuous) even the function* f *is continuous.*

Now, let $f : [0, T] \times R^n \to R^n$ and $r_1, r_2, ..., r_n \in R_+$. Then we can give the following definition

Definition 2.1. *The discontinuous dynamical system is the problem of retarded functional equation*

$$x(t) \; = \; f(t, x(t - r_1), x(t - r_2), \cdots, x(t - r_n)), \quad t \in (0, T], \tag{2.6}$$

$$x(t) \; = \; x_0, \quad t \leq 0 \tag{2.7}$$

Definition 2.2. *The equilibrium points of the discontinuous dynamical system (2.6)-(2.7) is the solutions of the equation,*

$$x(t) \; = \; f(t, x, x, \cdots, x).$$

Consider now the discontinuous dynamical system of the Logistic retarded functional equation with two different delays $r_1, r_2 > 0$

$$x(t) = \rho x(t - r_1)[1 - x(t - r_2)], \quad t \in (0, T], \tag{2.8}$$

$$x(t) = x_0, \quad t \leq 0. \tag{2.9}$$

We study here the existence of a unique continuously dependent solution $x \in L^1[0, T]$ of the problem (2.8)−(2.9). The asymptotic stability (see [1]- [9]) at the equilibrium points will be studied. We study the chaos and bifurcation for different values of r_1, r_2 and T and we compare the results with the results of the discrete dynamical system of the Logistic difference equations,

$$x_n = \rho \, x_{n-1}(1 - x_{n-1}), \quad n = 1, 2, \cdots . \tag{2.10}$$

and

$$x_n = \rho \, x_{n-1}(1 - x_{n-2}), \quad n = 1, 2, \cdots . \tag{2.11}$$

3 Existence and Uniqueness

Let $L^1 = L^1[0,T]$, $T < \infty$ be the class of Lebesgue integrable functions on $[0,T]$ with norm

$$\|f\| = \int_0^T |f(t)|\,dt, \qquad f \in L^1.$$

Let $D = \{x \in R : 0 \le x(t) \le 1,\ t \in (0,T] \quad and \quad x(0) = x_0,\ t \le 0\}$.

Definition 3.3. *By a solution of the problem* $(2.8)-(2.9)$ *we mean a function* $x \in L^1$ *satisfying the conditions* $(2.8)-(2.9)$.

Theorem 3.2. *The problem* $(2.8)-(2.9)$ *has a unique solution* $x \in L^1$.

Proof. Define, on D, the operator $F : L^1 \longrightarrow L^1$ by

$$Fx(t) = \rho x(t-r_1)[1 - x(t-r_2)].$$

The operator F makes sense, indeed for $x \in D$ we have

$$|Fx(t)| \le \rho\,|x(t-r_1)|$$

and

$$\|Fx\| \le \rho(x_0 r_1 + \|x\|).$$

Now for $x, y \in D$,we can obtain

$$\begin{aligned}|Fx - Fy| &= |\rho x(t-r_1)(1-x(t-r_2)) - \rho y(t-r_1)(1-y(t-r_2))| \\ &\le \rho\,|x(t-r_1) - y(t-r_1)| + \rho\,|x(t-r_2) - y(t-r_2)|\end{aligned}$$

which implies that

$$\begin{aligned}\|Fx - Fy\| &\le \rho \int_0^T |x(t-r_1) - y(t-r_1)|\,dt + \rho \int_0^T |x(t-r_2) - y(t-r_2)|\,dt = \\ &= \rho\left[\int_0^{r_1} |x(t-r_1) - y(t-r_1)|\,dt + \int_{r_1}^T |x(t-r_1) - y(t-r_1)|\,dt + \right. \\ &\left. + \int_0^{r_2} |x(t-r_2) - y(t-r_2)|\,dt + \int_{r_2}^T |x(t-r_2) - y(t-r_2)|\,dt\right] = \\ &= \rho\left[\int_{r_1}^T |x(t-r_1) - y(t-r_1)|\,dt + \int_{r_2}^T |x(t-r_2) - y(t-r_2)|\,dt\right] \\ &\le \rho\left[\int_0^{T-r_1} |x(\theta) - y(\theta)|\,d\theta + \int_0^{T-r_2} |x(\varphi) - y(\psi)|\,d\varphi\right] \\ &\le \rho\left[\int_0^T |x(\theta) - y(\theta)|\,d\theta + \int_0^T |x(\varphi) - y(\psi)|\,d\varphi\right] \\ &\le 2\rho\,\|x - y\|\,.\end{aligned}$$

