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On extended $M-$ series

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Abstract

This paper deals with extended M-series, which is extension of the generalized M-series [12]. Mittag-Leffler function, $\omega-$ hypergeometric function, generalized $\omega-$ Gauss hypergeometric function, $\omega-$ confluent hypergeometric function, Bessel-Maitland function can be deduced as special cases of our finding. Moreover, we obtain some theorem for extended M-series by using fractional calculus operators and many results associated with Riemann-Liouville, Weyl and Erdelyi-Kober operators. We begin our study from the following definitions.

Keywords: Extended M-series, Saigo- Meada operators, Pathway fractional integral operator.

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1 Introduction

Fractional calculus operators $(I_{0+}^{\alpha,\beta,\eta}f)(x), (I_{-}^{\alpha,\beta,\eta}f)(x), (D_{0+}^{\alpha,\beta,\eta}f)(x)$ and $(D_{-}^{\alpha,\beta,\eta}f)(x)$ be defined for and complex $\alpha, \beta, \eta \in C$ and $x \in \Re_+$; by Saigo [10].

$$\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt \tag{1.1}$$

$(\Re(\alpha) > 0)$;

$$= \frac{d^n}{dx^n}\left(I_{0+}^{\alpha+n,\beta-n,\eta-n}f\right)(x) \tag{1.2}$$

$(\Re(\alpha) \leq 0; n = [\Re(-\alpha)] + 1)$;

$$\left(I_{-}^{\alpha,\beta,\eta}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)dt \tag{1.3}$$

$(\Re(\alpha) > 0)$;

$$= (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{\alpha+n,\beta-n,\eta}f\right)(x) \tag{1.4}$$

$(\Re(\alpha) \leq 0; n = [\Re(-\alpha)] + 1)$ and

$$\left(D_{0+}^{\alpha,\beta,\eta}f\right)(x) = \left(I_{0+}^{-\alpha,-\beta,\alpha+\eta}f\right)(x) = \frac{d^n}{dx^n}\left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n}f\right)(x) \tag{1.5}$$

$(\Re(\alpha) > 0; n = [\Re(\alpha)] + 1)$;

$$\left(D_{-}^{\alpha,\beta,\eta}f\right)(x) = \left(I_{-}^{-\alpha,-\beta,\alpha+\eta}f\right)(x) = (-1)^n \frac{d^n}{dx^n}\left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta}f\right)(x) \tag{1.6}$$

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$(\Re(\alpha) > 0; n = [\Re(\alpha)] + 1)$.
When $\beta = -\alpha$, (1.1) and (1.3) coincide with the classical Riemann-Liouville and Weyl fractional integral of order $\alpha \in C$ shown below

$$\left(R^{\alpha}_{0,x} f\right)(x) = \left(I^{\alpha,-\alpha,\eta}_{0+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, (\Re(\alpha) > 0); \tag{1.7}$$

$$= \frac{d^n}{dx^n} \left(R^{\alpha+n}_{0,x} f\right)(x) \tag{1.8}$$

$(0 < \Re(\alpha) + n \le 1; n = 1, 2, ...)$;

$$\left(W^{\alpha}_{x,\infty} f\right)(x) = \left(I^{\alpha,-\alpha,\eta}_{-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\, dt \tag{1.9}$$

$(\Re(\alpha) > 0)$;

$$= (-1)^n \frac{d^n}{dx^n} \left(W^{\alpha+n}_{x,\infty} f\right)(x) \tag{1.10}$$

$(0 < \Re(\alpha) + n \le 1; n = 1, 2, ...)$;
and equation (1.5) and (1.6) coincide with Riemann- Liouville fractional derivative of order $\alpha > 0$ is defined by

$$\left(D^{\alpha}_{0+} f\right)(x) = \left(D^{\alpha,-\alpha,\eta}_{0+} f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}} \tag{1.11}$$

$(n = [\Re(\alpha)] + 1)$;

$$\left(D^{\alpha}_{-} f\right)(x) = \left(D^{\alpha,-\alpha,\eta}_{-} f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^\infty \frac{f(t)dt}{(t-x)^{\alpha-n+1}} \tag{1.12}$$

$(n = [\Re(\alpha)] + 1)$.
While for $\beta = 0$, (1.1) and (1.3) coincide with the Erdelyi- Kober fractional calculus operators of order $\alpha \in C$

$$\left(E^{\alpha,\eta}_{0,x} f\right)(x) = \left(I^{\alpha,0,\eta}_{0+} f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, dt \tag{1.13}$$

$(\Re(\alpha) > 0)$;

$$\left(K^{\alpha,\eta}_{x,\infty} f\right)(x) = \left(I^{\alpha,0,\eta}_{-} f\right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\, dt \tag{1.14}$$

$(\Re(\alpha) > 0)$.
Now here the definition of the following generalized fractional integration and differentiation operators of any complex order involving Appell function $F_3(.)$ due to Saigo and Meada [11, p. 393, Eqs. (4.12) and (4.13)] in the kernal in the following form.

