Malaya Journal of Matematik 1(1)(2013) 98–105

Malaya Journal of Matematik
MJM
an international journal of mathematical sciences with computer applications...
www.malayajournal.org
ISSN : 2319-3786

Global stability of mutualistic interactions among three species population model with continuous time delay

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Abstract

This paper deals with the study on a mathematical model consisting of mutualistic interactions among three species with continuous time delay. The delay kernels are being convex combinations of suitable nonnegative and normalized functions, the linear chain trick gives an expanded system of ordinary differential equations with the same stability properties as the original integro-differential system. Global stability is discussed by constructing Lyapunov function. It has been shown that equilibrium state of the model is globally stable. Finally, numerical simulations supporting our theoretical results are also included.

Keywords: Mutualism model, local and global stabilities, Lyapunov function, population dynamics, time delay.

2010 MSC: 92B05, 92D25, 93A30, 93C15 93D99.

1 Introduction

The study of equations describing population growth is very interesting and challenging mathematically as well as biologically to discuss the problems on global stability. In the biological process of evolution, the population of one species does not respond instantaneously to interact with other species. To incorporate this role in a modeling approach, time delay models have been developed. Gopalsamy K. [5] and Kuang Y. [9] discussed the necessity of delay differential equation models, see also Beretta E. and Takeuchi [1], Busekros A. W. [2], Cushing J. M. [3], Gopalsamy K. [6], Hale J. K. and Waltman P. [7], Harlan S. W. [8], Mc Donald N. [10], and Solimano F. and Beretta E. [13]. Relatively less attention has been given to the study of three species model with continuous time delay and their dynamical behavior. This motivates the authors to study mutualistic interactions among three species population model with continuous time delay.

The main purpose of this paper is to establish global stability of three species mutualistic system with continuous time delay. In section 2, we introduce our mathematical model. In section 3, we discuss global stability about the biologically feasible equilibrium point of the model by constructing a Lyapunov functional. In section 4, we illustrate our results by some examples. We conclude with a short discussion in section 5.

2 Mathematical Model

In this section, we consider a mathematical model for three mutually interacting species with continuous time delay is given by the following integro-differential equations:

$$\frac{dN_1}{dt} = N_1\Big(a_1 - \alpha_{11}N_1 + \alpha_{12}\int_{-\infty}^{t} k_2(t-s)N_2(s)ds + \alpha_{13}\int_{-\infty}^{t} k_3(t-s)N_3(s)ds\Big),$$
$$\frac{dN_2}{dt} = N_2\Big(a_2 - \alpha_{22}N_2 + \alpha_{21}\int_{-\infty}^{t} k_1(t-s)N_1(s)ds + \alpha_{23}\int_{-\infty}^{t} k_3(t-s)N_3(s)ds\Big),$$

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$$\frac{dN_3}{dt} = N_3\Big(a_3 - \alpha_{33}N_3 + \alpha_{31}\int_{-\infty}^{t} k_1(t-s)N_1(s)ds + \alpha_{32}\int_{-\infty}^{t} k_2(t-s)N_2(s)ds\Big), \tag{2.1}$$

where $N_i, i = 1, 2, 3$ represent the population density of first, second and third species respectively, a_i represent the intrinsic growth rate of first, second and third species respectively, $\alpha_{ii}, i = 1, 2, 3$ represent the rate of decrease of first, second and third species due to limited resources, α_{12} is the mutual coefficient of second species to first species, α_{13} is the mutual coefficient of third species to first species, α_{21} is the mutual coefficient of first species to second species, α_{23} is the mutual coefficient of third species to second species, α_{31} is the mutual coefficient of first species to third species, α_{32} is the mutual coefficient of second species to third species, $k_i(t)$ called the delay kernels, are weighting factors which indicating how much emphasis should be given to the size of the population at earlier times to determine the present effect on resources availability. Here $a_i, \alpha_{ii}, i = 1, 2, 3,$ and $\alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{23}, \alpha_{31}, \alpha_{32}$ are assumed to be nonnegative constants. Usually the delay kernels are normalized so that

