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Oscillation criteria for third order neutral difference equations with distributed delay

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Abstract

In this paper we study the oscillatory behavior of third order neutral difference equation of the form

$$\Delta\Big(r(n)\Delta^2 z(n)\Big) + \sum_{s=c}^{d} q(n,s) f(x(n+s-\sigma)) = 0, n \geq n_0 \geq 0, \tag{0.1}$$

where $z(n) = x(n) + \sum_{s=a}^{b} p(n,s)x(n+s-\tau)$. We establish some sufficient conditions which ensure that every solution of the equation (0.1) oscillates or converges to zero by using a generalized Ricaati transformation and Philos - type technique. An example is given to illustrate the main result.

Keywords: Third order, oscillation, neutral difference equations, Philos - type.

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1 Introduction

In this paper we consider the oscillatory behavior of third order neutral difference equation of the form

$$\Delta\Big(r(n)\Delta^2 z(n)\Big) + \sum_{s=c}^{d} q(n,s) f(x(n+s-\sigma) = 0, n \in \mathbb{N}_0 \tag{1.1}$$

where

$$z(n) = x(n) + \sum_{s=a}^{b} p(n,s)x(n+s-\tau), \tag{1.2}$$

Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, $\mathbb{N}_0 = \{n_0,\ n_0+1,\ n_0+2,\ \ldots\}$, n_0 is a nonnegative integer, and $a,\ b,\ c,\ d \in \mathbb{N}_0$ subject to the following conditions:

(C_1) $\{r(n)\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{r(n)} = \infty$;

(C_2) $\{q(n,s)\}$ and $\{p(n,s)\}$ are nonnegative real sequences with $0 \leq p(n) \equiv \sum_{s=a}^{b} p(n,s) \leq P < 1$;

(C_3) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\frac{f(u)}{u} \geq L > 0$, for $u \neq 0$.

By a solution of equation (1.1) we mean a real sequence $\{x(n)\}$ and satisfying equation (1.1) for all $n \in \mathbb{N}_0$. We consider only those solution $\{x(n)\}$ of equation (1.1) which satisfy $sup\{|x(n)| : n \geq N\} > 0$ for all $N \in \mathbb{N}_0$. A solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years there is a great interest in studying the oscillatory behavior of third order difference equations, see for example [1–5, 7–14] and the references cited therein. Motivated by this observation, in this paper

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we obtain some sufficient conditions for the oscillation of all solution of equation (1.1).

In Section 2, we present some preliminary lemmas and in Section 3, we establish some sufficient conditions which ensure that all solutions of equation (1.1) are either oscillatory or converges to zero. An example is given to illustrate the main result.

2 Preliminary Lemmas

In this section, we present some lemmas which will be useful to prove our main results.

Lemma 2.1. *Let $\{x(n)\}$ be a positive solution of equation* (1.1) *and $\{z(n)\}$ be defined as in* (1.2). *Then $\{z(n)\}$ satisfies only of the following two cases eventually*

(*I*) $z(n) > 0, \quad \Delta z(n) > 0, \quad \Delta^2 z(n) > 0$;

(*II*) $z(n) > 0, \quad \Delta z(n) < 0, \quad \Delta^2 z(n) > 0.$

Proof. Assume that $\{x(n)\}$ is a positive solution of equation (1.1). By definition of $\{z(n)\}$ we have $z(n) > x(n) > 0$ for all $n \geq n_0$. From the equation (1.1), we have

$$\Delta(r(n)\Delta^2 z(n)) = -\sum_{s=c}^{d} q(n,s) f(x(n+s-\sigma) < 0.$$

Thus $r(n)\Delta^2 z(n)$ is a nonincreasing function and therefore eventually of one sign. So $\Delta^2 z(n)$ is either eventually positive or eventually negative for $n \geq n_1 \geq n_0$. If $\Delta^2 z(n) < 0$, then there is constant $M > 0$ such that

$$r(n)\Delta^2 z(n) \leq -M < 0, \ n \geq n_1.$$

Summing the last inequality from n_1 to $n-1$, we obtain

$$\Delta z(n) \leq \Delta z(n_1) - M \sum_{s=n_1}^{n-1} \frac{1}{r(s)}.$$

Letting $n \to \infty$, then using condition (C_1), we have $\Delta z(n) \to -\infty$, and therefore $\Delta z(n) < 0$. Since $\Delta^2 z(n) < 0$ and $\Delta z(n) < 0$, we have $z(n) < 0$, which is a contradiction to our assumption. This proves that $\Delta^2 z(n) > 0$ and we have only Case (I) or (II) for $\{z(n)\}$. This completes the proof. □

