Malaya Journal of Matematik 1(1)(2013) 70–85

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Existence and controllability results for damped second order impulsive neutral functional differential systems with state-dependent delay in Banach spaces

N.Y. Nadaf[a] and M. Mallika Arjunan[b,*]

[a] *Department of Mathematics, Anjuman Institute of Technology and Management, Bhatkal–581320, Karnataka, India.*

[b] *Department of Mathematics, C. B. M. College, Kovaipudur, Coimbatore–641 042, Tamil Nadu, India.*

Abstract

In this paper, we investigate the existence and controllability of mild solutions for a damped second order impulsive neutral functional differential equation with state-dependent delay in Banach spaces. The results are obtained by using Sadovskii's fixed point theorem combined with the theories of a strongly continuous cosine family of bounded linear operators. Finally, an example is provided to illustrate the main results.

Keywords: Damped second order differential equations, impulsive neutral differential equations, controllability, state-dependent delay, cosine function, mild solution, fixed point.

2010 MSC: 34A37, 93B05.

1 Introduction

In this paper, we are interested to study the existence and controllability of mild solutions for a damped second order impulsive neutral functional differential equation with state-dependent delay in Banach spaces. First, we consider the following class of damped second order impulsive neutral functional differential equation with state-dependent delay in the form:

$$\frac{d}{dt}[x'(t) - g(t, x_t)] = Ax(t) + \mathcal{D}x'(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = \eta \in X, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, \ldots, n, \tag{1.3}$$

$$\Delta x'(t_i) = J_i(x_{t_i}), \quad i = 1, 2, \ldots, n, \tag{1.4}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operator $(C(t))_{t\in\mathbb{R}}$ defined on a Banach space X; the function $x_s : (-\infty, 0] \to X$, $x_s(\theta) = x(s + \theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically; $\mathcal{D}$ is a bounded linear operator on a Banach space X; $0 < t_1 < \cdots < t_n < a$ are prefixed numbers; $f, g : I \times \mathcal{B} \to X$, $\rho : I \times \mathcal{B} \to (-\infty, a]$, $I_i(\cdot) : \mathcal{B} \to X$, $J_i(\cdot) : \mathcal{B} \to X$ are appropriate functions and the symbol $\Delta\xi(t)$ represents the jump of the function $\xi(\cdot)$ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the total duration of the process, such changes can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. These process tend to be more suitably modeled by impulsive differential equations, which

*Corresponding author.
E-mail addresses: nadaf_nabisab@yahoo.com (N. Y. Nadaf) and arjunphd07@yahoo.co.in (M. Mallika Arjunan).

allow for discontinuities in the evolution of the state. For more details on this theory and on its applications, we refer to the monographs of Lakshmikantham et al. [1], Samoilenko and Perestyuk [2], Bainov and Simeonov [3], and the papers of [4, 5, 6, 7, 8, 9, 10, 11] and the references therein. Ordinary differential equations of first and second order with impulses have been treated in several works, see for instance [12, 13]. Abstract partial differential equations with impulses have been studied by Liu [9], Rogovchenko [10, 11], Chang et al. [4, 43], and Hernández et al. [27, 28].

In control theory, one of the most important qualitative properties of dynamical systems is controllability. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to final state, where the initial and final states may vary over the entire space. Many authors has been studied the controllability of nonlinear systems with and without impulses, see for instance [14, 15, 16, 17, 18, 19, 20]. In dynamical systems damping is another important issue; it may be mathematically modelled as a force synchronous with the velocity of the object but opposite in direction to it. Concerning first and second order differential equations with damped term we cite [21, 22, 23, 24, 25] among some works.

On the other hand, functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received much attention in the recent years. The reader is referred to [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42] and the references therein for some examples and applications. The literature related to second order impulsive differential system with state-dependent delay is very limited, and related to this matter we only cite [43, 44]. To the best of our knowledge, the study of the existence and controllability system described in the abstract form (1.1)-(1.4) is an untreated problem, and this fact is the main motivation of this paper.

This paper is organized as follows. In Section 2, we recall some notations, definitions and preliminary facts which will be used throughout this paper. In Section 3, we establish sufficient conditions for the existence of mild solutions for the problem (1.1)-(1.4) by using Sadovskii's fixed point theorem combined with the theories of a cosine family of bounded linear operators. In Section 4, we study controllability results for the problem (1.1)-(1.4). In Section 5, we present some examples to show the application of the results.

2 Preliminaries

In this section, we recall briefly some notations, definitions and lemmas needed to establish our main results.

Throughout this paper, A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on Banach space $(X, \|\cdot\|)$.

Definition 2.1. *A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family iff*

(i) $C(s+t)+C(s-t)=2C(s)C(t)$ *for all* $s,t\in\mathbb{R}$,

(ii) $C(0)=I$;

(iii) $C(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x\in X$.

We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by $S(t)x=\int_0^t C(s)x ds, x\in X, t\in\mathbb{R}$ and we always assume that N and $\overline{N}$ are positive constants such that $\|C(t)\|\le N$ and $\|S(t)\|\le\overline{N}$, for every $t\in I$. The infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ is the operator $A: X\to X$ defined by

$$Ax=\frac{d^2}{dt^2}C(t)x|_{t=0},\quad x\in D(A),$$

where $D(A)=\{x\in X: C(t)x$ is twice differentiable in $t\}$. Define $E=\{x\in X: C(t)x$ is once continuously differentiable in $t\}$.

The following properties are well known [45]:

(i) If $x\in X$ then $S(t)x\in E$ for every $t\in\mathbb{R}$.

(ii) If $x\in E$ then $S(t)x\in D(A)$, $(\frac{d}{dt})C(t)x=AS(t)x$ and $(\frac{d^2}{dt^2})S(t)x=S(t)x$ for every $t\in\mathbb{R}$.

(iii) If $x \in D(A)$ then $C(t)x \in D(A)$, and $(\frac{d^2}{dt^2})C(t)x = AC(t)x = C(t)Ax$ for every $t \in \mathbb{R}$.

(iv) If $x \in D(A)$ then $S(t)x \in D(A)$, and $(\frac{d^2}{dt^2})S(t)x = AS(t)x = S(t)Ax$ for every $t \in \mathbb{R}$.