$$\int_0^{\infty} k_i(u)du = 1 \qquad i = 1, 2, 3.$$

We assume that every kernel k_i appearing in system (2.1) is a normalized convex combination of functions

$$k(u) = \frac{\beta^n u^{n-1} e^{-\beta u}}{(n-1)!} \qquad n = 1, 2, ..$$

with $\beta > 0$ is a constant, n an integer. When $n = 1$, the kernel is $k(u) = \beta e^{-\beta u}$. Therefore, the system (2.1) becomes

$$\begin{aligned}
\frac{dN_1}{dt} &= N_1\Big(a_1 - \alpha_{11}N_1 + \alpha_{12}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_2(s)ds + \alpha_{13}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_3(s)ds\Big),\\
\frac{dN_2}{dt} &= N_2\Big(a_2 - \alpha_{22}N_2 + \alpha_{21}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_1(s)ds + \alpha_{23}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_3(s)ds\Big),\\
\frac{dN_3}{dt} &= N_3\Big(a_3 - \alpha_{33}N_3 + \alpha_{31}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_1(s)ds + \alpha_{32}\int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_2(s)ds\Big),
\end{aligned} \tag{2.2}$$

where using linear chain trick, define

$$\begin{aligned}
P_1(t) &= \int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_1(s)ds,\\
P_2(t) &= \int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_2(s)ds,\\
P_3(t) &= \int_{-\infty}^{t} \beta e^{-\beta(t-s)}N_3(s)ds.
\end{aligned}$$

Therefore, the system (2.2) is equivalent to the following system of six ordinary differential equations.

$$\begin{aligned}
\frac{dN_1}{dt} &= N_1(a_1 - \alpha_{11}N_1 + \alpha_{12}P_2 + \alpha_{13}P_3),\\
\frac{dN_2}{dt} &= N_2(a_2 - \alpha_{22}N_2 + \alpha_{21}P_1 + \alpha_{23}P_3),\\
\frac{dN_3}{dt} &= N_3(a_3 - \alpha_{33}N_3 + \alpha_{31}P_1 + \alpha_{32}P_2),\\
\frac{dP_1}{dt} &= \beta(N_1 - P_1),\\
\frac{dP_2}{dt} &= \beta(N_2 - P_2),\\
\frac{dP_3}{dt} &= \beta(N_3 - P_3).
\end{aligned} \tag{2.3}$$

3 Stability Analysis

In this section, the existence of the unique positive biologically feasible equilibrium point of the system (2.3) and local and global stabilities are investigated. The equilibrium point $E_1(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*)$ exists if and only if there is a unique positive solution to the following equations.

$$
\begin{aligned}
-\alpha_{11}N_1 + \alpha_{12}P_2 + \alpha_{13}P_3 &= -a_1,\\
-\alpha_{22}N_2 + \alpha_{21}P_1 + \alpha_{23}P_3 &= -a_2,\\
-\alpha_{33}N_3 + \alpha_{31}P_1 + \alpha_{32}P_2 &= -a_3,\\
\beta(N_1 - P_1) &= 0,\\
\beta(N_2 - P_2) &= 0,\\
\beta(N_3 - P_3) &= 0,
\end{aligned}
$$

provided that the four conditions

(C_1) $a_1\alpha_{22}\alpha_{33} + a_2(\alpha_{12}\alpha_{33} + \alpha_{13}\alpha_{32}) + a_3(\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{22}) > a_1\alpha_{23}\alpha_{32}$,

(C_2) $a_1(\alpha_{21}\alpha_{33} + \alpha_{23}\alpha_{31}) + a_2\alpha_{11}\alpha_{33} + a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21}) > a_2\alpha_{13}\alpha_{31}$,

