

Difference entire sequence spaces of fuzzy numbers

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Abstract

In the present paper we introduce difference entire sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions. We also make an effort to study some topological properties and inclusion relations between these spaces.

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1 Introduction and Preliminaries

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [16] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [11] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. For more details about sequence spaces of fuzzy numbers see ([1], [4], [7], [12], [13], [14], [15]) and references therein.

The notion of difference sequence spaces was introduced by Kızmaz [8], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [6] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let r, s be non-negative integers, then for $Z = l_\infty, c, c_0$ we have sequence spaces

$$Z(\Delta_s^r) = \{x = (x_k) \in w : (\Delta_s^r x_k) \in Z\},$$

where $\Delta_s^r x = (\Delta_s^r x_k) = (\Delta_s^{r-1} x_k - \Delta_s^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_s^r x_k = \sum_{v=0}^r (-1)^v \binom{r}{v} x_{k+sv}.$$

Taking $s = 1$, we get the spaces which were introduced and studied by Et and Çolak [6]. Taking $r = s = 1$, we get the spaces which were introduced and studied by Kızmaz [8].

An Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

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Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [9] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($p \geq 1$). An Orlicz function M satisfies Δ_2 -condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M , is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Let D be the set of all bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line \mathbb{R} . For $A, B \in D$, define $A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$, $d(A, B) = \max\{\underline{A} - \underline{B}, \overline{A} - \overline{B}\}$.

Then it can be easily see that d defines a metric on D and (D, d) is complete metric space (see [5]).

A fuzzy number is fuzzy subset of the real line \mathbb{R} which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semi continuous and have compact support, i.e. if $X \in L(\mathbb{R})$ then for any $\alpha \in [0, 1]$, X^α is compact where

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha, & \text{if } 0 < \alpha \leq 1, \\ t : X(t) > 0, & \text{if } \alpha = 0. \end{cases}$$

For each $0 < \alpha \leq 1$, the α -level set X^α is a non-empty compact subset of \mathbb{R} . The linear structure of $L(\mathbb{R})$ includes addition $X + Y$ and scalar multiplication λX , (λ a scalar) in terms of α -level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda[X]^\alpha,$$

for each $0 \leq \alpha \leq 1$.

Define a map $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

For $X, Y \in L(\mathbb{R})$ define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0, 1]$. It is known that $(L(\mathbb{R}), \bar{d})$ is a complete metric space (see [11]).

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set \mathbb{N} of natural numbers into $L(\mathbb{R})$. The fuzzy number X_n denotes the value of the function at $n \in \mathbb{N}$ and is called the n^{th} term of the sequence.

In this paper we define difference entire sequence spaces of fuzzy numbers by using regular matrices $A = (a_{nk})$, ($n, k = 1, 2, 3, \dots$). By the regularity of A we mean that the matrix which transform convergent sequence into a convergent sequence leaving the limit (see [10]). We denote by $w(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

Let $X = (X_k)$ be a sequence of fuzzy numbers, $A = (a_{nk})$ $n, k = 1, 2, 3, \dots$ be a non-negative regular matrix and $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. Now, we define the following sequence spaces in this paper :

$$\left\{ X = (X_k) : \sum_k a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for some } \rho > 0 \right\}$$

and

$$\Lambda_{\mathcal{M}}(F, A, p, \Delta_s^r) =$$

$$\left\{ X = (X_k) : \sup_n \left(\sum_k a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

If $A = I$, the unit matrix, we get the above spaces as follows :

$$\Gamma_{\mathcal{M}}(F, p, \Delta_s^r) =$$

$$\left\{ X = (X_k) : \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for some } \rho > 0 \right\}$$

and

$$\Lambda_{\mathcal{M}}(F, p, \Delta_s^r) =$$

$$\left\{ X = (X_k) : \sup_n \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

If we take $\mathcal{M}(x) = x$, we get

$$\Gamma(F, A, p, \Delta_s^r) =$$

$$\left\{ X = (X_k) : \sum_k a_{nk} \left[\bar{d} \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for some } \rho > 0 \right\}$$

and

$$\Lambda(F, A, p, \Delta_s^r) =$$

$$\left\{ X = (X_k) : \sup_n \left(\sum_k a_{nk} \left[\bar{d} \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right]^{p_k} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

If we take $p = (p_k) = 1 \ \forall k$, we get

$$\Gamma_{\mathcal{M}}(F, A, \Delta_s^r) =$$

$$\left\{ X = (X_k) : \sum_k a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right] \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for some } \rho > 0 \right\}$$

and

$$\Lambda_{\mathcal{M}}(F, A, \Delta_s^r) =$$

$$\left\{ X = (X_k) : \sup_n \left(\sum_k a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right] \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

If $A = (a_{nk})$ is a Cesaro matrix of order 1, i.e.