In this paper, $[D(A)]$ stands for the domain of the operator A endowed with the graph norm $\|x\|_A = \|x\|+\|Ax\|$, $x \in D(A)$. Moreover, in this work, E is the space formed by the vectors $x \in X$ for which $C(\cdot)x$ is of class C^1 on $\mathbb{R}$. It was proved by Kisinsky [46] that E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0\le t\le 1} \|AS(t)x\|, \quad x \in E, \tag{2.1}$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows from this that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0,\ t \to 0,$ for each $x \in E$. Furthermore, if $x : [0,\infty) \to X$ is a locally integrable function, then $z(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function. This is a consequence of the fact that

$$\int_0^t G(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\,ds, & \int_0^t C(t-s)x(s)\,ds \end{bmatrix}^T$$

defines an $E \times X$-valued continuous function.

The existence of solutions for the second order abstract Cauchy problem

$$x''(t) = Ax(t) \ + \ h(t), \ \ 0 \le t \le a, \tag{2.2}$$

$$x(0) = z, \ \ x'(0) = w, \tag{2.3}$$

where $h : I \to X$ is an integrable function has been discussed in [45]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [47]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)z + S(t)w + \int_0^t S(t-s)h(s)ds, \quad 0 \le t \le a, \tag{2.4}$$

is called mild solution of (2.2)-(2.3), and that when $z \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)z + C(t)w + \int_0^t C(t-s)h(s)ds, \quad 0 \le t \le a. \tag{2.5}$$

For additional details about cosine function theory, we refer to the reader to [45, 47].

To consider the impulsive conditions (1.3)-(1.4), it is convenient to introduce some additional concepts and notations.

A function $u : [\sigma,\tau] \to X$ is said to be a normalized piecewise continuous function on $[\sigma,\tau]$ if u is piecewise continuous and left continuous on $(\sigma,\tau]$. We denote by $\mathcal{PC}([\sigma,\tau],X)$ the space of normalized piecewise continuous functions from $[\sigma,\tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all normalized piecewise continuous functions $u : [0,a] \to X$ such that u is continuous at $t \ne t_i$, $i = 1,\dots,n$. It is clear that $\mathcal{PC}$ endowed with the norm $\| u \|_{\mathcal{PC}} = \sup_{s\in I} \| u(s) \|$ is a Banach space.

In what follows, we set $t_0 = 0$, $t_{n+1} = a$, and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_i$, for $i = 0,1,...,n-1$, the function $\tilde{u}_i \in C([t_i,t_{i+1}];X)$ given by $\widetilde{u}_i(t) = u(t)$ for $t \in (t_i,t_{i+1}]$ and $\widetilde{u}_i(t_i) = \lim_{t\to t_i^+} u(t)$. Moreover, for a set $B \subseteq \mathcal{PC}$, we denote by $\widetilde{B}_i$, for $i = 0,1,...,n-1$, the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$.

Lemma 2.1. *[48] A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, each set $\widetilde{B}_i$, $i = 0,1,\cdots,n-1$, is relatively compact in $C([t_i,t_{i+1}],X)$.*

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$, which has been used in [48] and suitably modified to treat retarded impulsive differential equations. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and we will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma + b] \to X$, $b > 0$, is such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b], X)$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$,

(iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

For more details about phase space axioms and examples, we refer the reader to [40].

Additional terminologies and notations used in the sequel are standard in functional analysis. In particular, for Banach spaces $(Z, \| \cdot \|_Z)$, $(W, \| \cdot \|_W)$, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate to $\mathcal{L}(Z)$ whenever $Z = W$. Additionally, $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z.

Our main results are based upon the following well-known result.

Lemma 2.2. *[49, Sadovskii's Fixed Point Theorem] Let G be a condensing operator on a Banach space X. If $G(S) \subset S$ for a convex, closed and bounded set S of X, then G has a fixed point in S.*

3 Existence Results

In this section we discuss the existence of mild solutions for the abstract system (1.1)-(1.4). We also suppose that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is a continuous function. Additionally, we introduce following conditions.

(H_φ) Let $\mathcal{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in I \times \mathcal{B},\ \rho(s, \psi) \leq 0\}$. The function $t \to \varphi_t$ is well defined from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\| \varphi_t \|_{\mathcal{B}} \leq J^\varphi(t) \| \varphi \|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

(H_1) The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions:

(i) Let $x : (-\infty, a] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. The function $t \to f(t, x_{\rho(t,x_t)})$ is measurable on I and the function $t \to f(s, x_t)$ is continuous on $\mathcal{R}(\rho^-) \cup I$ for every $s \in I$.

(ii) For each $t \in I$, the function $f(t, \cdot) : \mathcal{B} \to X$ is continuous.

(iii) There exist an integrable function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that for every $(t, \psi) \in I \times \mathcal{B}$

$$\| f(t, \psi) \| \quad \leq \quad m(t)W(\| \psi \|_{\mathcal{B}}), \ \liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} = \Lambda < \infty.$$

(H_2) The function $g : I \times \mathcal{B} \to X$ is continuous and there exists $L_g > 0$ such that

$$\|g(t, \psi_1) - g(t, \psi_2)\| \leq L_g \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad (t, \psi_i) \in I \times \mathcal{B},\ i = 1, 2.$$

(H_3) There exist positive constants c_1, c_2 such that $\|g(t, \psi)\| \leq c_1 \|\psi\|_{\mathcal{B}} + c_2$, for every $(t, \psi) \in I \times \mathcal{B}$.

(H_4) There are positive constants L_{I_i}, L_{J_i} such that

$$\|I_i(\psi_1) - I_i(\psi_2)\| \leq L_{I_i} \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad \psi_j \in \mathcal{B}, \quad i = 1, 2, \ldots, n,$$
$$\|J_i(\psi_1) - J_i(\psi_2)\| \leq L_{J_i} \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad \psi_j \in \mathcal{B}, \quad i = 1, 2, \ldots, n.$$

(H_5) The maps $I_i, J_i : \mathcal{B} \to X$, $i = 1, 2, \ldots, n$ are completely continuous and there exist continuous nondecreasing functions $\lambda_i, \mu_i : [0, \infty) \to (0, \infty)$, $i = 1, 2, \ldots, n$, such that

$$\|I_i(\psi)\| \leq \lambda_i(\|\psi\|_{\mathcal{B}}), \quad \liminf_{\zeta \to +\infty} \frac{\lambda_i(\zeta)}{\zeta} = \zeta_i < \infty, \quad \text{and}$$
$$\|J_i(\psi)\| \leq \mu_i(\|\psi\|_{\mathcal{B}}), \quad \liminf_{\zeta \to +\infty} \frac{\mu_i(\zeta)}{\zeta} = \eta_i < \infty.$$