Lemma 2.2. *Let $\{x(n)\}$ be a positive solution of equation* (1.1), *and let the corresponding function $\{z(n)\}$ satisfies the Case (II) of Lemma 2.1. If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \left[\frac{1}{r(s)} \sum_{t=s}^{\infty} \sum_{j=c}^{d} q(t,j) \right] = \infty, \tag{2.1}$$

then $\lim_{n\to\infty} x(n) = \lim_{n\to\infty} z(n) = 0$.

Proof. Let $\{x(n)\}$ be a positive solution of equation (1.1), and $\{z(n)\}$ satisfies Case (II) of Lemma 2.1. Then there exists $\ell \geq 0$ such that $\lim_{n\to\infty} z(n) = \ell$. We shall prove that $\ell = 0$. Assume that $\ell > 0$, then we have $\ell + \epsilon < z(n) < \ell$ for all $\epsilon > 0$ and $n \geq n_1 \geq n_0$. Choosing $0 < \epsilon < \frac{\ell(1-P)}{P}$. From (1.2), we have

$$\begin{aligned} x(n) &= z(n) - \sum_{s=a}^{b} p(n,s) x(n+s-\tau) \\ &> \ell - \sum_{s=a}^{b} p(n,s) z(n+s-\tau) \\ &> \ell - P(\ell+\epsilon) \\ &= \frac{\ell - P(\ell+\epsilon)}{\ell+\epsilon}(\ell+\epsilon) \\ &> k z(n) \end{aligned} \tag{2.2}$$

where $k = \frac{\ell - P(\ell+\epsilon)}{\ell+\epsilon}$. From the equation (1.1), we have

$$\begin{aligned}\Delta(r(n)\Delta^2 z(n)) &= -\sum_{s=c}^{d} q(n,s) f(x(n+s-\sigma) \\ &\leq -\sum_{s=c}^{d} q(n,s) L x(n+s-\sigma).\end{aligned}$$

Now using (2.2), we obtain

$$\Delta(r(n)\Delta^2 z(n)) \leq -kL \sum_{s=c}^{d} q(n,s) z(n+s-\sigma).$$

Summing the last inequality from n to ∞, we have

$$-r(n)\Delta^2 z(n) \leq -kL \sum_{t=n}^{\infty}\sum_{s=c}^{d} q(t,s) z(t+s-\sigma)$$

or

$$\Delta^2 z(n) \geq kL\ell \frac{1}{r(n)} \sum_{t=n}^{\infty}\sum_{s=c}^{d} q(t,s).$$

Summing again from n to ∞, we have

$$-\Delta z(n) \geq kL\ell \sum_{s=n}^{\infty} \Big[\frac{1}{r(s)} \sum_{t=s}^{\infty}\sum_{j=c}^{d} q(t,j)\Big].$$

Summing the last inequality from n_1 to ∞, we obtain

$$z(n_1) \geq kL\ell \sum_{n=n_1}^{\infty}\sum_{s=n}^{\infty} \Big[\frac{1}{r(s)} \sum_{t=s}^{\infty}\sum_{j=c}^{d} q(t,j)\Big]$$

which contradicts condition (2.1). Thus $\ell = 0$. Moreover, the inequality $0 < x(n) \leq z(n)$ implies that $\lim_{n\to\infty} x(n) = 0$. The proof is now complete. □