(C_3) $a_1(\alpha_{22}\alpha_{31} + \alpha_{21}\alpha_{32}) + a_2(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31}) + a_3\alpha_{11}\alpha_{22} > a_3\alpha_{12}\alpha_{21}$,

(C_4) $\alpha_{11}\alpha_{22}\alpha_{33} > \alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{22}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32}$,

hold, where

$$
\begin{aligned}
N_1^* = P_1^* = &\Big[a_1(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + a_2(\alpha_{12}\alpha_{33} + \alpha_{13}\alpha_{32} + a_3(\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{22})\Big]\Big/\Big[\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32}\\
&- \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}\Big],\\
N_2^* = P_2^* = &\Big[a_1(\alpha_{21}\alpha_{33} + \alpha_{23}\alpha_{31}) + a_2(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}) + a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21})\Big]\Big/\Big[\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32}\\
&- \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}\Big],\\
N_3^* = P_3^* = &\Big[a_1(\alpha_{22}\alpha_{31} + \alpha_{21}\alpha_{32}) + a_2(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31}) + a_3(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\Big]\Big/\Big[\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32}\\
&- \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}\Big].
\end{aligned}
$$

We note that the equilibrium point E_1 of the system (2.3) is also an equilibrium point of the system (2.1) with the kernel $\beta e^{-\beta u}$. To discuss the local stability of the system (2.3), we compute variational matrix about equilibrium point E_1 as

$$
J_1(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*) =
\begin{bmatrix}
-\alpha_{11}N_1^* & 0 & 0 & 0 & \alpha_{12}N_1^* & \alpha_{13}N_1^*\\
0 & -\alpha_{22}N_2^* & 0 & \alpha_{21}N_2^* & 0 & \alpha_{23}N_2^*\\
0 & 0 & -\alpha_{33}N_3^* & \alpha_{31}N_3^* & \alpha_{32}N_3^* & 0\\
\beta & 0 & 0 & -\beta & 0 & 0\\
0 & \beta & 0 & 0 & -\beta & 0\\
0 & 0 & \beta & 0 & 0 & -\beta
\end{bmatrix}
$$

The characteristic equation of the above variational matrix about equilibrium point E_1 is

$$\lambda^6 + k_1\lambda^5 + k_2\lambda^4 + k_3\lambda^3 + k_4\lambda^2 + k_5\lambda + k_6 = 0,$$

where,

$$
\begin{aligned}
k_1 =& 3\beta + \alpha_{11}N_1^* + \alpha_{22}N_2^* + \alpha_{33}N_3^*\\
k_2 =& 3\beta^2 + 3\Big(\alpha_{11}N_1^* + \alpha_{22}N_2^* + \alpha_{33}N_3^*\Big)\beta + \alpha_{11}\alpha_{22}N_1^*N_2^* + \alpha_{22}\alpha_{33}N_2^*N_3^* + \alpha_{11}\alpha_{33}N_1^*N_3^*
\end{aligned}
$$

$$\begin{aligned}
k_3 =& \beta^3 + 3\Big(\alpha_{11}N_1^* + \alpha_{22}N_2^* + \alpha_{33}N_3^*\Big)\beta^2 + 3\Big(\alpha_{11}\alpha_{22}N_1^*N_2^* + \alpha_{22}\alpha_{33}N_2^*N_3^* + \alpha_{11}\alpha_{33}N_1^*N_3^*\Big)\beta \\
&+ \alpha_{11}\alpha_{22}\alpha_{33}N_1^*N_2^*N_3^* \\
k_4 =& \Big(\alpha_{11}N_1^* + \alpha_{22}N_2^* + \alpha_{33}N_3^*\Big)\beta^3 + \Big[\Big(3\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\Big)N_1^*N_2^* + \Big(3\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}\Big)N_2^*N_3^* \\
&+ \Big(3\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\Big)N_1^*N_3^*\Big]\beta^2 + 3\alpha_{11}\alpha_{22}\alpha_{33}N_1^*N_2^*N_3^*\beta \\
k_5 =& \Big[\Big(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\Big)N_1^*N_2^* + \Big(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}\Big)N_2^*N_3^* + \Big(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\Big)N_1^*N_3^*\Big]\beta^3 \\
&+ \Big(3\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} - \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{13}\alpha_{22}\alpha_{31}\Big)N_1^*N_2^*N_3^*\beta^2 \\
k_6 =& \Big(\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} - \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}\Big)N_1^*N_2^*N_3^*\beta^3.
\end{aligned}$$