Remark 3.1. The condition H_φ is frequently satisfied by functions that are continuous and bounded. In fact, assume that the space of continuous and bounded functions $C_b((-\infty,0],X)$ is continuously included in $\mathcal{B}$. Then, there exists $L>0$ such that

$$\| \varphi_t \|_{\mathcal{B}} \leq L \frac{\sup_{\theta\leq 0} \| \varphi(\theta) \|}{\| \varphi \|_{\mathcal{B}}} \| \varphi \|_{\mathcal{B}}, \qquad t\leq 0,\ \varphi\neq 0,\ \varphi\in C_b((-\infty,0]:X).$$

It is easy to see that the space $C_b((-\infty,0],X)$ is continuously included in $\mathcal{PC}_g(X)$ and $\mathcal{PC}_g^0(X)$. Moreover, if $g(\cdot)$ verifies (g-5)-(g-6) in [?] and $g(\cdot)$ is integrable on $(-\infty,-r]$, then the space $C_b((-\infty,0],X)$ is also continuously included in $\mathcal{PC}_r\times L^p(g;X)$. For complementary details related this matter, see Proposition 7.1.1 and Theorems 1.3.2 and 1.3.8 in [50].

If $x(\cdot)$ is a solution of (1.1)-(1.4), then from (2.4), we adopt the following concept of mild solution,

$$\begin{aligned} x(t) &= C(t)\varphi(0)+S(t)[\eta-g(0,\varphi)]+\int_0^t C(t-s)g(s,x_s)ds+\int_0^t S(t-s)[\mathcal{D}x'(s)+f(s,x_{\rho(s,x_s)})]ds \\ &\quad +\sum_{0<t_i<t} C(t-t_i)I_i(x_{t_i})+\sum_{0<t_i<t} S(t-t_i)J_i(x_{t_i}), \quad t\in I. \end{aligned}$$

Inspired from the above expression, we present the following definition.

Definition 3.1. *A function $x:(-\infty,a]\to X$ is called a mild solution of the abstract Cauchy problem (1.1)-(1.4) if $x_0=\varphi, x_{\rho(s,x_s)}\in\mathcal{B}$ for every $s\in I; x(\cdot)|_I\in\mathcal{PC}$ and*

$$\begin{aligned} x(t) &= C(t)\varphi(0)+S(t)[\eta-g(0,\varphi)]+\int_0^t C(t-s)g(s,x_s)ds+\sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}x(t_{i+1}^-)-S(t-t_i)\mathcal{D}x(t_i^+)] \\ &\quad -S(t-t_j)\mathcal{D}x(t_j^+)+\int_0^c C(t-s)\mathcal{D}x(s)ds+\int_0^t S(t-s)f(s,x_{\rho(s,x_s)})ds \\ &\quad +\sum_{0<t_i<t} C(t-t_i)I_i(x_{t_i})+\sum_{0<t_i<t} S(t-t_i)J_i(x_{t_i}), \quad t\in I. \end{aligned}$$

Remark 3.2. In the rest of this paper, $y:(-\infty,a]\to X$ is the function defined by $y(t)=\varphi(t)$ on $(-\infty,0]$ and $y(t)=C(t)\varphi(0)+S(t)\zeta$ for $t\in I$. In addition, $\| y \|_a$, M_a, K_a, and J_0^φ are the constants defined by $\| y \|_a=\sup\limits_{s\in[0,a]} \| y(s) \|$, $M_a=\sup\limits_{s\in[0,a]} M(s)$, $K_a=\sup\limits_{s\in[0,a]} K(s)$ and $J_0^\varphi=\sup\limits_{t\in\mathcal{R}(\rho^-)} J^\varphi(t)$.

Lemma 3.1. *[51, Lemma 2.1] Let $x:(-\infty,a]\to X$ be a function such that $x_0=\varphi$ and $x|_I\in\mathcal{PC}$. Then*

$$\| x_s \|_{\mathcal{B}} \leq (M_a+J_0^\varphi)\| \varphi \|_{\mathcal{B}} + K_a\sup\{\| x(\theta) \|;\ \theta\in[0,\max\{0,s\}]\}, \quad s\in\mathcal{R}(\rho^-)\cup I.$$

Theorem 3.1. *Let conditions $(H_\varphi),(H_1)-(H_4)$ be hold and assume that $S(t)$ is compact for every $t\in\mathbb{R}$. If*

$$K_a\Big[aNL_g+\frac{1}{K_a}(3\overline{N}+aN)\|\mathcal{D}\|+\overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds+\sum_{i=1}^n(NL_{I_i}+\overline{N}L_{J_i})\Big]<1,$$

then the problem (1.1)-(1.4) has at least one mild solution on $(-\infty,a]$.

Proof. On the space $Y=\{x\in\mathcal{PC}: u(0)=\varphi(0)\}$ endowed with the uniform convergence topology, we define the operator $\Gamma:Y\to Y$ by

$$\begin{aligned} \Gamma x(t) &= C(t)\varphi(0)+S(t)[\eta-g(0,\varphi)]+\int_0^t C(t-s)g(s,\bar{x}_s)ds+\sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-)-S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] \\ &\quad -S(t-t_j)\mathcal{D}\bar{x}(t_j^+)+\int_0^c C(t-s)\mathcal{D}\bar{x}(s)ds+\int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds+\sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) \\ &\quad +\sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t\in I, \end{aligned}$$

where $\bar{x}:(-\infty,a]\to X$ is such that $\bar{x}_0=\varphi$ and $\bar{x}=x$ on I. From the axiom (A) and our assumptions on φ, we infer that $\Gamma x\in\mathcal{PC}$.