Let $\alpha, \alpha', \beta, \beta', \gamma \in C, x > 0$, then the generalized fractional calculus operators involving the Appell function F_3 are defined by the following equations:

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x t^{-\alpha'} (x-t)^{\gamma-1}$$

$$\times F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t)\, dt, (\Re(\gamma) > 0); \tag{1.15}$$

$$= \frac{d^n}{dx^n} \left(I^{\alpha,\alpha',\beta+n,\beta',\gamma+n}_{0+} f\right)(x) \tag{1.16}$$

$(\Re(\gamma) \le 0; n = [-\Re(\gamma)] + 1)$;

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{-} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty t^{-\alpha} (t-x)^{\gamma-1}$$

$$\times F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x}\right) f(t)\, dt, (\Re(\gamma) > 0); \tag{1.17}$$

$$= (-1)^n \frac{d^n}{dx^n} \left(I_{-}^{\alpha,\alpha',\beta,\beta'+n,\gamma+n} f \right)(x) \tag{1.18}$$

$(\Re(\gamma) \leq 0; n = [-\Re(\gamma)] + 1)$ and

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f \right)(x) \tag{1.19}$$

$$= \frac{d^n}{dx^n} \left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n} f \right)(x); \tag{1.20}$$

$(\Re(\gamma) > 0; n = [\Re(\gamma)] + 1)$;

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f \right)(x) \tag{1.21}$$

$$= (-1)^n \frac{d^n}{dx^n} \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n} f \right)(x) \tag{1.22}$$

$(\Re(\gamma) > 0; n = [\Re(\gamma)] + 1)$.

These operators reduce to that in (1.15)-(1.22) as the following.

$$\left(I_{0+}^{\alpha,0,\beta,\beta',\gamma} f \right)(x) = \left(I_{0+}^{\gamma,\alpha-\gamma,-\beta} f \right)(x)(\gamma \in C); \tag{1.23}$$

$$\left(I_{-}^{\alpha,0,\beta,\beta',\gamma} f \right)(x) = \left(I_{-}^{\gamma,\alpha-\gamma,-\beta} f \right)(x)(\gamma \in C); \tag{1.24}$$

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f \right)(x) = \left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f \right)(x)(\Re(\gamma) > 0); \tag{1.25}$$

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma} f \right)(x) = \left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f \right)(x)(\Re(\gamma) > 0). \tag{1.26}$$

Our results are based on a preliminary assertion giving composition formulas of generalized fractional integrals (1.15) and (1.17) with a power function established by Saigo and Meada [11, p. 394, eqs. (4.18) and (4.19)], we also have

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} \right)(x) = \frac{\Gamma(\rho)\,\Gamma\left(\rho+\gamma-\alpha-\alpha'-\beta\right)\Gamma\left(\rho+\beta'-\alpha'\right)}{\Gamma(\rho+\gamma-\alpha-\alpha')\,\Gamma(\rho+\gamma-\alpha'-\beta)\,\Gamma(\rho+\beta')} \times x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{1.27}$$

where $\Re(\gamma) > 0, \Re(\rho) > \max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')]$, and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} \right)(x) = \frac{\Gamma\left(1+\alpha+\alpha'-\gamma-\rho\right)\Gamma\left(1+\alpha+\beta'-\gamma-\rho\right)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\,\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\,\Gamma(1+\alpha-\beta-\rho)} \times x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{1.28}$$

where $\Re(\gamma) > 0, \Re(\rho) < 1 + \min[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)]$.

For fractional integrals (1.1) and (1.3) with a power function established by Saigo [10], given below

(a) If $\alpha, \beta, \eta, \rho \in C$ are such that

$$\Re(\alpha) > 0, \Re(\rho) > \max[0, \Re(\beta-\eta)], \tag{1.29}$$

then

$$\left(I_{0+}^{\alpha,\beta,\eta} x^{\rho-1} \right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\beta)}{\Gamma(\rho-\beta)\Gamma(\rho+\alpha+\eta)} x^{\rho-\beta-1} (x > 0). \tag{1.30}$$

(b) If $\alpha, \beta, \eta, \rho \in C$ are such that

$$\Re(\alpha) > 0, \Re(\rho) > -\min[\Re(\beta), \Re(\eta)], \tag{1.31}$$

then

$$\left(I_{-}^{\alpha,\beta,\eta} x^{-\rho} \right)(x) = \frac{\Gamma(\rho+\beta)\Gamma(\rho+\eta)}{\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\eta)} x^{-\rho-\beta} (x > 0). \tag{1.32}$$

