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Relative controllability of fractional stochastic dynamical systems with multiple delays in control

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Abstract

This paper is concerned with the global relative controllability of fractional stochastic dynamical systems with multiple delays in control for finite dimensional spaces. Sufficient conditions for controllability results are obtained using Banach fixed point theorem and the controllability Grammian matrix which is defined by the Mittag-Leffler matrix function. An example is provided to illustrate the theory.

Keywords: Control delay, Relative controllability, Stochastic systems, Fractional differential equations, Mittag-Leffler matrix function.

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1 Introduction

Control theory is an important area of application oriented mathematics which deals with the design and analysis of control systems. In particular, the concept of controllability plays an important role in both the deterministic and the stochastic control theory. In recent years, controllability problems for various types of nonlinear dynamical systems in infinite dimensional spaces by using different kinds of approaches have been considered in many publications. An extensive list of these publications can be found (see [2, 3, 6, 17] and the references therein). Moreover, the exact controllability enables to steer the system to arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. Klamka [8] derived a set of sufficient conditions for the exact controllability of semilinear systems. Further, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications. The approximate controllability of systems represented by nonlinear evolution equations has been investigated by several authors [9, 13, 14, 18], in which the authors effectively used the fixed point approach. Fu and Mei [6] studied the approximate controllability of semilinear neutral functional differential systems with finite delay. The conditions are established with the help of semigroup theory and fixed point technique under the assumption that the linear part of the associated nonlinear system is approximately controllable.

Stochastic differential equations have many applications in economics, ecology and finance. In recent years, the controllability problems for stochastic differential equations have become a field of increasing interest, (see [10, 11, 19] and references therein). The extensions of deterministic controllability concepts to stochastic control systems have been discussed only in a limited number of publications.

We would like to mention that controllability and approximate controllability of fractional dynamical systems with or without delay in control have been considered by a few authors (see, for instance [1, 5, 20]). As for the stochastic systems, there are less number of papers on the controllability and the approximate controllability of fractional stochastic dynamical systems with delay in control. Recently, Sakthivel et al. [16] established a set of sufficient conditions for obtaining the approximate controllability of semilinear fractional differential systems in

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Hilbert spaces. The same author in [15] prove the approximate controllability of nonlinear fractional stochastic control system under the assumptions that the corresponding linear system is approximately controllable. More recently, the approximate controllability of neutral stochastic fractional integro-differential equation with infinite delay in a Hilbert space by using Krasnoselskii's fixed point theorem and stochastic analysis theory has been discussed in [18]. The authors derived a new set of sufficient conditions for the approximate controllability of nonlinear fractional stochastic system under the assumption the corresponding linear system is approximately controllable. Shen [21] studied the relative controllability of stochastic nonlinear systems with delay in control. However, to the best of our knowledge, there are no relevant reports on the relative controllability of fractional stochastic dynamical systems with multiple delay in control as treated in the current paper. Motivated by this consideration, in this article we will study the global relative controllability problem for fractional stochastic dynamical systems with multiple delays in control variables for finite dimensional spaces. Sufficient conditions for the controllability results are obtained by using the Banach fixed point theorem and fractional calculus. The paper is organized as follows: In Section 2, some well known fractional operators and special functions, along with a set of properties are defined and the solution representation of linear fractional stochastic differential equations are derived using Laplace transform for further discussion. In Section 3, the linear and nonlinear stochastic fractional dynamical systems with multiple delays in control are proposed and the controllability condition is established using the controllability Grammian matrix which is defined by means of the Mittag-Leffler matrix function. In Section 4, example is discussed to illustrate the effectiveness of our results. Finally, concluding remarks are given in Section 5.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Let $\alpha, \beta > 0$, with $n-1 < \alpha < n$, $n-1 < \beta < n$ and $n \in \mathbb{N}$, D is the usual differential operator. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space, $\mathbb{R}_+ = [0, \infty)$, and suppose $f \in L^1(\mathbb{R}_+)$. The following definitions and properties are well known, for $\alpha, \beta > 0$ and f as a suitable function (see, for instance, [7]):

(a) Riemann-Liouville fractional operators:

$$\begin{aligned} (I^{\alpha}_{0+}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \\ (D^{\alpha}_{0+}f)(x) &= D^n (I^{n-\alpha}_{a+} f)(x). \end{aligned}$$

(b) Caputo fractional derivative:

$$({}^cD^{\alpha}_{0+}f)(x) = (I^{n-\alpha}_{0+} D^n f)(x),$$

in particular $I^{\alpha}_{0+}\,{}^cD^{\alpha}_{0+} f(t) = f(t) - f(0)$, $(0 < \alpha < 1)$.