Next, we prove that there exists $r>0$ such that $\Gamma(B_r(y_{|I},Y))\subseteq B_r(y_{|I},Y)$. If we assume this property is false, then for every $r>0$ there exist $x^r\in B_r(y_{|I},Y)$ and $t^r\in I$ such that $r<\|\Gamma x^r(t^r)-y(t^r)\|$. Then, from Lemma 3.1, we get

$$\begin{aligned}
r &< \|\Gamma x^r(t^r)-y(t^r)\|\\
&\leq NH\|\varphi\|_{\mathcal{B}}+\overline{N}[\|\eta\|+\|g(0,\varphi)\|]+NL_gK_a\int_0^{t^r}\|\overline{x^r}-y\|_s ds+N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}}+c_2)ds\\
&\quad +3\overline{N}\|\mathcal{D}\|_r+aN\|\mathcal{D}\|_r+\overline{N}\int_0^{t^r}m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds\\
&\quad +\sum_{i=1}^{n}N(L_{I_i}\|\bar{x}_{t_i}-y_{t_i}\|_{\mathcal{B}}+\|I_i(y_{t_i})\|)+\sum_{i=1}^{n}\overline{N}(L_{J_i}\|\bar{x}_{t_i}-y_{t_i}\|_{\mathcal{B}}+\|J_i(y_{t_i})\|)\\
&\leq NH\|\varphi\|_{\mathcal{B}}+\overline{N}[\|\eta\|+\|g(0,\varphi)\|]+NL_gK_ar+N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}}+c_2)ds+3\overline{N}\|\mathcal{D}\|_r+aN\|\mathcal{D}\|_r\\
&\quad +\overline{N}W((M_a+J_0^{\varphi})\|\varphi\|_{\mathcal{B}}+K_ar+K_a\parallel y\parallel_a)\int_0^a m(s)ds\\
&\quad +\sum_{i=1}^{n}N(L_{I_i}K_ar+\|I_i(y_{t_i})\|)+\sum_{i=1}^{n}\overline{N}(L_{J_i}K_ar+\|J_i(y_{t_i})\|),
\end{aligned}$$

and hence

$$1\leq K_a\Big[aNL_g+\frac{1}{K_a}(3\overline{N}+aN)\|\mathcal{D}\|+\overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds+\sum_{i=1}^{n}(NL_{I_i}+\overline{N}L_{J_i})\Big],$$

which is contrary to our assumption.

Let $r>0$ be such that $\Gamma(B_r(y_{|I},Y))\subset B_r(y_{|I},Y)$. In order to prove that Γ is a condensing map on $B_r(y_{|I},Y)$ into $B_r(y_{|I},Y)$. We introduce the decomposition $\Gamma=\Gamma_1+\Gamma_2$ where

$$\begin{aligned}
\Gamma_1x(t) &= C(t)\varphi(0)+S(t)[\eta-g(0,\varphi)]+\int_0^t C(t-s)g(s,\bar{x}_s)ds+\sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-)-S(t-t_i)\mathcal{D}\bar{x}(t_i^+)]\\
&\quad -S(t-t_j)\mathcal{D}\bar{x}(t_j^+)+\int_0^a C(t-s)\mathcal{D}\bar{x}(s)ds+\sum_{0<t_i<t}C(t-t_i)I_i(\bar{x}_{t_i})+\sum_{0<t_i<t}S(t-t_i)J_i(\bar{x}_{t_i}).\\
\Gamma_2x(t) &= \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds.
\end{aligned}$$

From the proof of [39, Theorem 3.4], we conclude that Γ_2 is completely continuous. Moreover, from the estimate

$$\begin{aligned}
\|\Gamma_1x-\Gamma_1z\|_{\mathcal{PC}} &\leq aNL_gK_a\|x-z\|_{\mathcal{PC}}+3\overline{N}\|\mathcal{D}\|\|x-z\|_{\mathcal{PC}}+aN\|\mathcal{D}\|\|x-z\|_{\mathcal{PC}}\\
&\quad +K_a\sum_{i=1}^{n}(NL_{I_i}+\overline{N}L_{J_i})\|x-z\|_{\mathcal{PC}}\\
&\leq K_a[aNL_g+\frac{1}{K_a}(3\overline{N}+aN)\|\mathcal{D}\|+\sum_{i=1}^{n}(NL_{I_i}+\overline{N}L_{J_i})]\|x-z\|_{\mathcal{PC}}.
\end{aligned}$$

It follows that Γ_1 is contraction on $B_r(y_{|I},Y)$, which implies that Γ is a condensing operator on $B_r(y_{|I},Y)$.

Finally, from Lemma 2.2, we infer that there exists a mild solution of (1.1)-(1.4). The completes the proof. □

Theorem 3.2. *Let conditions $(H_\varphi),(H_1)-(H3)$ and $(H5)$ be hold and assume that $S(t)$ is compact for every $t\in\mathbb{R}$. If*

$$K_a\Big[aNL_g+\frac{1}{K_a}(3\overline{N}+aN)\|\mathcal{D}\|+\overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds+\sum_{i=1}^{n}(N\zeta_i+\overline{N}\eta_i)\Big]<1,$$

then there exists a mild solution of (1.1)-(1.4).

Proof. On the space $Y = \{x \in \mathcal{PC} : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology, we define the operator $\Gamma : Y \to Y$ by

$$\begin{aligned}\Gamma x(t) &= C(t)\varphi(0) + S(t)[\eta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] \\ &\quad -S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^c C(t-s)\mathcal{D}\bar{x}(s)ds + \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) \\ &\quad + \sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t \in I,\end{aligned}$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From axiom (A) and our assumptions on φ, we infer that $\Gamma x \in \mathcal{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(y_{|I}, Y)) \subseteq B_r(y_{|I}, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(y_{|I}, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - y(t^r)\|$. Then, from Lemma 3.1 we get

$$\begin{aligned} r &< \|\Gamma x^r(t^r) - y(t^r)\| \\ &\le NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\ &\quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + N\sum_{i=1}^n\|I_i(\bar{x}_{t_i})\| + \overline{N}\sum_{i=1}^n\|J_i(\bar{x}_{t_i})\| \\ &\le NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\ &\quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}W((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_ar + K_a \parallel y \parallel_a)\int_0^a m(s)ds \\ &\quad +N\sum_{i=1}^n\lambda_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}) + \overline{N}\sum_{i=1}^n\mu_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}).\end{aligned}$$

Since λ_i and μ_i are nondecreasing operators, we have

$$\begin{aligned} r &< NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\ &\quad +3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}W((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_ar + K_a \parallel y \parallel_a)\int_0^a m(s)ds \\ &\quad +N\sum_{i=1}^n\lambda_i(r^*) + \overline{N}\sum_{i=1}^n\mu_i(r^*),\end{aligned}$$

where $\|\bar{x}_{t_i}\|_{\mathcal{B}} \le r^* = (M_a + J_0^{\phi})\|\varphi\|_{\mathcal{B}} + K_a(r + \|y\|_a)$
and hence

$$1 \le K_a\Big[aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^n(N\zeta_i + \overline{N}\eta_i)\Big],$$

which contradicts to our assumption.