Lemma 2.3. *Assume that $y(n) > 0$, $\Delta y(n) \geq 0$, $\Delta^2 y(n) \leq 0$ for all $n \geq n_0$. Then for each $\alpha \in (0,1)$ there exists a $N \in \mathbb{N}_0$ such that*

$$\frac{y(n-\sigma)}{n-\sigma} \geq \alpha \frac{y(n+1)}{n+1} \quad \text{for all } n \geq N. \tag{2.3}$$

Proof. From the monotonicity property of $\{\Delta y(n)\}$, we have

$$y(n+1) - y(n-\sigma) = \sum_{s=n-\sigma}^{n} \Delta y(s) \leq (\sigma+1)\Delta y(n-\sigma)$$

or

$$\frac{y(n+1)}{y(n-\sigma)} \leq 1 + \frac{(\sigma+1)\Delta y(n-\sigma)}{y(n-\sigma)}. \tag{2.4}$$

Also,

$$y(n-\sigma) \geq y(n-\sigma) - y(n_0) \geq (n-\sigma-n_0)\Delta y(n-\sigma).$$

So, for each $\alpha \in (0,1)$, there is a $N \in \mathbb{N}_0$ such that

$$\frac{y(n-\sigma)}{\Delta y(n-\sigma)} \geq \alpha(n-\sigma), \quad n \geq N. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\frac{y(n+1)}{y(n-\sigma)} \leq 1 + \frac{(\sigma+1)}{\alpha(n-\sigma)} \leq \frac{\alpha n - \alpha\sigma + \sigma + 1}{\alpha(n-\sigma)}$$

or

$$\frac{y(n+1)}{y(n-\sigma)} \leq \frac{(n+1)}{\alpha(n-\sigma)}.$$

This completes the proof. □

Lemma 2.4. *Assume that* $z(n) > 0,\ \Delta z(n) > 0,\ \Delta^2 z(n) > 0,\ \Delta^3 z(n) \le 0$ *for all* $n \ge N$. *Then*

$$\frac{z(n)}{\Delta z(n)} \ge \frac{n-N}{2} \ \textit{for all } n \ge N. \tag{2.6}$$

Proof. From the monotonicity property of $\{\Delta^2 z(n)\}$, we have

$$\Delta z(n) = \Delta z(N) + \sum_{s=N}^{n-1} \Delta^2 z(n) \ge (n-N)\Delta^2 z(n).$$

Summing from N to $n-1$, we obtain

$$\begin{aligned} z(n) &\ge z(N) + \sum_{s=N}^{n-1} (s-N)\Delta^2 z(s) \\ &= z(N) + (n-N)\Delta z(n) - z(n+1) + z(N). \end{aligned}$$

Hence $z(n) \ge \frac{(n-N)}{2}\Delta z(n),\quad n \ge N$. This completes the proof. □

3 Main Results

In this section, we obtain new oscillation criteria for the equation (1.1) by using the generalized Riccati transformation and Philos type technique.

Theorem 3.1. *Assume that condition* (2.1) *holds. If there exists a positive nondecreasing real sequence* $\{\rho(n)\}$ *such that*

$$\lim_{n\to\infty} \sum_{s=N}^{n-1} \left[Q(s) - \frac{(\Delta\rho(s))^2}{4\rho(s+1)r(s)} \right] = \infty \tag{3.1}$$

where

$$Q(n) = \rho(n) q_1(n) \frac{\alpha(n-\sigma)(n+c-\sigma-N)}{2(n+1)}, \tag{3.2}$$

and

$$q_1(n) = L(1-P)\sum_{s=c}^{d} q(n,s), \tag{3.3}$$

then every solution of equation (1.1) *is either oscillatory or converges to zero.*

Proof. Assume that $\{x(n)\}$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(n) > 0,\ x(n+s-\tau) > 0$ for $n \ge n_1 \ge n_0 \in \mathbb{N}_0$ and $\{z(n)\}$ is defined as in (1.2). Then $\{z(n)\}$ satisfies two cases of Lemma 2.1.