The following is a well known relation, for finite interval $[a, b] \in \mathbb{R}_+$

$$(D^{\alpha}_{a+}f)(x) = ({}^cD^{\alpha}_{a+}f)(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha}, \quad n = \mathcal{R}(\alpha) + 1.$$

The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\{{}^cD^{\alpha}_{0+}f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+)s^{\alpha-1-k}.$$

The Riemann-Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann-Liouville sense require initial conditions of special form lacking physical interpretation. To overcome this difficulty Caputo introduced a new definition of fractional derivative but in general, both the Riemann-Liouville and the Caputo fractional operators possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. Due to this fact, the concept of sequential fractional differential equations are discussed in [7].

(c) Linear Sequential Derivative:

For $n \in \mathbb{N}$ the sequential fractional derivative for suitable function f is defined by

$$f^{(k\alpha)} := (\mathbf{D}^{k\alpha} f)(x) = (\mathbf{D}^{\alpha}\mathbf{D}^{(k-1)\alpha} f)(x),$$

where $k = 1, \ldots, n$, $(\mathbf{D}^{\alpha} f)(x) = f(x)$, and $\mathbf{D}^{\alpha}$ is any fractional differential operator, here we mention it as ${}^{c}D^{\alpha}_{0+}$.

(d) Mittag-Leffler Function

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k\alpha+\beta)}, \quad \alpha, \beta > 0.$$

The general Mittag-Leffler function satisfies

$$\int_0^{\infty} e^{-t} t^{\beta-1} E_{\alpha,\beta}(t^{\alpha} y) dt = \frac{1}{1-y}, \quad |y| < 1.$$

The Laplace transform of $E_{\alpha,\beta}(y)$ follows from the integral

$$\int_0^{\infty} e^{-st} t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha}) dt = \frac{s^{\alpha-\beta}}{(s \mp a)}.$$

That is

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha})\} = \frac{s^{\alpha-\beta}}{(s \mp a)},$$

for $\mathcal{R}(s) > |a|^{1/\alpha}$ and $\mathcal{R}(\beta) > 0$. In particular, for $\beta = 1$,

$$E_{\alpha,1}(\lambda y^{\alpha}) = E_{\alpha}(\lambda y^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k y^{k\alpha}}{\Gamma(\alpha k+1)}, \quad \lambda, y \in \mathbf{C}$$

have the interesting property ${}^{c}D^{\alpha} E_{\alpha}(\lambda t^{\alpha}) = \lambda E_{\alpha}(\lambda t^{\alpha})$ and

$$\mathcal{L}\{E_{\alpha}(\pm a t^{\alpha})\} = \frac{s^{\alpha-1}}{(s \mp a)}, \quad \text{for } \beta = 1.$$

For brevity of notation let us take I^q_{0+} as I^q and ${}^{c}D^q_{0+}$ as ${}^{c}D^q$ and the fractional derivative is taken as Caputo sense.

Let us consider the linear fractional stochastic differential equation of the form

$$\begin{aligned} {}^{c}D^q x(t) &= Ax(t) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in [0, T], \\ x(0) &= x_0, \end{aligned} \tag{2.1}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$, A is an $n \times n$ matrix, $w(t)$ is a given l-dimensional Wiener process with the filtration $\mathcal{F}_t$ generated by $w(s)$, $0 \le s \le t$ and $\sigma : [0, T] \to \mathbb{R}^{n\times l}$ is appropriate function. In order to find the solution, apply Laplace transform on both sides and use the Laplace transform of Caputo derivative, we get

$$s^q X(s) - s^{q-1} x(0) = AX(s) + \Sigma(s)\frac{dw(s)}{ds}.$$

Apply inverse Laplace transform on both sides (see [4]) we have

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{s^{q-1}(s^q I - A)^{-1}\} x_0 + \mathcal{L}^{-1}\{\Sigma(s)\tfrac{dw(s)}{ds}\} * \mathcal{L}^{-1}\{(s^q I - A)^{-1}\}.$$