Arguing as in the proof of Theorem 3.1, we can prove that $\Gamma(\cdot)$ is a condensing map on $B_r(y|_I, Y)$ and, from Lemma 2.2, we conclude that there exists a mild solution $x(\cdot)$ for (1.1)-(1.4). The proof is now complete. □

4 Controllability results

In this section, we shall establish sufficient conditions for the controllability of mild solutions for a damped second order impulsive neutral functional differential equation with state-dependent delay. More precisely, we consider the following abstract control system in the form:

$$\frac{d}{dt}[x'(t) - g(t,x_t)] = Ax(t) + \mathcal{D}x'(t) + Bu(t) + f(t,x_{\rho(t,x_t)}), \quad t \in I = [0,a], \tag{4.1}$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = \eta \in X, \tag{4.2}$$
$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, \ldots, n, \tag{4.3}$$
$$\Delta x'(t_i) = J_i(x_{t_i}), \quad i = 1, 2, \ldots, n, \tag{4.4}$$

where $A, \mathcal{D}, f, I_i$ and J_i are defined as in equations (1.1)-(1.4), the control function $u(\cdot)$ given in $L^2(I, U)$, a Banach space of admissible control functions with U as a Banach space and $B : U \to X$ is a bounded linear operator on a Banach space X with $D(\mathcal{D}) \subset D(A)$.

Furthermore, we assume the following conditions:

$(H_1)'$ The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions:

(i) The function $f : I \times \mathcal{B} \to X$ is completely continuous.

(ii) For every positive constant r, there exists an $\alpha_r \in L^1(r)$ such that

$$\sup_{\|\psi\| \le r} \|f(t, \psi)\| \le \alpha_r(t).$$

(H_6) B is continuous operator from U to X and the linear operator $W : L^2(I, U) \to X$, defined by

$$Wu = \int_0^a S(a-s)Bu(s)ds,$$

has a bounded invertible operator, W^{-1} which takes the values in $L^2(I, U)/KerW$ such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$ for some positive integers M_1, M_2.

Definition 4.1. *The system (4.1)-(4.4) is said to be controllable on the interval* $[0, a]$ *if for every* $x_0 = \varphi \in \mathcal{B}, x'(0) = \eta \in X$ *and* $x_1 \in X$*, there exists a control* $u \in L^2(J, U)$ *such that the mild solution* $x(t)$ *of (4.1)-(4.4) satisfies* $x(a) = x_1$*.*

Definition 4.2. *A functions* $x : (-\infty, a] \to X$ *is called a mild solution of the abstract Cauchy problem (4.1)-(4.4) if* $x_0 = \varphi, x_{\rho(s,x_s)} \in \mathcal{B}$ *for every* $s \in I; x(\cdot)|_I \in \mathcal{PC}$ *and*

$$\begin{aligned} x(t) &= C(t)\varphi(0) + S(t)[\eta - g(0,\varphi)] + \int_0^t C(t-s)g(s,x_s)ds + \sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}x(t_{i+1}^-) - S(t-t_i)\mathcal{D}x(t_i^+)] \\ &- S(t-t_j)\mathcal{D}x(t_j^+) + \int_0^t C(t-s)\mathcal{D}x(s)ds + \int_0^t S(t-s)\big[Bu(s) + f(s, x_{\rho(s,x_s)})\big]ds \\ &+ \sum_{0<t_i<t} C(t-t_i)I_i(x_{t_i}) + \sum_{0<t_i<t} S(t-t_i)J_i(x_{t_i}), \quad t \in I. \end{aligned}$$

Theorem 4.1. *Let conditions* $(H_\varphi), (H_1) - (H_6)$ *and* $(H_1)'$ *be hold. Then the system (4.1)-(4.4) is controllable on* $(-\infty, a]$ *provided that*

$$(1 + a\overline{N}M_1M_2)\Big[K_a\Big(aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^n (NL_{I_i} + \overline{N}L_{J_i})\Big)\Big] < 1,$$

Proof. Consider the space $Y = \{x \in \mathcal{PC}; u(0) = \varphi(0)\}$ endowed with the uniform convergence topology. Using the assumption (H_6), for an arbitrary function $x(\cdot)$, we define the control

$$\begin{aligned} u(t) = W^{-1}\Big[&x_1 - C(a)\varphi(0) - S(t)[\eta - g(0,\varphi)] - \int_0^a C(a-s)g(s,x_s)ds - \sum_{i=0}^{j-1}[S(a-t_{i+1})\mathcal{D}x(t_{i+1}^-) \\ &- S(a-t_i)\mathcal{D}x(t_i^+)] + S(a-t_j)\mathcal{D}x(t_j^+) - \int_0^a C(a-s)\mathcal{D}x(s)ds - \int_0^a S(a-s)f(s, x_{\rho(s,x_s)})ds \\ &- \sum_{0<t_i<a} C(a-t_i)I_i(x_{t_i}) - \sum_{0<t_i<a} S(a-t_i)J_i(x_{t_i})\Big](t). \end{aligned}$$

Using this control, we shall show that the operator $\Gamma : Y \to Y$ defined by

$$\Gamma x(t) = C(t)\varphi(0) + S(t)[\eta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-)$$

$$-S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] - S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^t C(t-s)\mathcal{D}\bar{x}(s)ds + \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds$$
$$+ \int_0^t S(t-\xi)BW^{-1}\Big[x_1 - C(a)\varphi(0) - S(a)[\eta - g(0,\varphi)] - \int_0^a C(a-s)g(s,\bar{x}_s)ds$$
$$-\sum_{i=0}^{j-1}[S(a-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(a-t_i)\mathcal{D}\bar{x}(t_i^+)] + S(a-t_j)\mathcal{D}\bar{x}(t_j^+) - \int_0^a C(a-s)\mathcal{D}\bar{x}(s)ds$$
$$-\sum_{0<t_i<a} C(a-t_i)I_i(\bar{x}_{t_i}) - \sum_{0<t_i<a} S(a-t_i)J_i(\bar{x}_{t_i})\Big](\xi)d\xi + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i})$$
$$+\sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t \in I,$$