Case(I). Let $\{z(n)\}$ satisfies Case (I) of Lemma 2.1. From (1.2), we have

$$\begin{aligned} x(n) &\ge z(n) - \sum_{s=a}^{b} p(n,s) z(n+s-\tau) \\ &\ge \Big(1 - \sum_{s=a}^{b} p(n,s)\Big) z(n) \\ &\ge (1-P)z(n). \end{aligned} \tag{3.4}$$

Using condition (C_3) in equation (1.1), we have

$$\Delta(r(n)\Delta^2 z(n)) \le -\sum_{s=c}^{d} q(n,s) L x(n+s-\sigma). \tag{3.5}$$

Now using (3.4) in inequality (3.5), we obtain

$$\Delta(r(n)\Delta^2 z(n)) \le -L(1-P)\sum_{s=c}^{d} q(n,s) z(n+s-\sigma)$$

$$\leq \quad -q_1(n)z(n+c-\sigma). \tag{3.6}$$

Define

$$w(n) = \rho(n)\frac{r(n)\Delta^2 z(n)}{\Delta z(n)}, \quad n \geq n_1. \tag{3.7}$$

Then $w(n) > 0$ for all $n \geq n_1$ and from (3.6), we have

$$\begin{aligned}\Delta w(n) \quad &\leq \quad -\rho(n)\frac{q_1(n)z(n+c-\sigma)}{\Delta z(n+1)} + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ &\qquad - w(n+1)\frac{\Delta^2 z(n)}{\Delta z(n)} \\ &\leq \quad -\rho(n)\frac{q_1(n)z(n+c-\sigma)}{\Delta z(n+1)} + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ &\qquad - \frac{w^2(n+1)}{\rho(n+1)r(n)}. \end{aligned}\tag{3.8}$$

By Lemma 2.3 with $y(n) = \Delta z(n)$, we have

$$\frac{1}{\Delta z(n+1)} \leq \frac{\alpha(n-\sigma)}{n+1}\frac{1}{\Delta z(n-\sigma)} \quad for\ all \quad n \geq N. \tag{3.9}$$

Unsing (3.9) in (3.8), we obtain

$$\begin{aligned}\Delta w(n) \quad &\leq \quad -\rho(n)q_1(n)\frac{\alpha(n-\sigma)}{n+1}\frac{z(n+c-\sigma)}{\Delta z(n-\sigma)} + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ &\qquad - \frac{w^2(n+1)}{\rho(n+1)r(n)}.\end{aligned}$$

Now applying Lemma 2.4 in the last inequality, we obtain

$$\begin{aligned}\Delta w(n) \quad &\leq \quad -\rho(n)q_1(n)\frac{\alpha(n-\sigma)}{n+1}\frac{(n+c-\sigma-N)}{2} \\ &\qquad + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) - \frac{w^2(n+1)}{\rho(n+1)r(n)} \\ &\leq \quad -Q(n) + A(n)w(n+1) - B(n)w^2(n+1)\end{aligned}$$

or

$$Q(n) \leq -\Delta w(n) + A(n)w(n+1) - B(n)w^2(n+1) \tag{3.10}$$

where

$$A(n) = \frac{\Delta\rho(n)}{\rho(n+1)}, \qquad B(n) = \frac{1}{\rho(n+1)r(n)}.$$

Now, using completing the square, we have

$$Q(n) - \frac{(A(n))^2}{4B(n)} \leq -\Delta w(n).$$

Summing the last inequality from N to $n-1$, we have

$$\sum_{s=N}^{n-1}\left(Q(s) - \frac{(\Delta\rho(s))^2}{4\rho(s+1)r(s)}\right) \leq w(N) - w(n) \leq w(N).$$

Letting $n \to \infty$, we obtain a contradiction to (3.1).
If $\{z(n)\}$ satisfies Case (II) of Lemma 2.1, then by condition (2.1) we have $\lim_{n\to\infty} x(n) = 0$. This completes the proof. □

Before stating the next theorem, we define functions $h,\ H : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ such that

(i) $H(n,n) = 0$ for $n \geq n_0 \geq 0$;

(*ii*) $H(n,s)>0$ for $n>s\geq n_0$;

(*iii*) $\Delta_2 H(n,s)=H(n,s+1)-H(n,s)\leq 0$ for $n>s\geq n_0$ and there exists a positive real sequence $\{\rho(n)\}$ such that

$$\Delta_2 H(n,s)+\frac{\Delta\rho(s)}{\rho(s+1)}H(n,s)=-h(n,s)\sqrt{H(n,s)}$$

for $n>s\geq n_0$.