Finally, substituting Laplace transformation of the Mittag-Leffler function, we get the solution of the given system

$$x(t) = E_q(At^q) x_0 + \int_0^t (t-s)^{q-1} \left(\int_0^{\tau} \sigma(\theta) dw(\theta) \right) E_{q,q}(A(t-s)^q) ds$$

where $E_q(At^q)$ is the matrix extension of the mentioned Mittag-Leffler functions with the following representation:

$$E_q(At^q) = \sum_{k=0}^{\infty} \frac{A^k t^{kq}}{\Gamma(1+kq)}$$

with the property ${}^{c}D^q E_q(At^q) = AE_q(At^q)$.

3 Controllability results

Let $L^2_{\mathcal{F}_t}(J \times \Omega, \mathbb{R}^n)$ be the Banach space of all $\mathcal{F}_t$-measurable square integrable processes $x(t)$ with norm $\|x\|^2_{L^2} = \sup_{t\in J} \mathbb{E}\|x(t)\|^2$, where $\mathbb{E}(.)$ denotes the expectation with respect to the measure $\mathbb{P}$. Let $C = C([0,T]; L^2_{\mathcal{F}_t})$ be the Banach space of continuous maps from $[0,T]$ into $L^2_{\mathcal{F}_t}(J \times \Omega, \mathbb{R}^n)$ satisfying $\sup_{t\in J} \mathbb{E}\|x(t)\|^2 < \infty$. Consider the linear fractional stochastic dynamical system with multiple delays in control represented by the fractional stochastic differential equation of the form

$$\begin{aligned} {}^cD^q x(t) &= Ax(t) + \sum_{k=1}^{M} B_k u(h_k(t)) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in J := [0,T] \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^l$, A is an $n \times n$ matrix, B_k are $n \times l$ matrices, for $k = 0, 1, \ldots, M$, $w(t)$ is a given l-dimensional Wiener process with the filtration $\mathcal{F}_t$ generated by $w(s)$, $0 \le s \le t$ and $\sigma : [0,T] \to \mathbb{R}^{n\times l}$ is appropriate function.

Let us assume the following assumptions:

(**i**) Assume the function $h_k : J \to \mathbb{R}$, $k = 0, 1, \ldots, M$ are twice continuously differentiable and strictly increasing in J. Moreover,

$$h_k(t) \le t \quad \text{for } t \in J, i = 0, 1, \ldots, M. \tag{3.2}$$

(**ii**) Introduce the time lead functions $r_k(t) : [h_k(0), h_k(T)] \to J$, $k = 0, 1, \ldots, M$ such that $r_k(h_k(t)) = t$ for $t \in J$. Further assume that $h_0(t) = t$ and for $t = T$, the following inequalities hold

$$h_M(T) \le h_{M_1}(T) \le \ldots h_{M_{m+1}}(T) \le 0 = h_m(T) < h_{m-1}(T) = \ldots h_1(T) = h_0(T) = T. \tag{3.3}$$

(**iii**) let $h > 0$ be given. For functions $u : [-h, T] \to \mathbb{R}^l$ and $t \in J$, we use the symbol u_t to denote the function on $[-h, 0]$, defined by $u_t(s) = u(t+s)$ for $s \in [-h, 0)$.

The following definitions of complete state of the system (2) at time t and relative controllability are assumed.

Definition 3.1. *The set $\phi(t) = \{x(t), u_t\}$ is the complete state of the system* (2) *at time t.*

Definition 3.2. *System* (2) *is said to be globally relatively controllable on J if for every complete state $\phi(0)$ and every vector $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on J such that the corresponding trajectory of the system* (2) *satisfies $x(T) = x_1$.*

Note that the solution of system (2) ca be expressed in the following form

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \sum_{k=0}^{M} B_k u(h_k(s)) ds \\ &+ \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta) dw(\theta) \right) E_{q,q}(A(t-s)^q) ds. \end{aligned}$$

Taking into account the time lead functions $r_k(t)$, this solution can be further changed into