has a fixed point $x(\cdot)$. This fixed point $x(\cdot)$ is then a mild solution of the system (4.1)-(4,4). Clearly, $(\Gamma x)(a) = x_1$, which means that the control u steers the system from the initial state φ to x_1 in time a, provided we obtain a fixed point of the operator which implies that the system is controllable. Here $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From the axiom (A) and our assumptions on φ, we infer that $\Gamma x \in \mathcal{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(y_{|I}, Y)) \subseteq B_r(y_{|I}, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(y_{|I}, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - y(t^r)\|$. Then, from Lemma 3.1, we get

$$\begin{aligned}
r &< \|\Gamma x^r(t^r) - y(t^r)\| \\
&\le NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
&+3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + \overline{N}M_1M_2\int_0^{t^r}\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} \\
&+\overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
&+3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + \sum_{i=1}^n N(L_{I_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|I_i(y_{t_i})\|) \\
&+\sum_{i=1}^n \overline{N}(L_{J_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|J_i(y_{t_i})\|)\Big] + \sum_{i=1}^n N(L_{I_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|I_i(y_{t_i})\|) \\
&+\sum_{i=1}^n \overline{N}(L_{J_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|J_i(y_{t_i})\|) \\
r &\le NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
&+3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}W\Big((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_ar + K_a\|y\|_a\Big)\int_0^a m(s)ds \\
&+\overline{N}M_1M_2\int_0^{t^r}\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds \\
&+N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds + 3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds \\
&+\sum_{i=1}^n N(L_{I_i}K_ar + \|I_i(y_{t_i})\|) + \sum_{i=1}^n \overline{N}(L_{J_i}K_ar + \|J_i(y_{t_i})\|)\Big] \\
&+\sum_{i=1}^n N(L_{I_i}K_ar + \|I_i(y_{t_i})\|) + \sum_{i=1}^n \overline{N}(L_{J_i}K_ar + \|J_i(y_{t_i})\|),
\end{aligned}$$

and hence

$$1 \le (1 + a\overline{N}M_1M_2)\Big[K_a\Big(aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^n(NL_{I_i} + \overline{N}L_{J_i})\Big)\Big],$$

which contradicts to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(y_{|I}, Y)) \subset B_r(y_{|I}, Y)$. In order to prove that Γ is a condensing map on $B_r(y_{|I}, Y)$ into $B_r(y_{|I}, Y)$. We introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned}\Gamma_1 x(t) &= C(t)\varphi(0) + S(t)[\eta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] \\ &\quad -S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^a C(t-s)\mathcal{D}\bar{x}(s)ds + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) + \sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}). \\ \Gamma_2 x(t) &= \int_0^t S(t-s)\Big[f(s,\bar{x}_{\rho(s,\bar{x}_s)}) + Bu(s)\Big]ds.\end{aligned}$$

Now

$$\begin{aligned}\|Bu(s)\| &\le \|B\|\|W^{-1}\|\Big[\|x_1\| + \|C(a)\|\|\varphi(0)\| + \|S(t)\|\big[\|\eta\| + \|g(0,\varphi)\|\big] + \int_0^a \|C(a-s)\|\|g(s,\bar{x}_s)ds\| \\ &\quad + \sum_{i=0}^{j-1}[\|S(a-t_{i+1})\|\|\mathcal{D}\|\|\bar{x}(t_{i+1}^-)\| + \|S(a-t_i)\|\|\mathcal{D}\|\|\bar{x}(t_i^+)]\| + \|S(a-t_j)\|\|\mathcal{D}\|\|\bar{x}(t_j^+)\| \\ &\quad + \int_0^a \|C(a-s)\|\|\mathcal{D}\|\|\bar{x}(s)\|ds + \int_0^a \|S(a-s)\|\|f(s,\bar{x}_{\rho(s,\bar{x}_s)})\|ds + \sum_{0<t_i<a}\|C(a-t_i)\|\|I_i(\bar{x}_{t_i})\| \\ &\quad + \sum_{0<t_i<a}\|S(a-t_i)\|\|J_i(\bar{x}_{t_i})\|\Big] \\ &\le M_1M_2\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \overline{N}\big[\|\eta\| + c_1\|\varphi\| + c_2\big] + N\int_0^a (c_1\|\bar{x}_s\| + c_2)ds + 3\overline{N}\|D\|r + aN\|\mathcal{D}\|r \\ &\quad + \overline{N}\int_0^a \alpha_r(s)ds + N\sum_{i=1}^n \lambda_i\|\bar{x}_{t_i}\| + \sum_{i=1}^n \mu_i\|\bar{x}_{t_i}\|\Big] \\ &\le M_1M_2\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \overline{N}\big[\|\eta\| + c_1\|\varphi\| + c_2\big] + aN(c_1r + c_2) + 3\overline{N}\|D\|r + aN\|\mathcal{D}\|r \\ &\quad + \overline{N}\int_0^a \alpha_r(s)ds + \sum_{i=1}^n r(N\lambda_i + \overline{N}\mu_i)\Big] = A_0.\end{aligned}$$

Here by applying the same technique that is used in the proof of [16, Lemma 3.1], we arrived that Γ_2 is completely continuous.

Next, we show that Γ_1 is contraction on $B_r(y|_I, Y)$. Indeed, $x, z \in B_r(y|_I, Y)$, we have

$$\begin{aligned}\|\Gamma_1 x - \Gamma_1 z\|_{\mathcal{PC}} &\le aN\|\mathcal{D}\|\|x-z\|_{\mathcal{PC}} + aNL_gK_a\|x-z\|_{\mathcal{PC}} + 3\overline{N}\|\mathcal{D}\|\|x-z\|_{\mathcal{PC}} + \sum_{i=1}^n NL_{I_i}K_a\|x-z\|_{\mathcal{PC}} \\ &\quad + \sum_{i=1}^n \overline{N}L_{J_i}K_a\|x-z\|_{\mathcal{PC}} \\ &\le K_a\Big[aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \sum_{i=1}^n (NL_{I_i} + \overline{N}L_{J_i})\Big]\|x-z\|_{\mathcal{PC}}.\end{aligned}$$

It follows that Γ_1 is a contraction on $B_r(y|_I, Y)$ which implies that Γ is a condensing operator on $B_r(y|_I, Y)$.