If $\rho < \frac{1}{2}$ we deduce that

$$\|Fx - Fy\| < \|x - y\|$$

and then the problem $(2.8)-(2.9)$ has, on D, a unique solution $x \in L^1$. □

4 Continuous dependence on initial conditions

Consider the problem

$$x(t) = \rho x(t-r_1)[1 - x(t-r_2)], \quad t \in (0,T],$$

$$x(t) = x_0^*, \quad t \le 0. \tag{4.12}$$

For the continuous dependence of The solution of $(2.8)-(2.9)$ on the initial data we have the following theorem.

Theorem 4.3. *The solution of the discontinuous dynamical system represents the problem of the logistic retarded functional equation with two different delays is continuously dependent on the initial data.*

Proof. Let $x(t)$ and $x^*(t)$ be the solution of the two problems (2.8) – (2.9) and (2.8) – (4.12) respectively, then

$$|x(t) - x^*(t)| \le \rho\,|x(t-r_1) - x^*(t-r_1)| + \rho\,|x(t-r_2) - x^*(t-r_2)|$$

which implies that

$$\begin{aligned} \|x(t) - x^*(t)\| &\le \rho \int_0^T |x(t-r_1) - x^*(t-r_1)|\,dt + \rho \int_0^T |x(t-r_2) - x^*(t-r_2)|\,dt = \\ &= \rho\left[\int_0^{r_1} |x(t-r_1) - x^*(t-r_1)|\,dt + \int_{r_1}^T |x(t-r_1) - x^*(t-r_1)|\,dt + \right. \\ &+ \left. \int_0^{r_2} |x(t-r_2) - x^*(t-r_2)|\,dt + \int_{r_2}^T |x(t-r_2) - x^*(t-r_2)|\,dt\right] = \\ &= \rho\left[|x_0 - x_0^*| \int_0^{r_1} dt + \|x - x^*\| + |x_0 - x_0^*| \int_0^{r_2} dt + \|x - x^*\|\right] \\ &\le \rho(r_1 + r_2)\,|x_0 - x_0^*| + 2\rho\,\|x - x^*\| \end{aligned}$$

which implies

$$\|x - x^*\| \le \frac{\rho(r_1 + r_2)}{1 - 2\rho}\,|x_0 - x_0^*|$$

and prove that

$$|x_0 - x_0^*| \le \delta \qquad \Rightarrow \qquad \|x - x^*\| \le \varepsilon = \frac{\rho(r_1 + r_2)}{1 - 2\rho}\delta$$

and the theorem is proved. □

5 Equilibrium Points and their asymptotic stability

The equilibrium points of (2.8) are the solution of the equation

$$\rho\, x_{eq}\,(1 - x_{eq}) = x_{eq}$$

which are

$$\begin{aligned} (x_{eq})_1 &= 0, \\ (x_{eq})_2 &= 1 - \frac{1}{\rho}. \end{aligned}$$

The equilibrium point of (2.8) is locally asymptotically sable if all the roots λ of the equation,

$$1 = \rho\left[(1 - x_{eq})\,\lambda^{-r_1} - x_{eq}\,\lambda^{-r_2}\right], \tag{5.13}$$

satisfy $|\lambda| < 1$ (see [10]).
Then the equilibrium point $x_{eq} = 0$ is locally asymptotically sable if $\rho < 1$, while the second equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$ is locally asymptotically sable if all the roots λ of the equation,

$$\lambda^{r_2} - \lambda^{r_2 - r_1} + (\rho - 1) = 0. \tag{5.14}$$

satisfy $|\lambda| < 1$.
The equilibrium point $x_{eq} = 0$ is locally asymptotically sable if $\rho < 1$, which is the same as in the discrete case (2.10). Also, when $r_2 = r_1 = 1$, we deduce that the equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$, $\rho > 1$ is locally asymptotically sable if $1 < \rho < 3$, which is the same as in the discrete case (2.10).
In studying (2.8) – (2.9) it may be useful to study the difference equations (2.10) and (2.11).