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^{M} \int_{h_k(0)}^{h_k(t)} (t - r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) u(s) ds \\ &+ \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta) dw(\theta) \right) E_{q,q}(A(t-s)^q) ds. \end{aligned} \tag{3.4}$$

Using the inequalities (4), the above equation becomes,

$$\begin{aligned} x(t) &= E_q(At^q)x_0 + \sum_{k=0}^{m}\int_{h_k(0)}^{0}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \sum_{k=0}^{m}\int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u(s)ds \\ &+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds. \end{aligned} \tag{3.5}$$

For brevity, let us introduce the following notation:

$$\begin{aligned} \varphi(t) &= \sum_{k=0}^{m}\int_{h_k(0)}^{0}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \end{aligned} \tag{3.6}$$

and

$$\chi(t) = \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds.$$

Recall the controllability Grammian matrix

$$\psi_0^T = \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)][E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)]^\star ds$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and the $\star$ denotes the matrix transpose.

Theorem 3.3. *The linear stochastic control system* (2) *is relatively controllable on* $[0,T]$ *if and only if the controllability Grammian matrix* ψ_0^T *is positive definite for some* $T > 0$.

Proof. Since ψ is positive definite, it is non-singular and therefore its inverse is well defined. Define the control function as,

$$u(t) = [B_k^\star E_{q,q}(A^\star(T-r_k(t))^q)r_k'(t)]\psi^{-1}[x_1 - E_q(At^q)x_0 - \varphi(T) - \chi(T)], \quad k=0,1,\ldots,m \tag{3.7}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily. Inserting (8) in (6) and using (7) we get

$$\begin{aligned} x(T) &= E_q(At^q)x_0 + \varphi(T) + \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)] \\ &\times [B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)]\psi^{-1}[x_1 - E_q(AT^q)x_0 - \varphi(T) - \chi(T)]ds \\ &+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\ &= x_1. \end{aligned}$$

Thus the control $u(t)$ transfers the initial state $\phi(0)$ to the desired vector $x_1 \in \mathbb{R}^n$ at time T. Hence the system (2) is controllable.

On the other hand, if it is not positive definite, there exists a nonzero ϕ such that $\phi^\star\psi\phi = 0$, that is

$$\begin{aligned} &\phi^\star\sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)][E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)]^\star\phi ds = 0 \\ &\phi^\star\sum_{k=0}^{m}(T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)] = 0, \end{aligned}$$

on $[0,T]$. Let $x_0 = [E_q(AT^q)]^{-1}\phi$. By assumption, there exists a control u such that it steers the complete initial state $\phi(0) = \{x(0), u_0(s)\}$ to the origin in the interval $[0,T]$. It follows that

$$\begin{aligned}
x(T) &= E_q(At^q)x_0 + \varphi(T) + \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)] \\
&\times [B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)]\psi^{-1}[x_1 - E_q(AT^q)x_0 - \varphi(T) - \chi(T)]ds \\
&+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\
&= \phi + \varphi(T) + \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)] \\
&\times [B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)]\psi^{-1}[x_1 - E_q(AT^q)x_0 - \varphi(T) - \chi(T)]ds \\
&+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\
&= 0.
\end{aligned}$$

Thus,

$$0 = \phi^\star\phi + \sum_{k=0}^{m}\int_0^T \phi^\star(T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)]u(s)ds + \phi^\star(\varphi(T) + \chi(T)).$$

But the second and third term are zero leading to the conclusion $\phi^\star\phi = 0$. This is a contradiction to $\phi \neq 0$. Thus ψ is positive definite. Hence the desired result. □

Consider a nonlinear fractional stochastic dynamical system with multiple delays in control represented by the fractional stochastic differential equation of the form

$$\begin{aligned}
{}^cD^q x(t) &= Ax(t) + \sum_{k=1}^{M} B_k u(h_k(t)) + f(t,x(t)) + \sigma(t,x(t))\frac{dw(t)}{dt}, \quad t \in J := [0,T] \\
x(0) &= x_0,
\end{aligned} \tag{3.8}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^l$, A, B_k are defined as above and $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : J \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are appropriate functions. Then the solution of the system (9) ca be expressed in the following form

$$\begin{aligned}
x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\sum_{k=0}^{M} B_k u(h_k(s))ds \\
&+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds + \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds.
\end{aligned}$$