2 Extended M-Series

Extended M-series is the Special case of the generalized Wright function [9] as remarked by Saxena [16]. Since

$$
\begin{aligned}
{}_{p+2}\overset{\omega}{M}_{q+2}\left[\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu) \end{array} \mid z\right] &= \kappa_{p+2}\Psi_{q+2}\left[\begin{array}{c} (a_1,1),...,(a_p,1),(1,1),(\tau,\omega) \\ (b_1,1),...,(b_q,1),(\delta,\omega),(\xi,\mu) \end{array} \mid z\right] \\
&= \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{z^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}z^k, \qquad (2.1)
\end{aligned}
$$

where ${}_{p+2}\overset{\omega}{M}_{q+2}(.)$ is called omega M-series ($\omega - M$ series) and $\kappa = \frac{\Pi_{j=1}^{q}\Gamma(b_j)_k}{\Pi_{j=1}^{p}\Gamma(a_j)_k}; \tau,\xi,\mu,\delta \in C, \Re(\mu) > 0, \Re(\omega) > 0, p \leq q+1$.

3 Special Cases

(i) If $\delta = \tau$ then equation (2.1) can be written in the following form

$$
{}_{p}\overset{\xi,\mu}{M}_{q}(z) = \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k\Gamma(\xi+\mu k)}z^k, \qquad (3.1)
$$

where z, $\xi,\mu \in C, \Re(\mu) > 0, p \leq q+1$ is known as generalized M-Series [12].
(ii) If we put $\xi = 1$ then from the above equation (3.1) called the M-series [12].

$$
{}_{p}\overset{\mu}{M}_{q}(z) = \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k\Gamma(1+\mu k)}z^k, \qquad (3.2)
$$

where $\mu \in C, p \leq q+1$.
(iii) The ω-confluent hypergeometric function [13, 14]: when $p = q = 0$ and $\xi = \mu = 1$, we have

$$
\frac{\Gamma(\tau)}{\Gamma(\delta)}{}_1\overset{\omega}{\Phi}_1(\tau;\delta;z) = \sum_{k=0}^{\infty}\frac{\Gamma(\tau+\omega k)}{\Gamma(\delta+\omega k)\Gamma(1+k)}z^k = \sum_{k=0}^{\infty}\frac{\Gamma(\tau+\omega k)}{\Gamma(\delta+\omega k)k!}z^k, \qquad (3.3)
$$

where $|z| < \infty, \omega > 0, (\delta+\omega k) \neq 0,-1,-2,...$.
(iv) The ω-hypergeometric function [14]: For $p = 1, q = 0,\ \xi = \mu = 1$, we have

$$
\frac{\Gamma(\tau)}{\Gamma(\delta)}{}_2\overset{\omega}{R}_1(a,\tau;\delta;z) = \sum_{k=0}^{\infty}\frac{(a)_k(1)_k\Gamma(\tau+\omega k)}{\Gamma(\delta+\omega k)\Gamma(1+k)}\frac{z^k}{k!} = \sum_{k=0}^{\infty}\frac{(a)_k\Gamma(\tau+\omega k)}{\Gamma(\delta+\omega k)k!}z^k, \qquad (3.4)
$$

where $|z| < 1, \omega > 0$.
(v) The generalized ω-Gauss hypergeometric function [21]: If we take $p = 2, q = 1,\ \xi = \mu = 1$, then we have

$$
\frac{\Gamma(\tau)}{\Gamma(\delta)}{}_3\overset{\omega}{R}_2(a_1,\underline{a_2},\tau;\underline{b_1},\delta;z) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\Gamma(\tau+\omega k)}{(b_1)_k\Gamma(\delta+\omega k)k!}z^k, \qquad (3.5)
$$

where $\underline{a}$ is defined to be $\frac{\Gamma(a+\omega k)}{\Gamma(a)}$ and $|z| < 1$.
(vi) When $p = 0, q = 1,\ \tau = \delta, b = 1, \xi = \xi+1$ and z is replaced by -z, the function $\phi(\mu,\xi+1;-z)$ is denoted by $J_{\eta}^{\mu}(z)$:

$$
J_{\xi}^{\mu}(z) \equiv \phi(\mu,\xi+1;-z) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\xi+1+\mu k)}\frac{(-z)^k}{k!} \qquad (3.6)
$$

and such a function is known as the Bessel-Maitland function, or the Wright generalized Bessel function, See [20, p. 352] and [15, 8.3].