6 Bifurcation and Chaos

In this section, some numerical simulations results are presented to show that dynamics behaviors of the discontinuous dynamical system (2.8) − (2.9) change for different values of r_1, r_2 and T . To do this, we will use the bifurcation diagrams as follow:-

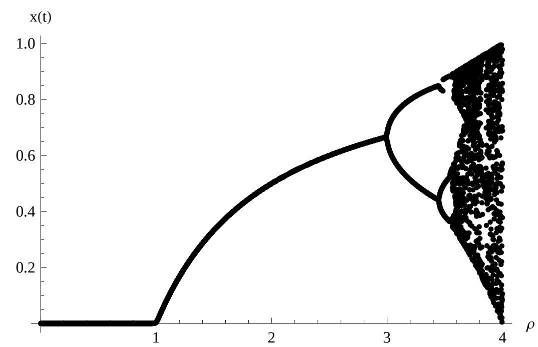

Figure 6.1
Bifurcation diagram of map (2.8)-(2.9) with respect to ρ, $r_1 = r_2 = 1$ and $t \in [0, 200]$.

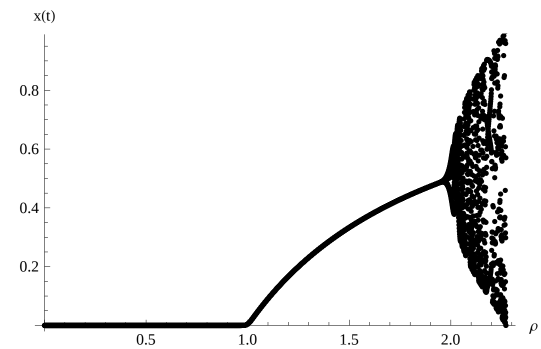

Figure 6.2
Bifurcation diagram of map (2.8)-(2.9) with respect to ρ, $r_1 = 1, r_2 = 2$ and $t \in [0, 200]$.

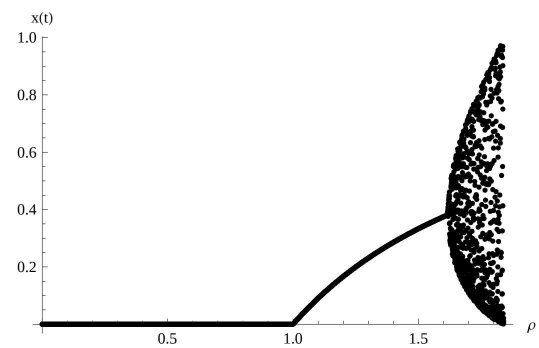

Figure 6.3
Bifurcation diagram of map (2.8)-(2.9) with respect to ρ, $r_1 = 0.1, r_2 = 0.3$ and $t \in [0, 200]$.

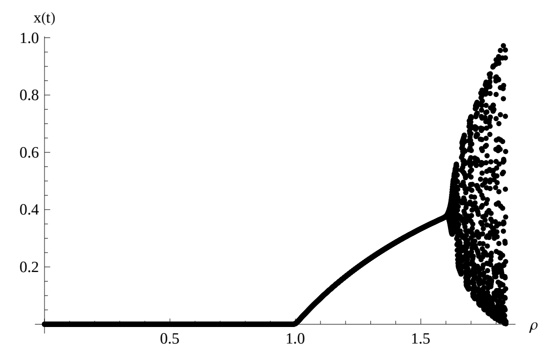

Figure 6.4
Bifurcation diagram of map (2.8)-(2.9) with respect to ρ, $r_1 = 0.25, r_2 = 0.75$ and $t \in [0, 200]$.