Theorem 3.2. *Assume that* (2.1) *holds. If there exists a positive real sequence* $\{\rho(n)\}$ *such that*

$$\lim_{n\to\infty} sup\frac{1}{H(n,n_0)}\sum_{s=n_0}^{n-1}\left[H(n,s)Q(s)-\frac{1}{4}\rho(s+1)r(s)h^2(n,s)\right]=\infty, \tag{3.11}$$

then every solution of equation (1.1) *is either oscillatory or converges to zero.*

Proof. Assume that $\{x(n)\}$ is a nonoscillatory solution of equation (1.1). Proceeding as the proof of Theorem 3.1, we have (3.10). Now multiplying the inequality (3.10) by $H(n,s)$, then summing the resulting inequality from n_2 to $n-1$ for all $n\geq n_2\geq n_0$, we have

$$\begin{aligned}\sum_{s=n_2}^{n-1}H(n,s)Q(s)\leq &-\sum_{s=n_2}^{n-1}\Delta w(s)H(n,s)\\ &+\sum_{s=n_2}^{n-1}(A(s)w(s+1)-B(s)w^2(s+1))H(n,s).\end{aligned}$$

By summation by parts, we obtain

$$\begin{aligned}&\sum_{s=n_2}^{n-1}H(n,s)Q(s)\\ &\leq H(n,n_2)w(n_2)+\sum_{s=n_2}^{n-1}w(s+1)\Delta_2 H(n,s)\\ &\quad+\sum_{s=n_2}^{n-1}A(s)w(s+1)H(n,s)-\sum_{s=n_2}^{n-1}B(s)w^2(s+1)H(n,s)\\ &\leq H(n,n_2)w(n_2)+\sum_{s=n_2}^{n-1}\left[\Delta_2 H(n,s)+\frac{\Delta\rho(s)}{\rho(s+1)}H(n,s)\right]\times\\ &\quad w(s+1)-\sum_{s=n_2}^{n-1}B(s)w^2(s+1)H(n,s).\end{aligned}\tag{3.12}$$

Using completing the square in the last inequality, we obtain

$$\sum_{s=n_2}^{n-1}\left[H(n,s)Q(s)-\frac{1}{4}\rho(s+1)r(s)h^2(n,s)\right]\leq H(n,n_2)w(n_2)$$

or

$$\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\left[H(n,s)Q(s)-\frac{1}{4}\rho(s+1)r(s)h^2(n,s)\right]\leq w(n_2).$$

Letting $n\to\infty$, we obtain a contradiction to (3.1).
If $\{z(n)\}$ satisfies Case (II) of Lemma 2.1, then by condition (2.1) we have $\lim_{n\to\infty}x(n)=0$. This completes the proof. □

Corollary 3.1. *If* $H(n,s)=(n-s)^\beta$ *for all* $n\geq s\geq 0$ *and*

$$\lim_{n\to\infty} sup\frac{1}{n^\beta}\sum_{s=n_0}^{n-1}\left[(n-s)^\beta Q(s)-\frac{1}{4}\rho(s+1)r(s)(n-s)^{\beta-2}\right]=\infty, \tag{3.13}$$

for every $\beta\geq 1$, *then every solution of equation* (1.1) *is oscillatory.*

Corollary 3.2. *If* $H(n,s) = \left(log\frac{n+1}{s+1}\right)^{\beta}$ *for all* $n \geq s \geq 0$ *and*

$$\lim_{n\to\infty} sup(log(n+1))^{-\beta}\frac{1}{n^{\alpha}}\sum_{s=n_0}^{n-1}\Big[\Big(log\frac{n+1}{s+1}\Big)^{\beta}Q(s) - \frac{1}{4(s+1)^2}\rho(s+1)r(s)\Big(log\frac{n+1}{s+1}\Big)^{\beta-2}\Big] = \infty, \tag{3.14}$$

for every $\beta \geq 1$, *then every solution of equation* (1.1) *is oscillatory.*

The proof of Corollary 3.1 and 3.2 follows from Theorem 3.2 and hence the details are omitted.