(vii) If we put $p=1, q=1$ and $\tau=\delta$, $b=1$ in (2.1), then we have

$$E^{a}_{\xi,\mu}(z)=\sum_{k=0}^{\infty}\frac{(a)_k}{\Gamma(\xi+\mu k)}\frac{z^k}{k!}, \tag{3.7}$$

where $\xi,\mu\in C, \Re(\xi)>0, \Re(\mu)>0$ and $|z|<1$ is called generalized Mitteg- leffer function introduced by Prabhakar [19] and studied by Killbas. et. al. [1] and [3].

(viii) For $\xi=\mu=1$ and $\tau=\delta$, we obtain

$$ {}_pF_q\left[\begin{array}{c} a_1,...,a_p \\ b_1,...,b_q \end{array} \mid z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k\Gamma(1+k)}z^k, \tag{3.8}$$

where $p\le q+1$ and $|z|<1$ and ${}_pF_q(.)$ is known as generalized hypergeometric function [3].

(ix) H-Function [2, 4, 8]: $\omega-M$ series can be represented as a special case of the Fox H-function

$$ {}_{p+2}\overset{\omega}{M}_{q+2}\left[\begin{array}{c} a_1...a_p,(1,1),(\tau,\omega) \\ b_1...b_q,(\delta,\omega),(\xi,\mu) \end{array} \mid z\right]$$

$$=kH^{1,n+2}_{p+2,q+2}\left[\begin{array}{c} (1-a_1,1),...,(1-a_p,1),(0,1),(1-\tau,\omega) \\ (1-b_1,1),...,(1-b_q,1),(0,1),(1-\delta,\omega),(1-\xi,\mu) \end{array} \mid (-z)\right], \tag{3.9}$$

where $k=\frac{\prod_{j=1}^{q}\Gamma(b_j)_r}{\Gamma_{j=1}^{p}\Pi(a_j)_r}$.

4 Left-Side Generalized Fractional Integration and Differentiation of Extended M-Series

Theorem 4.1. *Let $\alpha,\alpha^{'},\beta,\beta^{'},\gamma\in C$ be a complex number such that $\Re(\gamma)>0$ and let $\rho,\delta,\xi,\tau,\mu\in C, \Re(\rho)>0, p\le q+1$ and $|x|<1$. If the condition*

$$\Re(\rho)>\max\left[0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\right]$$

is satisfied then

$$\left[I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)=x^{\rho+\gamma-\alpha-\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\gamma-\alpha-\alpha',1), \end{array}\right.$$

$$\left.\begin{array}{c}(\rho+\gamma-\alpha-\alpha'-\beta,1),(\rho+\beta'-\alpha',1) \\ (\rho+\gamma-\alpha'-\beta,1),(\rho+\beta',1)\end{array};x\right). \tag{4.1}$$

Proof. From the equations (1.15) and (2.1), we have

$$\left[I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)k!}\left[I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}(t^{\rho+k-1})\right](x). \tag{4.2}$$

Now using equation (1.27), we obtained

$$=x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\rho+k)}{(b_1)_k...(b_q)_k\Gamma(\rho+\gamma-\alpha-\alpha'+k)}$$

$$\times\frac{\Gamma(\rho+\gamma-\alpha-\alpha'-\beta+k)\Gamma(\rho+\beta'-\alpha'+k)\Gamma(\tau+\omega k)}{\Gamma(\rho+\gamma-\alpha'-\beta+k)\Gamma(\rho+\beta'+k)\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{x^k}{k!}, \tag{4.3}$$

which is the required result. □

Corollary 4.1. *Let $\alpha,\beta,\eta,\rho\in C$ be such that condition (1.29) is satisfied, and further let $\delta,\xi,\tau,\mu\in C,\Re(\rho)>0$ and $|x|<1$. Then by relation (1.23) and (1.30) there hold the formula*

$$\left[I_{0+}^{\alpha,\beta,\eta}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$=x^{\rho-\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1),(\rho-\beta+\eta,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-\beta,1),(\rho+\alpha+\eta,1)\end{array};x\right). \tag{4.4}$$

Corollary 4.2. *Let $\alpha,\rho\in C$ be such that $\Re(\alpha)>0$ and $\Re(\rho)>0$. Further let $\delta,\xi,\tau,\mu\in C$, and $|x|<1$ then the relation (1.7) indicates that equation (4.4) reduces to the following result*

$$\left[R_{0,x}^{\alpha}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$=x^{\rho+\alpha-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\alpha,1)\end{array};x\right). \tag{4.5}$$

Corollary 4.3. *Let $\alpha,\eta,\rho\in C$ be such that $\Re(\alpha)>0$ and $\Re(\rho)>0$. Further let $\delta,\xi,\tau,\mu\in C$, and $|x|<1$ then the relation (1.13) indicates that equation (4.4) reduces to the following result*

$$\left[E_{0,x}^{\alpha,\eta}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$=x^{\rho-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(\rho+\eta,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\alpha+\eta,1)\end{array};x\right). \tag{4.6}$$