Using the time lead functions $r_k(t)$ the solution becomes,

$$\begin{aligned}
x(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^{M}\int_{h_k(0)}^{h_k(t)} (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u(s)ds \\
&+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds + \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds.
\end{aligned} \tag{3.9}$$

Now using the inequalities (4), the above equation for $t = T$ can be expressed as

$$\begin{aligned}
x(T) &= E_q(A(T)^q)x_0 + \sum_{k=0}^{m}\int_{h_k(0)}^{0} (T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\
&+ \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u(s)ds \\
&+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)} (T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\
&+ \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\
&+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds.
\end{aligned} \tag{3.10}$$

For brevity, let us introduce the following notation using (7)

$$\begin{aligned} \Upsilon(\phi(0),x_1;x) &= x_1 - E_q(A(T)^q)x_0 - \varphi(T) - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\ &- \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds. \end{aligned} \tag{3.11}$$

Now let us define the controllability Grammian matrix and the control function

$$\psi_0^T = \sum_{k=0}^m \int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)][E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)]^\star ds \tag{3.12}$$

$$u(t) = [B_k^\star E_{q,q}(A^\star(T-r_k(t))^q)r_k'(s)]\psi^{-1}\Upsilon(\phi(0),x_1;x), \quad \text{for } k=0,1,\ldots,m \tag{3.13}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and $\star$ denotes the matrix transpose. Inserting (14) in (11) by using (12) and (13), it is easy to verify that the control $u(t)$ transfers the initial complete state $\phi(0)$ to the desired vector x_1 at time T for each fixed x. Now observing (12) and substituting (14) in (10), we have

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^m \int_{h_k(0)}^0 (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \sum_{k=0}^m \int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s) \\ &\times B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)\psi^{-1}\Upsilon(\phi(0),x_1;x)ds \\ &+ \sum_{k=m+1}^M \int_{h_k(0)}^{h_k(t)} (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\ &+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds. \end{aligned} \tag{3.14}$$

Now, we impose the following conditions on data of the problem:

(**iv**) The linear fractional stochastic dynamical system (2) is globally relatively controllable.
(**v**) f and σ satisfy Lipschitz and linear growth conditions. That is, there exists some constants $N,\tilde{N},L,\tilde{L}>0$ such that

$$\begin{aligned} \|f(t,x)-f(t,y)\|^2 &\le N\|x-y\|^2, \quad \|f(t,x)\|^2 \le \tilde{N}(1+\|x\|^2) \\ \|\sigma(t,x)-\sigma(t,y)\|^2 &\le L\|x-y\|^2, \quad \|\sigma(t,x)\|^2 \le \tilde{L}(1+\|x\|^2). \end{aligned}$$

For our convenience, let us introduce the following notations.

$$\begin{aligned} a_1 &= \max\{\|E_q(At^q)\|^2; t\in J\}, \quad a_2 = \max\{\|u_0(t)\|^2; t\in J\}, \quad r_k = \max\{\|r_k'(t)\|^2; t\in J\} \\ b_k &= \max\{\|E_{q,q}(A(t-r_k(s))^q)\|^2; s\in[0,T]\}, \quad c_k = \int_0^T (T-r_k(s))^{2(q-1)}ds \\ \tilde{c}_k &= \int_{h_k(0)}^0 (T-r_k(s))^{2(q-1)}ds, \quad \hat{c}_k = \int_{h_k(0)}^{h_k(T)} (T-r_k(s))^{2(q-1)}ds \end{aligned}$$

We claim that if (**iv**) holds, the operator ψ_0^T is strictly positive definite and thus the inverse linear operator $(\psi_0^T)^{-1}$ is bounded, say, by l, (see [10] for more details).