Theorem 3.3. *Assume that all conditions of Theorem 3.2 are satisfied except condition* (3.11). *Also let*

$$0 < inf_{s\geq n_0}\Big[\lim_{n\to\infty} inf\frac{H(n,s)}{H(n,n_0)}\Big] \leq \infty \tag{3.15}$$

and

$$\lim_{n\to\infty} sup\frac{1}{H(n,n_0)}\sum_{s=n_0}^{n-1}\rho(s+1)r(s)h^2(n,s) < \infty \tag{3.16}$$

hold. If there exists a positive sequence $\{\psi(n)\}$ *such that*

$$\lim_{n\to\infty} sup\sum_{s=n_0}^{n-1}\frac{(\psi(n))^2}{\rho(s+1)r(s)} = \infty \tag{3.17}$$

and

$$\lim_{n\to\infty} sup\frac{1}{H(n,N)}\sum_{s=N}^{n-1}\Big[H(n,s)Q(s) - \frac{1}{4}\rho(s+1)r(s)h^2(n,s)\Big] \geq \psi(N), \tag{3.18}$$

then every solution of equation (1.1) *is either oscillatory or converges to zero.*

Proof. Proceeding as in the proof of Theorem 3.2, we obtain (3.12). Using completing the square in (3.12) and rearranging we obtain

$$\lim_{n\to\infty} sup\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\Big[H(n,s)Q(s) - \frac{h^2(n,s)}{4B(s)}\Big] \leq w(n_2) - \lim_{n\to\infty} inf\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\Big[\sqrt{H(n,s)B(s)}w(s+1) + \frac{h(n,s)}{2\sqrt{B(s)}}\Big]^2$$

for $n \geq n_2$. It follow from (3.18) that

$$w(n_2) \geq \psi(n_2) + \lim_{n\to\infty} inf\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\Big[\sqrt{H(n,s)B(s)}w(s+1) + \frac{h(n,s)}{2\sqrt{B(s)}}\Big]^2,$$

which means that,

$$w(n_2) \geq \psi(n_2) \quad for \quad n \geq N \tag{3.19}$$

and

$$\lim_{n\to\infty} inf\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\Big[\sqrt{H(n,s)B(s)}w(s+1) + \frac{h(n,s)}{2\sqrt{B(s)}}\Big]^2 < \infty.$$

Therefore

$$\lim_{n\to\infty} inf\Big[\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}H(n,s)B(s)w^2(s+1)$$

$$+\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}h(n,s)\sqrt{H(n,s)}w(s+1)$$
$$+\frac{1}{4}\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\frac{h^2(n,s)}{B(s)}\Big] < \infty.$$

Then

$$\lim_{n\to\infty} inf\Big[\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}H(n,s)B(s)w^2(s+1)$$
$$+\frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}h(n,s)\sqrt{H(n,s)}w(s+1)\Big] < \infty. \tag{3.20}$$

Define the functions

$$U(n) = \frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}H(n,s)B(s)w^2(s+1)$$
$$V(n) = \frac{1}{H(n,n_2)}\sum_{s=n_2}^{n-1}\sqrt{H(n,s)}h(n,s)w(s+1)$$

Then, the inequality (3.20), implies that

$$\lim_{n\to\infty} inf[U(n)+V(n)] < \infty. \tag{3.21}$$

The rest of the proof is similar to that of Theorem 2 of [6], and hence the details are omitted.
If $\{z(n)\}$ satisfies Case (II) of Lemma 2.1, then by condition (2.1) we have $\lim_{n\to\infty} x(n) = 0$. This completes the proof. □