Theorem 4.2. *Let $\alpha,\alpha^{'},\beta,\beta^{'},\gamma\in C$ be a complex number such that $\Re(\gamma)>0$ and let $\rho,\delta,\xi,\tau,\mu\in C$, $\Re(\rho)>0,p\le q+1$ and $|x|<1$. If the condition*

$$\Re(\rho)>\max\left[0,\Re(\gamma-\alpha-\alpha'-\beta'),\Re(\beta-\alpha)\right]$$

is satisfied then

$$\left[D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)=x^{\rho-\gamma+\alpha+\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{p+5}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1),\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-\gamma+\alpha+\alpha',1),\end{array}\right.$$

$$\left.\begin{array}{c}(\rho-\gamma+\alpha+\alpha'+\beta',1),(\rho-\beta+\alpha,1)\\ (\rho-\gamma+\alpha+\beta',1),(\rho-\beta,1)\end{array};x\right). \tag{4.7}$$

Proof. By using equations (1.20) and (2.1), we have

$$\left[D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$=\left(\frac{d}{dx}\right)^m\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)k!}$$

$$\times\left[I_{0+}^{-\alpha',-\alpha,-\beta'+m,-\beta,-\gamma+m}(t^{\rho+k-1})\right](x). \tag{4.8}$$

Now using equation (1.27), we obtained

$$=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{1}{k!}$$

$$\times\frac{\Gamma(\rho+k)\Gamma(\rho+k-\gamma+\alpha+\alpha'+\beta')\Gamma(\rho+k-\beta+\alpha)}{\Gamma(\rho+k-\gamma+m+\alpha+\alpha')\Gamma(\rho+k-\gamma+\alpha+\beta')\Gamma(\rho+k-\beta)}$$

$$\times\left(\frac{d}{dx}\right)^m x^{\rho+k-\gamma+m+\alpha+\alpha'-1}.$$

Using the formula $\frac{d^m x^n}{dx^m} = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m}, n \geq m$, we have

$$= x^{\rho-\gamma+\alpha+\alpha'-1} \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k (1)_k \Gamma(\rho+k)}{(b_1)_k...(b_q)_k \Gamma(\rho-\gamma+\alpha+\alpha'+k)}$$

.

$$\times \frac{\Gamma(\rho-\gamma+\alpha+\alpha'+\beta'+k)\Gamma(\rho-\beta+\alpha+k)\Gamma(\tau+\omega k)}{\Gamma(\rho-\gamma+\alpha+\beta'+k)\Gamma(\rho-\beta+k)\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)} \frac{x^k}{k!}. \tag{4.9}$$

Which is the required result. □

If we set $\alpha = 0$ in (4.7) we arrive at

Corollary 4.4. *Let $\alpha, \beta, \eta, \rho \in C$ be such that $\Re(\alpha) \geq 0$,*

$$\Re(\rho) > -\min\left[0, \Re(\alpha+\beta+\eta)\right],$$

and further let $\delta, \xi, \tau, \mu \in C, \Re(\rho) > 0$. Then by the relation (1.25) there hold the formula

$$\left[D_{0+}^{\alpha,\beta,\eta}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\rho+\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1),(\rho+\alpha+\beta+\eta,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\beta,1),(\rho+\eta,1) \end{array};x\right). \tag{4.10}$$

Corollary 4.5. *Let $\alpha, \rho \in C$ be such $\Re(\alpha) \geq 0$, and further let $\delta, \xi, \tau, \mu \in C$, $\Re(\rho) > 0$. Then by the relation (1.11) there hold the formula*

$$\left[D_{0+}^{\alpha}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\rho-\alpha-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-\alpha,1) \end{array};x\right). \tag{4.11}$$

5 Right -Side Generalized Fractional Integration and Differentiation of Extended M-Series

Theorem 5.1. *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$ be a complex number such that $\Re(\gamma) > 0$ and further let $\tau, \delta, \xi, \mu \in C, \Re(\rho) > 0, p \leq q+1$ and $|x| < 1$. If the condition*

$$\Re(\rho) < 1 + \min\left[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\right]$$

is satisfied then

$$\left[I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\tfrac{1}{t}\right)\right)\right](x) = x^{-\rho+\gamma-\alpha-\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho-\gamma+\alpha+\alpha',1), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\rho-\beta,1),(1+\rho-\gamma+\beta'+\alpha,1) \\ (1+\rho+\alpha-\beta,1),(1+\rho+\alpha+\alpha'+\beta'-\gamma,1) \end{array};\frac{1}{x}\right). \tag{5.1}$$