Theorem 3.4. *Under the conditions* (**iv**) *and* (**v**), *the nonlinear system (9) is globally relatively controllable on J.*

Proof. Firstly, from the definition (14) we can write the control function u as

$$\begin{aligned}
u(t) &= B_k^\star E_{q,q}(A^\star(T-r_k(t))^q)r_k'(t)\psi^{-1} \\
&\times \Bigg[x_1 - E_q(A(T)^q)x_0 - \sum_{k=0}^{m}\int_{h_k(0)}^{0}(T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\
&- \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\
&- \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\
&- \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds\Bigg].
\end{aligned}$$

Secondly, we define the operator $\mathcal{P}: C \to C$ by

$$\begin{aligned}
\mathcal{P}(x)(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^{m}\int_{h_k(0)}^{0}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\
&+ \sum_{k=0}^{m}\int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s) \\
&\times B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)\psi^{-1}\Upsilon(\phi(0),x_1;x)ds \\
&+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds \\
&+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\
&+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds.
\end{aligned}$$

In order to prove the global relative controllability of the system (9) it is enough to show that $\mathcal{P}$ has a fixed point in C. To do this, we can employ the contraction mapping principle. To apply the principle, first we show that $\mathcal{P}$ maps C into itself. We have

$$\begin{aligned}
\mathbf{E}\|\mathcal{P}(x)(t)\|^2 &\le 6a_1\mathbf{E}\|x_0\|^2 + 6\sum_{k=0}^{m}\mathbf{E}\left\|\int_{h_k(0)}^{0}(T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)u_0(s)ds\right\|^2 \\
&+ 6\sum_{k=0}^{m}\mathbf{E}\Bigg\|\int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s) \\
&\times B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r_k'(s)\psi^{-1}\Upsilon(\phi(0),x_1;x)ds\Bigg\|^2 \\
&+ 6\sum_{k=m+1}^{M}\mathbf{E}\left\|\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k'(s)u_0(s)ds\right\|^2 \\
&+ 6\mathbf{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds\right\|^2 \\
&+ 6\mathbf{E}\left\|\int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds\right\|^2.
\end{aligned}$$

It follows from Lemma 2.5, in [15], and the above notation that:

$$\begin{aligned}
\mathbf{E}\|\mathcal{P}(x)(t)\|^2 &\le 6a_1\mathbf{E}\|x_0\|^2 + 6a_2\left(\sum_{k=0}^{m}\tilde{c}_k b_k r_k\|B_k\|^2 + \sum_{k=m+1}^{M}\hat{c}_k b_k r_k\|B_k\|^2\right) \\
&+ 6b\frac{t^{2q-1}}{2q-1}\int_0^t \mathbf{E}\|f(s,x(s))\|^2 ds + 6l^2\sum_{k=0}^{m}c_k b_k^2 r_k^2\|B_k\|^4\int_0^t \mathbf{E}\|\Upsilon(\phi(0),x_1;x)\|^2 ds \\
&+ 6L_\sigma b\frac{t^{2q-1}}{2q-1}\int_0^t\left(\int_0^\tau \mathbf{E}\|\sigma(\theta,x(\theta))\|^2 d\theta\right)ds.
\end{aligned}$$

Thus we have

$$\begin{aligned}\mathbf{E}\|\mathcal{P}(x)(t)\|^2 &\le 6a_1\mathbf{E}\|x_0\|^2+6a_2\beta+6b\frac{t^{2q-1}}{2q-1}\tilde{N}\int_0^t(1+\mathbf{E}\|x(s)\|^2)ds\\ &+ 6l^2\eta\Bigg[\mathbf{E}\|x_1\|^2+a_1\mathbf{E}\|x_0\|^2+a_2\beta+b\frac{T^{2q-1}}{2q-1}\tilde{N}\int_0^T(1+\mathbf{E}\|x(s)\|^2)ds\\ &+ L_\sigma b\frac{T^{2q-1}}{2q-1}\tilde{L}\int_0^T\left(\int_0^\tau(1+\mathbf{E}\|x(\theta)\|^2)d\theta\right)ds\Bigg]\\ &+ 6L_\sigma b\frac{t^{2q-1}}{2q-1}\tilde{L}\int_0^t\left(\int_0^\tau(1+\mathbf{E}\|x(\theta)\|^2)d\theta\right)ds.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{E}\|\mathcal{P}(x)(t)\|^2 &\le 6l^2\eta\mathbf{E}\|x_1\|^2+6a_1\mathbf{E}\|x_0\|^2(1+l^2\eta)+6a_2\beta(1+l^2\eta)\\ &+ 6b\frac{T^{2q-1}}{2q-1}\tilde{N}(1+l^2\eta)(1+\|x\|^2_{L^2})+6L_\sigma\tilde{L}b\frac{T^{2q-1}}{2q-1}(1+l^2\eta)(1+T\|x\|^2_{L^2}).\end{aligned}$$

It follows from from the above inequality and the condition (**v**) that there exists $c>0$ such that

$$\mathbf{E}\|\mathcal{P}(x)(t)\|^2\le c(1+\|x\|^2_{L^2}).$$

Therefore $\mathcal{P}$ maps C into itself.