Theorem 3.4. *Assume that all conditions of Theorem 3.3 are satisfied except condition* (3.16). *Also let*

$$\lim_{n\to\infty} inf\frac{1}{H(n,n_0)}\sum_{s=n_0}^{n-1}H(n,s)Q(s) < \infty \tag{3.22}$$

and

$$\lim_{n\to\infty} inf\frac{1}{H(n,N)}\sum_{s=N}^{n-1}\Big[H(n,s)Q(s)-\frac{1}{4}\rho(s+1)r(s)h^2(n,s)\Big] \geq \psi(N) \tag{3.23}$$

then every solution of equation (1.1) *is either oscillatory or converges to zero.*

Proof. The proof is similar to that of Theorem 3.3 and hence the details are omitted. □

Now, let us define

$$H(n,s) = (n-s)^\beta, \quad n \geq s \geq 0,$$

where $\beta \geq 1$ is a constant. Then $H(n,n) = 0$, for $n \geq 0$ and $H(n,s) > 0$ for $n > s \geq 0$. Clearly $\Delta_2 H(n,s) \leq 0$ for $n > s \geq 0$ and

$$h(n,s) = [(n-s)^\beta - (n-s-1)^\beta](n-s)^{-(\beta/2)} \leq \beta(n-s)^{(\beta-2)/2},$$

for $n > s \geq 0$. We see that (3.15) holds,

$$\lim_{n\to\infty}\frac{H(n,s)}{H(n,n_0)} = \lim_{n\to\infty}\frac{(n-s)^\beta}{n^\beta} = 1.$$

Hence, by Theorems 3.3 and 3.4, we have the following two corollaries.

Corollary 3.3. *Let $\beta \geq 1$ be a constant, and suppose that*

$$\lim_{n\to\infty} sup \frac{1}{n^{\beta}} \sum_{s=n_0}^{n-1} \beta\rho(s+1)r(s)(n-s)^{\beta-2} < \infty. \tag{3.24}$$

If there is a sequence $\{\psi(n)\}$ satisfying (3.17) *and*

$$\lim_{n\to\infty} sup \frac{1}{(n-N)^{\beta}} \sum_{s=N}^{n-1} \left[(n-s)^{\beta}Q(s) - \frac{\beta^2}{4}\rho(s+1)r(s)(n-s)^{\beta-2}\right] \geq \psi(N) \tag{3.25}$$

then every solution of equation (1.1) *is oscillatory or converges to zero.*

Proof. The proof follows from Theorem 3.3 and hence the details are omitted. □

Corollary 3.4. *Let $\beta \geq 1$ be a constant, and suppose that*

$$\lim_{n\to\infty} inf \frac{1}{n^{\beta}} \sum_{s=n_0}^{n-1} (n-s)^{\beta}Q(s) < \infty. \tag{3.26}$$

If there is a sequence $\{\psi(n)\}$ satisfying (3.17) *and*

$$\lim_{n\to\infty} inf \frac{1}{(n-N)^{\beta}} \sum_{s=N}^{n-1} \left[(n-s)^{\beta}Q(s) - \frac{\beta^2}{4}\rho(s+1)r(s)(n-s)^{\beta-2}\right] \geq \psi(N) \tag{3.27}$$

then every solution of equation (1.1) *is oscillatory or converges to zero.*

Proof. The proof follows from Theorem 3.4 and hence the details are omitted. □

We conclude this paper with the following example.

4 An example

Consider the difference equation

$$\Delta\Big(n\Delta^2\Big(x(n) + \sum_{s=1}^{2} \frac{1}{2}x(n+s-1)\Big)\Big) + \sum_{s=1}^{2}\Big(4n + \frac{4}{3}s\Big)x(n+s-1) = 0. \tag{4.1}$$

Here $r(n) = n,\ p(n,s) = \frac{1}{2},\ q(n,s) = 4n + \frac{4}{3}s,\ \sigma = \tau = 1,\ a = 1,\ b = 2,\ c = 1$ and $d = 2$. It is easy to see that all conditions of Theorem 3.1 are satisfied. Hence every solution of equation (4.1) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (4.1).

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