Proof. Proof of the theorem is similar to that of Theorem 1. □

Corollary 5.1. *Let $\alpha, \beta, \eta, \rho \in C$ be such that condition (1.31) is satisfied, and further let $\delta, \xi, \tau, \mu \in C, \Re(\rho) > 0$ and $|x| < 1$. Then by relation (1.24) and (1.32) there hold the formula*

$$\left[I_{-}^{\alpha,\beta,\eta}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho-\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\rho+\beta,1),(1+\rho+\eta,1) \\ (1+\rho,1),(1+\rho+\alpha+\beta+\eta,1) \end{array};\frac{1}{x}\right). \tag{5.2}$$

Corollary 5.2. *Let $\alpha, \rho \in C$ be such that $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Further let $\delta, \xi, \tau, \mu \in C$ and $|x| < 1$ then the relation (1.9) indicates that equation (5.2) reduces to the following result*

$$\left[W^{\alpha}_{x,\infty}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho+\alpha-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho-\alpha,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1) \end{array};\frac{1}{x}\right). \tag{5.3}$$

Corollary 5.3. *Let $\alpha, \eta, \rho \in C$ be such that $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Further let $\delta, \xi, \tau, \mu \in C$, and $|x| < 1$ then the relation (1.14) indicates that equation (5.2) reduces to the following result*

$$\left[K^{\alpha,\eta}_{x,\infty}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho+\eta,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho+\alpha+\eta,1) \end{array};\frac{1}{x}\right). \tag{5.4}$$

Theorem 5.2. *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$ be a complex number such that $\Re(\gamma) > 0$ and further let $\rho, \delta, \tau, \xi, \mu \in C, \Re(\rho) > 0, p \leq q+1$ and $|x| < 1$. If the condition*

$$\Re(\rho) < 1 + \min\left[\Re(\beta'), \Re(\gamma-\alpha-\alpha'-k), \Re(\gamma-\alpha'-\beta)\right]$$

is satisfied then

$$\left[D^{\alpha,\alpha',\beta,\beta',\gamma}_{-}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x) = x^{-\rho+\gamma-\alpha-\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\zeta,\omega),(1+\rho+\gamma-\alpha-\alpha',1), \\ b_1,...,b_q,(\delta,\omega),(\eta,\mu),(1+\rho,1), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\rho+\beta',1),(1+\rho+\gamma-\alpha'-\beta,1) \\ (1+\rho+\beta'-\alpha',1),(1+\rho-\alpha-\alpha'-\beta+\gamma,1) \end{array};\frac{1}{x}\right). \tag{5.5}$$

Proof. It is similar to the previous Theorem. □

Corollary 5.4. *Let $\alpha, \beta, \eta, \rho \in C$ be such $\Re(\alpha) \geq 0$,*

$$\Re(\rho) > -\min\left[\Re(-\beta-n), \Re(\alpha+\eta)\right],$$

$n = [\Re(\alpha)] + 1$, and further let $\delta, \xi, \tau, \mu \in C, \Re(\rho) > 0$. Then by the relation (1.26) there hold the formula

$$\left[D^{\alpha,\beta,\eta}_{-}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho+\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega), \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\rho-\beta,1),(1+\rho+\alpha+\eta,1) \\ (1+\rho,1),(1+\rho+\eta-\beta,1) \end{array};\frac{1}{x}\right). \tag{5.6}$$

Corollary 5.5. *Let $\alpha, \rho \in C$ be such $\Re(\alpha) \geq 0$ and further let $\delta, \xi, \tau, \mu \in C$, $\Re(\rho) > 0$. Then by the relation (1.12) there hold the formula*

$$\left[D^{\alpha}_{-}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$= x^{-\rho-\alpha-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{array}{c} a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho+\alpha,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1) \end{array};\frac{1}{x}\right). \tag{5.7}$$

6 Fractional Integro-Differentiation of Extended M Series

Theorem 6.1. *Let $\alpha,\alpha^{'},\beta,\beta^{'},\gamma\in C$ be a complex number such that $\Re(\gamma)>0$ and let $\rho,\delta,\xi,\tau,\mu\in C,\Re(\rho)>0,p\le q+1$ and $|x|<1$. If the condition*

$$\Re(\rho)>\max\left[0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\right]$$

is satisfied then

$$\left[I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)=x^{\rho+\gamma-\alpha-\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1),\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\gamma-\alpha-\alpha',1),\end{array}\right.$$

$$\left.\begin{array}{c}(\rho+\gamma-\alpha-\alpha'-\beta,1),(\rho+\beta'-\alpha',1)\\ (\rho+\gamma-\alpha'-\beta,1),(\rho+\beta',1)\end{array};x\right). \tag{6.1}$$