Secondly, we claim that $\mathcal{P}$ is a contraction mapping on C. For $x,y\in C$,

$$\begin{aligned}&\mathbf{E}\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\|^2\\ \le\ & 3\sum_{k=0}^m\mathbf{E}\Bigg\|\int_0^t(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_kr'_k(s)\\ \times\ & B_k^\star E_{q,q}(A^\star(T-r_k(s))^q)r'_k(s)\psi^{-1}[\Upsilon(\phi(0),x_1;x)-\Upsilon(\phi(0),x_1;y)]ds\Bigg\|^2\\ +\ & 3\mathbf{E}\left\|\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)(f(s,x(s))-f(s,y(s)))ds\right\|^2\\ +\ & 3\mathbf{E}\left\|\int_0^t(t-s)^{q-1}\left(\int_0^\tau(\sigma(\theta,x(\theta))-\sigma(\theta,y(\theta)))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds\right\|^2.\end{aligned}$$

Using Lemma 2.5, in [15], condition (**v**), and the above notations we get

$$\begin{aligned}&\mathbf{E}\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\|^2\\ \le\ & 3l^2\frac{T^{2q}}{2q-1}b\sum_{k=0}^m c_kb_k^2r_k^2\|B_k\|^4\Bigg[\int_0^T\mathbf{E}\|f(s,y(s))-f(s,x(s))\|^2ds\\ +\ & L_\sigma\int_0^\tau\mathbf{E}\|\sigma(\theta,y(\theta))-\sigma(\theta,x(\theta))\|^2d\theta\Bigg]\\ +\ & 3\frac{T^{2q-1}}{2q-1}b\int_0^t\mathbf{E}\|f(s,x(s))-f(s,y(s))\|^2ds\\ +\ & 3\frac{T^{2q-1}}{2q-1}bL_\sigma\int_0^t\left(\int_0^\tau\mathbf{E}\|\sigma(\theta,x(\theta))-\sigma(\theta,y(\theta))\|^2d\theta\right)ds.\\ \le\ & 3l^2b\eta\frac{T^{2q-1}}{2q-1}[N+LL_\sigma]\int_0^T\mathbf{E}\|x(s)-y(s)\|^2ds\\ +\ & 3b\frac{T^{2q-1}}{2q-1}[N+TLL_\sigma]\int_0^T\mathbf{E}\|x(s)-y(s)\|^2ds.\end{aligned}$$

It results that

$$\sup_{t\in[0,T]}\mathbf{E}\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\|^2\le\left[3l^2b\eta\frac{T^{2q-1}}{2q-1}[N+LL_\sigma]+3b\frac{T^{2q-1}}{2q-1}[N+TLL_\sigma]\right]\sup_{t\in[0,T]}\mathbf{E}\|x(t)-y(t)\|^2.$$

Therefore we conclude that if $3l^2b\eta\frac{T^{2q-1}}{2q-1}[N+LL_\sigma]+3b\frac{T^{2q-1}}{2q-1}[N+TLL_\sigma]<1$, then $\mathcal{P}$ is a contraction mapping on C, implies that the mapping $\mathcal{P}$ has a unique fixed point $x(\cdot)\in C$. Hence we have

$$\begin{aligned}
x(t) &= E_q(A(t)^q)x_0+\sum_{k=0}^{m}\int_{h_k(0)}^{0}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_kr_k'(s)u_0(s)ds\\
&+\sum_{k=0}^{m}\int_0^t(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_kr_k'(s)u(s)ds\\
&+\sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_kr_k'(s)u_0(s)ds\\
&+\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds\\
&+\int_0^t(t-s)^{q-1}\left(\int_0^\tau\sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds.
\end{aligned}$$

Thus $x(t)$ is the solution of the system (9), and it is easy to verify that $x(T)=x_1$. Further the control function u(t) steers the system (9) from initial complete state $\phi(0)$ to x_1 on J. Hence the system (9) is globally relatively controllable on J.