It is very difficult to find the roots or apply Routh-Hurwitz criteria. Therefore, we conclude that if all the roots have negative real part then the system (2.3) is stable (see numerical examples in Section 4).
Now we establishes the global stability of the system (2.3) by constructing a suitable Lyapunov function in the following theorem.

Theorem 3.1. *The positive equilibrium point $E_1(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*)$ of the system (2.3) is globally stable, if*

$$\begin{aligned}
2\alpha_{11} &> \alpha_{12}^2 + \alpha_{13}^2 + 4 \\
2\alpha_{22} &> \alpha_{21}^2 + \alpha_{23}^2 + 4 \\
2\alpha_{33} &> \alpha_{31}^2 + \alpha_{32}^2 + 4
\end{aligned}$$

holds.

Proof. The proof can be reached by using a Lyapunov stability theorem which gives a sufficient condition. Now, let us consider a positive definite function

$$V(N_1, N_2, N_3) = V_1(N_1) + V_2(N_2) + V_3(N_3) + V_4(P_1) + V_5(P_2) + V_6(P_3)$$

where,

$$\begin{aligned}
V_1(N_1) &= 2\Big(N_1 - N_1^* - N_1^* \ln\frac{N_1}{N_1^*}\Big), \\
V_2(N_2) &= 2\Big(N_2 - N_2^* - N_2^* \ln\frac{N_2}{N_2^*}\Big), \\
V_3(N_3) &= 2\Big(N_3 - N_3^* - N_3^* \ln\frac{N_3}{N_3^*}\Big), \\
V_4(P_1) &= \frac{2}{\beta}(P_1 - P_1^*)^2, \\
V_5(P_2) &= \frac{2}{\beta}(P_2 - P_2^*)^2, \\
V_6(P_3) &= \frac{2}{\beta}(P_3 - P_3^*)^2,
\end{aligned}$$

on $H = \{(N_1, N_2, N_3, P_1, P_2, P_3) \mid \quad N_1 > 0, N_2 > 0, N_3 > 0, P_1 > 0, P_2 > 0, P_3 > 0\}$. It is obvious that $V(N_1, N_2, N_3, P_1, P_2, P_3) \in C^1(H, R)$ and $V(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*) = 0$. The function $V(N_1, N_2, N_3, P_1, P_2, P_3)$ satisfies

$$V(N_1, N_2, N_3, P_1, P_2, P_3) > V(N_1^*, N_2^*, N_3^*, P_1^*, P_2^*, P_3^*) = 0$$

which holds for all $V(N_1, N_2, N_3, P_1, P_2, P_3) \in H - \{E_1\}$. Then the time derivative of $V(N_1, N_2, N_3, P_1, P_2, P_3)$ computed along the solution of the system (2.3) is

$$\frac{dV}{dt} = 2\Big[- \alpha_{11}(N_1 - N_1^*)^2 - \alpha_{22}(N_2 - N_2^*)^2 - \alpha_{33}(N_3 - N_3^*)^2$$