Proof. To prove (6.1) using equation (1.16) which represent integro-differentiation operator and applying the same reasoning similar to the Theorem 1. Therefore we omit detail. □

If we take $\alpha^{'}=0$ (6.1), we arrive at

Corollary 6.1. *Let $\alpha,\beta,\eta,\rho\in C$ be such that condition (1.29) is satisfied, and further let $\delta,\xi,\tau,\mu\in C,\Re(\rho)>0$ and $|x|<1$. Then by relation (1.2) and (1.30) there hold the formula*

$$\left[I_{0+}^{\alpha,\beta,\eta}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$=x^{\rho-\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1),(\rho-\beta+\eta,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-\beta,1),(\rho+\alpha+\eta,1)\end{array};x\right). \tag{6.2}$$

Theorem 6.2. *Let $\alpha,\alpha^{'},\beta,\beta^{'},\gamma\in C$ be a complex number such that $\Re(\gamma)>0$ and further let $\tau,\delta,\xi,\mu\in C,\Re(\rho)>0,p\le q+1$ and $|x|<1$. If the condition*

$$\Re(\rho)<1+\min\left[\Re(-\beta),\Re(\alpha+\alpha^{'}-\gamma),\Re(\alpha+\beta^{'}-\gamma)\right]$$

is satisfied then

$$\left[I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)=x^{-\rho+\gamma-\alpha-\alpha'-1}$$

$$\times{}_{p+5}\overset{\omega}{M}_{q+5}\left(\begin{array}{c}a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho-\gamma+\alpha+\alpha',1),\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1),\end{array}\right.$$

$$\left.\begin{array}{c}(1+\rho-\beta,1),(1+\rho-\gamma+\beta'+\alpha,1)\\ (1+\rho+\alpha-\beta,1),(1+\rho+\alpha+\alpha'+\beta'-\gamma,1)\end{array};\frac{1}{x}\right). \tag{6.3}$$

Proof. In view of (1.18) and (2.1), we have

$$\left[I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$=(-1)^n\frac{d^n}{dx^n}x^{-\alpha-\alpha'+\gamma+n-1}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\zeta+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\eta+\mu k)k!}$$

$$\times\left[I_{0+}^{\alpha,\alpha',\beta,\beta'+n,\gamma+n}\left(t^{1+\alpha+\alpha'-\gamma-n+\rho+k-1}\right)\right]\left(\frac{1}{x}\right). \tag{6.4}$$

With the help of equation (1.27) we arrive at

$$=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k}{(b_1)_k...(b_q)_k}\frac{1}{k!}$$

$$\times\frac{\Gamma(1+\rho+k+\alpha+\alpha'-\gamma-n)\Gamma(1+\rho+k-\beta)\Gamma(1+\rho+k+\alpha+\beta'-\gamma)\Gamma(\zeta+\omega k)}{\Gamma(1+\rho+k)\Gamma(1+\rho+k+\alpha-\beta)\Gamma(1+\rho+\alpha+\alpha'-\gamma+\beta')\Gamma(\delta+\omega k)\Gamma(\eta+\mu k)}$$

$$\times\left(1+\rho+k+\alpha+\alpha^{'}-\gamma-n\right)_n x^{-\rho-k-\alpha-\alpha^{'}+\gamma+n-1}.$$

Finally using formula $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}, a\neq 0$, the above expression becomes

$$=x^{-\rho+\gamma-\alpha-\alpha'-1}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k}{(b_1)_k...(b_q)_k}\frac{\Gamma(1+\rho+k+\alpha+\alpha'-\gamma)}{\Gamma(1+\rho+k)\Gamma(1+\rho+k+\alpha-\beta)}$$

$$\times\frac{\Gamma(1+\rho+k-\beta)\Gamma(1+\rho+k+\alpha+\beta'-\gamma)\Gamma(\zeta+\omega k)}{\Gamma(1+\rho+\alpha+\alpha'-\gamma+\beta')\Gamma(\delta+\omega k)\Gamma(\eta+\mu k)}\frac{x^{-k}}{k!}, \tag{6.5}$$

which is the required result. □

If we take $\alpha'=0$ in (6.3), then the following result holds:

Corollary 6.2. *Let $\alpha,\beta,\eta,\rho\in C$ be such that condition (1.31) is satisfied, and further let $\delta,\xi,\tau,\mu\in C, \Re(\rho)>0$ and $|x|<1$. Then by relation (1.4) there hold the formula*

$$\left[I_{-}^{\alpha,\beta,\eta}\left(t^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{t}\right)\right)\right](x)$$

$$=x^{-\rho-\beta-1}{}_{p+4}\overset{\omega}{M}_{q+4}\left(\begin{matrix}a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho+\beta,1),(1+\rho+\eta,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1),(1+\rho+\alpha+\beta+\eta,1)\end{matrix};\frac{1}{x}\right). \tag{6.6}$$