Finally, from the Sadovskii's Fixed Point Theorem, Γ has a fixed point on Y. This means that any fixed point of Γ is a mild solution of the problem (4.1)-(4.4). This completes the proof. □

Theorem 4.2. *Let conditions $(H_\varphi), (H_1) - (H_3), (H_5)$ and $(H_1)'$ be hold. Then the system (4.1)-(4.4) is controllable on $(-\infty, a]$ provided that*

$$(1 + a\overline{N}M_1M_2)\Big[K_a\Big(aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^n (N\zeta_i + \overline{N}\eta_i)\Big)\Big] < 1.$$

Proof. Consider the space $Y = \{x \in \mathcal{PC}; u(0) = \varphi(0)\}$ endowed with the uniform convergence topology. Using the assumption (H_6), for an arbitrary function $x(\cdot)$, we define the control

$$u(t) = W^{-1}\Big[x_1 - C(a)\varphi(0) - S(t)[\eta - g(0,\varphi)] - \int_0^a C(a-s)g(s,x_s)ds - \sum_{i=0}^{j-1}[S(a-t_{i+1})\mathcal{D}x(t_{i+1}^-)$$

$$- S(a-t_i)\mathcal{D}x(t_i^+)] + S(a-t_j)\mathcal{D}x(t_j^+) - \int_0^a C(a-s)\mathcal{D}x(s)ds - \int_0^a S(a-s)f(s,x_{\rho(s,x_s)})ds$$
$$- \sum_{0<t_i<a} C(a-t_i)I_i(x_{t_i}) - \sum_{0<t_i<a} S(a-t_i)J_i(x_{t_i})\Big](t).$$

Using this control, we shall show that the operator $\Gamma : Y \to Y$ defined by

$$\begin{aligned}
\Gamma x(t) \;\; = \;\; & C(t)\varphi(0) + S(t)[\eta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \sum_{i=0}^{j-1}[S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) \\
& -S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] - S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^t C(t-s)\mathcal{D}\bar{x}(s)ds + \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds \\
& + \int_0^t S(t-\xi)BW^{-1}\Big[x_1 - C(a)\varphi(0) - S(a)[\eta - g(0,\varphi)] - \int_0^a C(a-s)g(s,\bar{x}_s)ds \\
& - \sum_{i=0}^{j-1}[S(a-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(a-t_i)\mathcal{D}\bar{x}(t_i^+)] + S(a-t_j)\mathcal{D}\bar{x}(t_j^+) - \int_0^a C(a-s)\mathcal{D}\bar{x}(s)ds \\
& - \sum_{0<t_i<a} C(a-t_i)I_i(\bar{x}_{t_i}) - \sum_{0<t_i<a} S(a-t_i)J_i(\bar{x}_{t_i})\Big](\xi)d\xi + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) \\
& + \sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t \in I,
\end{aligned}$$

has a fixed point $x(\cdot)$. This fixed point $x(\cdot)$ is then a mild solution of the system (4.1)-(4,4). Clearly, $(\Gamma x)(a) = x_1$, which means that the control u steers the system from the initial state φ to x_1 in time a, provided we obtain a fixed point of the operator which implies that the system is controllable. Here $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From the axiom (A) and our assumptions on φ, we infer that $\Gamma x \in \mathcal{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(y_{|I}, Y)) \subseteq B_r(y_{|I}, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(y_{|I}, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - y(t^r)\|$. Then, from Lemma 3.1, we get

$$\begin{aligned}
& r < \|\Gamma x^r(t^r) - y(t^r)\| \\
& \quad \leq \;\; NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
& \qquad + 3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + \overline{N}M_1M_2\int_0^{t^r}\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} \\
& \qquad + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
& \qquad + 3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + N\sum_{i=1}^{n}\lambda_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}) \\
& \qquad + \overline{N}\sum_{i=1}^{n}\mu_i(\|\bar{x}_{t_i}\|_{\mathcal{B}})\Big] + N\sum_{i=1}^{n}\lambda_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}) + \overline{N}\sum_{i=1}^{n}\mu_i(\|\bar{x}_{t_i}\|_{\mathcal{B}}).
\end{aligned}$$

Since λ_i and μ_i are non-decreasing operators, we have

$$\begin{aligned}
r \;\; \leq \;\; & NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds \\
& + 3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}W\Big((M_a + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_ar + K_a\|y\|_a\Big)\int_0^a m(s)ds \\
& + \overline{N}M_1M_2\int_0^{t^r}\Big[\|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \overline{N}[\|\eta\| + \|g(0,\varphi)\|] + NL_gK_a\int_0^{t^r}\|\overline{x^r} - y\|_s ds \\
& + N\int_0^{t^r}(c_1\|y_s\|_{\mathcal{B}} + c_2)ds + 3\overline{N}\|\mathcal{D}\|_r + aN\|\mathcal{D}\|_r + \overline{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathcal{B}})ds + N\sum_{i=1}^{n}\lambda_i(r^*) \\
& + \overline{N}\sum_{i=1}^{n}\mu_i(r^*)\Big] + N\sum_{i=1}^{n}\lambda_i(r^*) + \overline{N}\sum_{i=1}^{n}\mu_i(r^*),
\end{aligned}$$

where $\|\bar{x}_{t_i}\|_{\mathcal{B}} \le r^* = (M_a + J_0^{\phi})\|\varphi\|_{\mathcal{B}} + K_a(r + \|y\|_a)$
and hence

$$1 \le (1 + a\overline{N}M_1M_2)\Big[K_a\Big(aNL_g + \frac{1}{K_a}(3\overline{N} + aN)\|\mathcal{D}\| + \overline{N}\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + \sum_{i=1}^{n}(N\zeta_i + \overline{N}\eta_i)\Big)\Big],$$

which contradicts to our assumption.

Arguing as in the proof of Theorem 4.1, we can prove that $\Gamma(\cdot)$ is a condensing map on $B_r(y|_I, Y)$ and, from Lemma 2.2, we conclude that there exists a mild solution $x(\cdot)$ for (4.1)-(4.4). The proof is now completed.

5 An example

In this section, we consider an application of our abstract results. We choose the space $X = L^2([0,\pi]), \mathcal{B} = \mathcal{PC}_0 \times L^2(g, X)$ is the space introduced in [50] and $A : D(A) \subset X \to X$ is the operator defined by $Au = u''$ with domain $D(A) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}$. It is well-known that A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on X. Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, for $n \in \mathbb{N}$, with corresponding eigenvectors $z_n(\tau) = \left(\frac{2}{\pi}\right)^{1/2}\sin(n\tau)$, and the following properties hold.

(a) The set of functions $\{z_n : n \in \mathbb{N}\}$ forms an orthonormal basis of X.