$$
\begin{aligned}
&+\alpha_{12}(N_1-N_1^*)(P_2-P_2^*)+\alpha_{13}(N_1-N_1^*)(P_3-P_3^*)\\
&\quad+\alpha_{21}(N_2-N_2^*)(P_1-P_1^*)+\alpha_{23}(N_2-N_2^*)(P_3-P_3^*)\\
&\quad\quad+\alpha_{31}(N_3-N_3^*)(P_1-P_1^*)+\alpha_{32}(N_3-N_3^*)(P_2-P_2^*)\Big]\\
&+4\Big[(N_1-N_1^*)(P_1-P_1^*)+(N_2-N_2^*)(P_2-P_2^*)\\
&\quad+(N_3-N_3^*)(P_3-P_3^*)-(P_1-P_1^*)^2-(P_2-P_2^*)^2-(P_3-P_3^*)^2\Big]\\
=&-(P_1-P_1^*)^2-(P_2-P_2^*)^2-(P_3-P_3^*)^2-\Big[2\alpha_{11}-\alpha_{12}^2\\
&\quad-\alpha_{13}^2-4\Big](N_1-N_1^*)^2-\Big[2\alpha_{22}-\alpha_{21}^2-\alpha_{23}^2-4\Big](N_2-N_2^*)^2\\
&\quad-\Big[2\alpha_{33}-\alpha_{31}^2-\alpha_{32}^2-4\Big](N_3-N_3^*)^2-\Big[\alpha_{12}(N_1-N_1^*)-(P_2-P_2^*)\Big]^2\\
&\quad-\Big[\alpha_{13}(N_1-N_1^*)-(P_3-P_3^*)\Big]^2-\Big[\alpha_{21}(N_2-N_2^*)-(P_1-P_1^*)\Big]^2\\
&\quad-\Big[\alpha_{23}(N_2-N_2^*)-(P_3-P_3^*)\Big]^2-\Big[\alpha_{31}(N_3-N_3^*)-(P_1-P_1^*)\Big]^2\\
&\quad-\Big[\alpha_{32}(N_3-N_3^*)-(P_2-P_2^*)\Big]^2\\
<&\quad 0
\end{aligned}
$$

This shows that $\frac{dV}{dt} < 0$ on H. Therefore, the function V is a Lyapunov function with respect to E_1. Hence, the equilibrium point E_1 is globally asymptotically stable on H. □

Consequently, we have the following result.

Theorem 3.2. *The equilibrium point (N_1^*, N_2^*, N_3^*) of the system (2.1) with a kernel $k(u) = \beta e^{-\beta u}$ is globally stable.*

4 Numerical Simulations

To check the feasibility of our analysis regarding stability conditions, we have conducted some numerical computation by choosing the following set of parameters values in model system (2.3) as

$$
\begin{aligned}
&a_1=1,\quad a_2=0.5,\quad a_3=2,\quad \alpha_{11}=1,\quad \alpha_{12}=0.1,\quad \alpha_{13}=0.3,\quad \alpha_{21}=0.2,\\
&\alpha_{22}=1.5,\quad \alpha_{23}=0.3,\quad \alpha_{31}=0.4\quad \alpha_{32}=0.6\quad \alpha_{33}=1.3,\quad \beta=8
\end{aligned}
$$

With the above parameter values, it follows that the system (2.3) is locally stable as shown in Figure 1. However, even if these parameter do not satisfy the conditions of Theorem 3.1, Figure 2 exhibits that the system (2.3) seems to be globally stable.
Consider the another set of parameters values in system (2.3) as

$$
\begin{aligned}
&a_1=2,\quad a_2=4,\quad a_3=3,\quad \alpha_{11}=2.5,\quad \alpha_{12}=0.1,\quad \alpha_{13}=0.3,\quad \alpha_{21}=0.2,\\
&\alpha_{22}=3.5,\quad \alpha_{23}=0.3,\quad \alpha_{31}=0.4\quad \alpha_{32}=0.6\quad \alpha_{33}=3.3
\end{aligned}
$$

From Theorem 3.1, under these parameters values the system (2.3) is globally stable as shown in the figure 3.