7 Usual Differentiation of the Extended M-Series

It is know that for the natural $\alpha=m\in N$, the Riemann- Liouville fractional derivative (1.11) is the usual derivative of order m, while (1.12) coincides with the usual derivative of order m with exactness to the multiplier $(-1)^m$ for example see [18, section 2 and 5]:

$$\left(D_{0+}^m f\right)(x)=\left(\frac{d}{dx}\right)^m f(x),$$

$$\left(D_{-}^m f\right)(x)=(-1)^m\left(\frac{d}{dx}\right)^m f(x)\,(x>0);$$

There hold the following result.

Theorem 7.1. *Let $m\in N$ and let $\delta,\xi,\tau,\mu\in C,\rho>0$. Then for $z\in C(z\neq 0)$ there hold the formula*

$$\left(\frac{d}{dx}\right)^m\left(z^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(z)\right)$$

$$=z^{\rho-m-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{matrix}a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho-m,1)\end{matrix};z\right), \tag{7.1}$$

and

$$\left(\frac{d}{dx}\right)^m\left(z^{-\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}\left(\frac{1}{z}\right)\right)$$

$$=(-1)^m z^{-\rho-m-1}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{matrix}a_1,...,a_p,(1,1),(\tau,\omega),(1+\rho+m,1)\\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(1+\rho,1)\end{matrix};\frac{1}{z}\right). \tag{7.2}$$

Proof. With the help of corollaries 4.5 and 5.5 we deduce the differentiation formulas for the extended M-series (2.1). Therefore these relations can be extended from $x>0$ to any complex $z\in C$, expect $z=0$, and the condition for their validity can be omitted. □

8 Pathway Fractional Integration of Extended M-Series

The Pathway model is introduced by Mathai [5] and studied further by Mathai and Haubold [6], [7] and Seema S. Nair [17]. Let $f(x) \in L(a,b), \eta \in C, \Re(\eta) > 0, a > 0$ and let us take a pathway parameter $\alpha < 1$. Then the pathway fractional integration operator, as an extension of (1.7) is defined as follows:

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)dt. \tag{8.1}$$

Theorem 8.1. *Let f(x)$\in L(a,b), \eta, \rho \in C, \Re(\eta) > 0, \Re(\rho) > 0, a > 0$ and pathway parameter $\alpha < 1$. Further let $\tau, \delta, \xi, \mu \in C, p \leq q+1$. Then for the pathway fractional integral $P_{0,+}^{(\eta,\alpha)}$ the following formula holds for the image of extended M series*

$$\left[P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= \frac{x^{\rho+\eta}}{(a(1-\alpha))^{\rho}}{}_{p+3}\overset{\omega}{M}_{q+3}\left(\begin{matrix} a_1,...,a_p,(1,1),(\tau,\omega),(\rho,1) \\ b_1,...,b_q,(\delta,\omega),(\xi,\mu),(\rho+\frac{\eta}{(1-\alpha)}+1,1)\end{matrix};\frac{x}{a(1-\alpha)}\right) \tag{8.2}$$

Proof. From Equation (8.1) and (2.1) we have

$$\left[P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= x^{\eta}\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}}$$

$$\times\left(\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{t^{\rho+k-1}}{k!}\right)dt.$$

Interchanging the order of integration and summation which is permissible under the condition and which is stated with the above theorem

$$= x^{\eta}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)}\frac{1}{k!}\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}}t^{\rho+k-1}dt.$$

If we substitute $\frac{a((1-\alpha)t)}{x} = u$ in the above integral, and using Type-1 beta family i.e. B(m, n), it reduced to

$$\left[P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}{}_{p+2}\overset{\omega}{M}_{q+2}(t)\right)\right](x)$$

$$= \frac{x^{\rho+\eta}}{(a(1-\alpha))^{\rho}}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k(1)_k\Gamma(\tau+\omega k)\Gamma(\rho+k)}{(b_1)_k...(b_q)_k\Gamma(\delta+\omega k)\Gamma(\xi+\mu k)\Gamma\left(\rho+\frac{\eta}{(1-\alpha)}+1+k\right)}$$

$$\times\frac{x^k}{(a(1-\alpha))^k}\frac{1}{k!}, \tag{8.3}$$

which is the required result. □

Remark 8.1. *When $\alpha = 0, a = 1$, then replacing η by $\eta - 1$ in (8.3) the integral operator get the form of equation (4.5).*

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