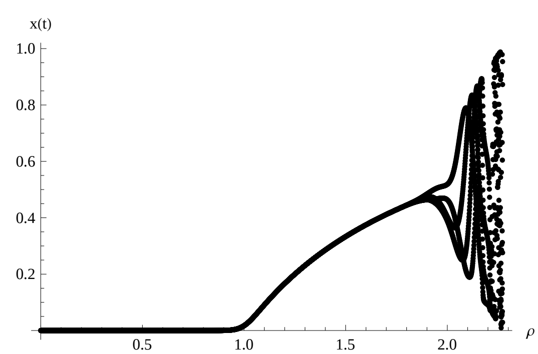

Figure 6.5
Bifurcation diagram of map (2.8)-(2.9) with respect to ρ, $r_1 = 1, r_2 = 2$ and $t \in [0, 50]$.

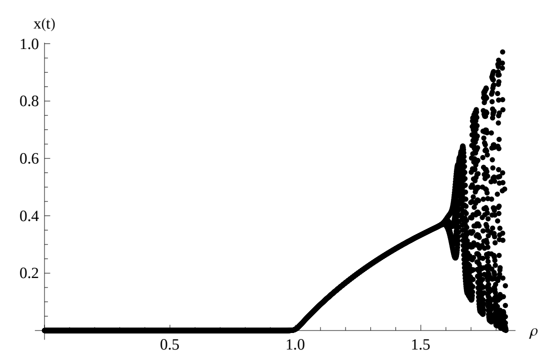

Figure 6.6
Bifurcation diagram of map (2.8)-(2.9) with respect to ρ, $r_1 = 0.25, r_2 = 0.75$ and $t \in [0, 50]$.

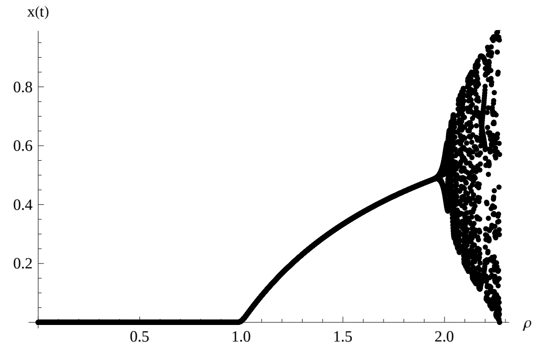

Figure 6.7
Bifurcation diagram of map (2.8)-(2.9) with respect to ρ, $r_1 = 0.5, r_2 = 1$ and $t \in [0, 100]$.

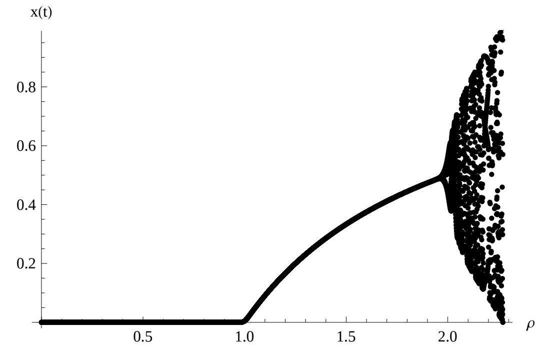

Figure 6.8
Bifurcation diagram of map (2.8)-(2.9) with respect to ρ, $r_1 = 0.1, r_2 = 0.2$ and $t \in [0, 20]$.

From Figures (6.1-6.8) we deduce that the change of r_1, r_2 and T effect of stability of the Logistic equation model, occurs of a bifurcation point, parameter sets for which aperiodic behavior occur and parameter sets for which a chaotic behavior occur.

7 Conclusions

Discrete dynamical system of the Logistic equation model describes the dynamical properties for the case $r_1 = r_2$ and the time is discrete $t = 1, 2, 3, 4, \cdots$.
On the other hand, discontinuous dynamical system of the Logistic equation model describes the dynamical properties for different values of the delayed parameters r_1 and r_2 and the time is continuous. Figures (6.1),(6.2) agrees with standard results. This confirms the correctness of our computation. The results of the other figures are new behavior (there is no analytic explanation for this behavior). From figures (6.2),(6.7) and (6.8), it locks like that there is a scale that gives identical chaos behavior.
This shows the richness of the models of discontinuous dynamical systems.

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