$\square$

4 An example

In this section, we apply the results obtained in the previous section for the following stochastic fractional dynamical systems with multiple delays in control which involves sequential Caputo derivative

$$\begin{aligned}
{}^cD^qx(t) &= Ax(t)+B_1u(t)+B_2u(t-h)+f(t,x(t))+\sigma(t,x(t))\frac{dw(t)}{dt};\ 0<q<1, t\in[0,T]\\
x(0) &= x_0,
\end{aligned}\tag{4.1}$$

where

$$A=\begin{pmatrix}-1 & 0\\ 3 & -2\end{pmatrix},\quad B_1=B_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$f(t,x(t))=\begin{pmatrix}x_1(t)\cos x_2(t)+3x_2(t)\\ x_2(t)\sin x_1(t)+2x_1(t)\end{pmatrix},\sigma(t,x(t))=\begin{pmatrix}(2t^2+1)x_1(t)e^{-t} & 0\\ 0 & x_2(t)e^{-t}\end{pmatrix}.$$

Let us introduce the variables $x_1(t)=x(t)$ and $x_2(t)={}^cD^{\frac{q}{2}}x_1(t)$. Then ${}^cD^{\frac{q}{2}}x_1(t)={}^cD^{\frac{q}{2}}x(t)=x_2$.

The Mittag-Leffler matrix of the given system is given by

$$\begin{pmatrix}E_q(-t^q) & 0\\ 3E_q(-t^q)-3E_q(-2t^q) & E_q(-2t^q)\end{pmatrix}.$$

Further

$$E_{q,q}(A(T-s)^q)=\begin{pmatrix}E_{q,q}(-(T-s)^q) & 0\\ 3E_{q,q}(-(T-s)^q)-3E_{q,q}(-2(T-s)^q) & E_{q,q}(-2(T-s)^q)\end{pmatrix},$$

$$E_{q,q}(A(T-(s+h))^q)=\begin{pmatrix}E_{q,q}(-(T-(s+h))^q) & 0\\ 3E_{q,q}(-(T-(s+h))^q)-3E_{q,q}(-2(T-(s+h))^q) & E_{q,q}(-2(T-(s+h))^q)\end{pmatrix}.$$

By simple matrix calculation one can see that the controllability matrix

$$\begin{aligned}
\psi_0^T &= \sum_{k=0}^{m}\int_0^T(T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_kr_k'(s)][E_{q,q}(A(T-r_k(s))^q)B_kr_k'(s)]^\star ds\\
&= \int_0^T\left[(T-s)^{q-1}\begin{pmatrix}a^2 & ac\\ ac & b^2+c^2\end{pmatrix}+(T-(s+h))^{q-1}\begin{pmatrix}\bar a^2 & \bar a\bar c\\ \bar a\bar c & \bar b^2+\bar c^2\end{pmatrix}\right]ds.
\end{aligned}$$

is positive definite for any $T > h$, where

$$\begin{aligned} a &= E_{q,q}(-(T-s)^q), \quad b = E_{q,q}(-2(T-(s+h))^q), \\ c &= 3E_{q,q}(-(T-s)^q) - 3E_{q,q}(-2(T-s)^q), \quad \bar{a} = E_{q,q}(-(T-(s+h))^q) \\ \bar{b} &= E_{q,q}(-2(T-(s+h))^q), \quad \bar{c} = 3E_{q,q}(-(T-(s+h))^q) - 3E_{q,q}(-2(T-(s+h))^q). \end{aligned}$$

Further the functions $f(t, x(t))$ and $\sigma(t, x(t))$ satisfies the hypothesis mentioned in Theorem 3.4., and so the fractional system (16) is globally relatively controllable on [0,T].

5 Conclusion

The article contains some controllability results for global relative controllability for the linear and nonlinear fractional stochastic dynamical systems with multiple delays in control function. The result shows that the Banach fixed point theorem can effectively be used to study the control problems for establishing sufficient conditions. Here it is proved that under some hypotheses together with the assumption that the linear stochastic system is globally relatively controllable, the nonlinear fractional stochastic system is also globally relatively controllable. An example is also included to illustrate the importance of the results obtained.

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