(b) If $x \in D(A)$, then $Ax = -\sum_{n=1}^{\infty} n^2\langle x, x_n\rangle x_n$, for $\varphi \in D(A)$.

(c) For $x \in X$, $C(t)x = \sum_{n=1}^{\infty}\cos(nt)\langle x, x_n\rangle x_n$ and the associated sine family is

$$S(t)x = \sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle x, x_n\rangle x_n,$$

which implies that the operator $S(t)$ is compact, for all $t \in \mathbb{R}$ and that $\|C(t)\| = \|S(t)\| = 1$, for all $t \in \mathbb{R}$.

(d) If G is the group of translations on X defined by $G(t)x(\zeta) = \tilde{x}(\zeta + t)$, where $\tilde{x}(\cdot)$ is the extension of $x(\cdot)$ with period 2π, then $C(t) = \frac{1}{2}\big[\Phi(t) + \Phi(-t)\big]$ and $A = B^2$, where B is the infinitesimal generator of Φ and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$ (see [52] for details).

5.1 Second order neutral system

Consider the following second order damped impulsive neutral differential system with state-dependent delay

$$\frac{\partial}{\partial t}\Big[\frac{\partial}{\partial t}w(t,\zeta) + \int_{-\infty}^{t}\int_0^{\pi} b(t-s,\eta,\zeta)w(s,\eta)d\eta ds\Big] = \frac{\partial^2}{\partial\zeta^2}w(t,\zeta) + \alpha\frac{\partial}{\partial t}w(t,\zeta) + \int_0^{\pi}\beta(s)\frac{\partial}{\partial t}w(t,s)ds$$
$$+ \int_{-\infty}^{t} k(s-t)w(s - \rho_1(t)\rho_2(\|w(t)\|),\zeta)ds, \quad t \in I,\ \zeta \in [0,\pi] \tag{5.1}$$

$$w(t,0) = w(t,\pi) = 0, \quad t \in I \tag{5.2}$$

$$\frac{\partial}{\partial t}w(0,\zeta) = \zeta(\pi), \tag{5.3}$$

$$w(\tau,\zeta) = \varphi(\tau,\zeta), \quad \tau \le 0, \quad 0 \le \zeta \le \pi \tag{5.4}$$

$$\triangle w(t_i)(\zeta) = \int_{-\infty}^{t_i} b_i(t_i - s)w(s,\zeta)ds, \quad i = 1,2,\ldots,n, \tag{5.5}$$

$$\triangle w'(t_i)(\zeta) = \int_{-\infty}^{t_i} \tilde{b}_i(t_i - s)w(s,\zeta)ds, \quad i = 1,2,\ldots,n, \tag{5.6}$$

where we assume that $\varphi \in \mathcal{B}$ with the identity $\varphi(s)(\zeta) = \varphi(s,\zeta), \varphi(0,\cdot) \in H^1([0,\pi])$ and $0 < t_1 < t_2 < \cdots < a$. Here α is a prefixed real number and $\beta \in L^2([0,\pi])$.

Let the functions $\rho_i : [0,\infty) \to [0,\infty), i = 1,2,;\ k : \mathbb{R} \to \mathbb{R}$ are continuous, $L_f = \Big(\int_{-\infty}^{0}\frac{(a^2(s))}{g(s)}ds\Big)^{\frac{1}{2}} < \infty$, and that the following conditions hold:

(a) The functions $b_i, \tilde{b}_i \in C(\mathbb{R},\mathbb{R})$ and $L_{I_i} := \left(\int_{-\infty}^{0} \frac{b_i^2(s)}{g(s)} ds\right)^{\frac{1}{2}}$, $L_{J_i} := \left(\int_{-\infty}^{0} \frac{\tilde{b}_i^2(s)}{g(s)} ds\right)^{\frac{1}{2}}$, $i = 1, \cdots, n$, are finite.

(b) The functions $b(s,\eta,\zeta)$, $\frac{\partial b(s,\eta,\zeta)}{\partial \zeta}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ and

$$L_g = \max\left\{ \left(\int_0^{\pi} \int_{-\infty}^{0} \int_0^{\pi} \frac{1}{g(s)} \left(\frac{\partial^i b(s,\eta,\zeta)}{\partial \zeta^i} \right)^2 d\eta ds d\zeta \right)^{\frac{1}{2}} : i = 0,1 \right\} < \infty.$$

Define the functions $\mathcal{D} : X \to X, g, f : J \times \mathcal{B} \to X,\ \rho : I \times \mathcal{B} \to X,\ I_i : \mathcal{B} \to X$ and $J_i : \mathcal{B} \to X$ by

$$\begin{aligned}
\mathcal{D}\psi(\zeta) &= \alpha\psi(t,\zeta) + \int_0^{\pi} \beta(s)\psi(t,s)ds, \\
f(\psi)(\zeta) &= \int_{-\infty}^{0} k(s)\psi(s,\zeta)ds, \\
g(\psi)(\zeta) &= \int_{-\infty}^{0} \int_0^{\pi} b(s,\nu,\zeta)\psi(s,\nu)d\nu ds, \\
\rho(s,\psi) &= s - \rho_1(s)\rho_2(\|\psi(0)\|), \\
I_i(\psi)(\zeta) &= \int_{-\infty}^{0} b_i(-s)\psi(s,\zeta)ds, \quad i = 1,2,\ldots,n, \\
J_i(\psi)(\zeta) &= \int_{-\infty}^{0} \tilde{b}_i(-s)\psi(s,\zeta)ds, \quad i = 1,2,\ldots,n.
\end{aligned}$$

With the choice of $A, \mathcal{D}, f, g, \rho, I_i$ and J_i, the system (1.1)-(1.4) is the abstract formulation of (5.1)-(5.6). Moreover, the maps $\mathcal{D}, g,\ f,\ I_i,\ J_i,\ i = 1,2,\ldots,n$ are bounded linear operators with

$$\|\mathcal{D}\|_{\mathcal{L}(X)} \le |\alpha| + \|\beta\|_{L^2(0,a)},\ \|g(t,\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \le L_g,\ \|f(t,\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \le L_f,\ \|I_i\|_{\mathcal{L}(\mathcal{B},X)} \le L_{I_i}, \|J_i\|_{\mathcal{L}(\mathcal{B},X)} \le L_{J_i}.$$

□

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Received: October 11, 2012; *Accepted*: January 7, 2013

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