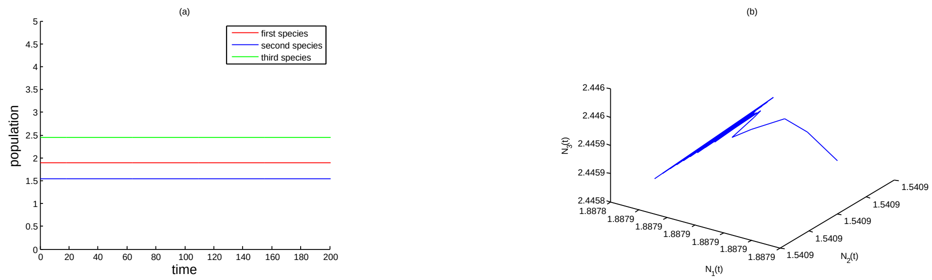

Figure 1: (a) Time series for $N_1(t), N_2(t)$ and $N_3(t)$. (b) The phase graph with initial condition $(1.8879, 1.5409, 2.4459, 1.8879, 1.5409, 2.4459)$.

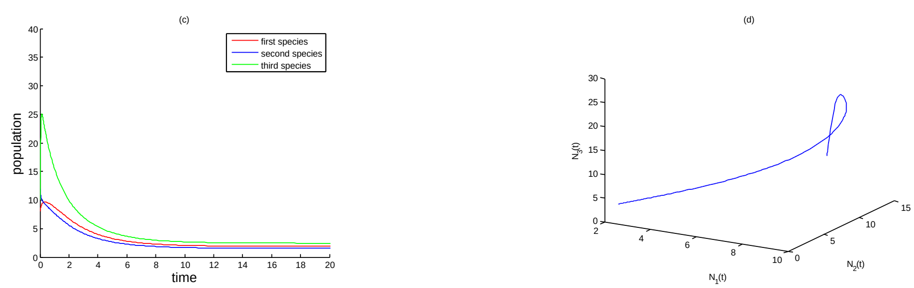

Figure 2: (c) Time series for $N_1(t), N_2(t)$ and $N_3(t)$. (d) The phase graph with initial condition $(8, 12, 10, 40, 30, 20)$.

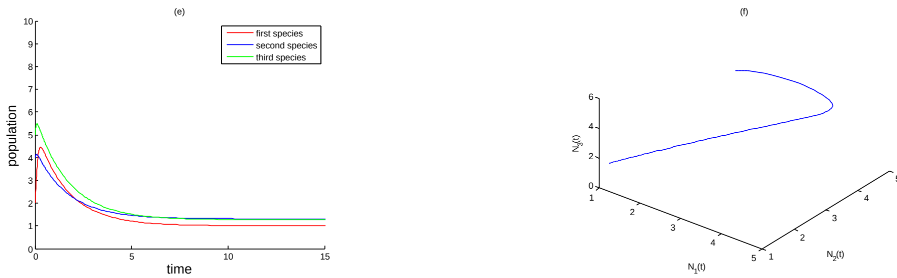

Figure 3: (e) Time series for $N_1(t), N_2(t)$ and $N_3(t)$. (f) The phase graph with initial condition $(2, 4, 5, 10, 20, 30)$.

5 Discussion

In this article, local and global stabilities of the three mutually interacting species with continuous time delay has been investigated. Our numerical simulation shows that even if time delay parameter vary for large value the system (2.3) remains stable. The approach of study in this article differs from Feng C. H. and Chao P. H. [4], Mukherjee D. [11], Shukla V. P. [12] and Xia Y. [14] in the sense that it studies two species mutualistic system with discrete delay. To the best of our knowledge, this paper is the first time to deal with the research for system (2.1) which belongs to a three species mutualism model with continuous time delay. There is a lot of work to do in this area. For example it would be interesting to see what the behavior of the model (2.1) would be when several delays occurs in this system. However less attention has been given to the study of mutualism as compared to the prey-predator and competition. Thus the present article contributes a few more results on mutualism model.

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Received: October 19, 2012; *Accepted*: February 3, 2013

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