$$a_{nk} = \begin{cases} \frac{1}{n}, & k \leq n, \\ 0, & k > n \end{cases}$$

then we get

$$\Gamma_{\mathcal{M}}(F, p, \Delta_s^r) =$$

$$\left\{ X = (X_k) : \frac{1}{n} \sum_{k=1}^n \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for some } \rho > 0 \right\}$$

and

$$\Lambda_{\mathcal{M}}(F, p, \Delta_s^r) =$$

$$\left\{ X = (X_k) : \sup_n \frac{1}{n} \sum_{k=1}^n \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

The space Γ is defined as follows:

$$\Gamma = \left\{ X = (X_k) : \frac{1}{n} \sum_{k=1}^n |X_k|^{\frac{1}{k}} \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for some } \rho > 0 \right\}.$$

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$ and let $K = \max\{1, 2^{H-1}\}$. Then for sequences $\{a_k\}$ and $\{b_k\}$ in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}).$$

The main purpose of this paper is to study difference entire sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions. We also studied some topological properties and interesting inclusion relations between the above defined sequence spaces.

2 Main Results

Proposition 2.1. *If \bar{d} is a translation invariant metric on $L(\mathbb{R})$ then*

$$(i) \quad \bar{d}(X + Y, 0) \leq \bar{d}(X, 0) + \bar{d}(Y, 0),$$

$$(ii) \quad \bar{d}(\lambda X, 0) \leq |\lambda| \bar{d}(X, 0), |\lambda| > 1.$$

Proof. It is easy to prove so we omit the details. □

Theorem 2.2. *If $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, then $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ is a complete metric space under the metric*

$$d(X, Y) = \sup_n \left[\frac{1}{n} \sum_{k=1}^n \bar{d} \left(M_k \left(\frac{|\Delta_s^r(X_k - Y_k)|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k}.$$

Proof. Let $X = (X_k)$, $Y = (Y_k) \in \Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$. Let $\{X^{(n)}\}$ be a Cauchy sequence in $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$. Then given any $\epsilon > 0$ there exists a positive integer N depending on ϵ such that $d(X^{(n)}, X^{(m)}) < \epsilon$, for all $n, m \geq N$. Hence

$$\sup_{(n)} \left[\frac{1}{n} \sum_{k=1}^n \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k^{(n)} - \Delta_s^r X_k^{(m)}|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} < \epsilon \quad \forall m, n \geq N.$$

Consequently $\{X_k^{(n)}\}$ is a Cauchy sequence in the metric space $L(\mathbb{R})$. But $L(\mathbb{R})$ is complete. So, $X_k^{(n)} \rightarrow X_k$ as $n \rightarrow \infty$. Hence there exists a positive integer n_0 such that

$$\left[\frac{1}{n} \sum_{k=1}^n \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k^{(n)} - \Delta_s^r X_k^{(m)}|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} < \epsilon \quad \forall n > n_0.$$

In particular, we have

$$\left[\frac{1}{n} \sum_{k=1}^n \bar{d} \left(M_k \left(\frac{|\Delta_s^r X^{(n_0)} - \Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} < \epsilon.$$

Now

$$\begin{aligned} \left[\frac{1}{n} \sum_{k=1}^n \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} &\leq \left[\frac{1}{n} \sum_{k=1}^n \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k - \Delta_s^r X_k^{(n_0)}|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \\ &\quad + \left[\frac{1}{n} \sum_{k=1}^n \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k^{(n_0)}|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \\ &\leq \epsilon + 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus

$$\left(\frac{1}{n} \sum_{k=1}^n \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right)^{p_k} < \epsilon \quad \text{as } n \rightarrow \infty.$$

This implies that $(X_k) \in \Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$. Hence $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ is a complete metric space. This completes the proof. □

Theorem 2.3. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $p = (p_k)$ be a bounded sequence of positive real numbers, the space $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ is a linear over the field of complex numbers \mathbb{C} .

Proof. Let $X = (X_k), Y = (Y_k) \in \Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist some positive numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_1}, 0 \right) \right) \right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty$$

and

$$\sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r Y_k|^{\frac{1}{k}}}{\rho_2}, 0 \right) \right) \right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Let $\frac{1}{\rho_3} = \min \left\{ \frac{1}{|\alpha|^p \rho_1}, \frac{1}{|\beta|^p \rho_2} \right\}$. Since M is non-decreasing and convex so by using inequality (1.1), we have

$$\begin{aligned} & \sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r (\alpha X_k + \beta Y_k)|^{\frac{1}{k}}}{\rho_3}, 0 \right) \right) \right]^{p_k} \\ & \leq \sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r \alpha X_k|^{\frac{1}{k}}}{\rho_3} + \frac{|\Delta_s^r \beta Y_k|^{\frac{1}{k}}}{\rho_3}, 0 \right) \right) \right]^{p_k} \\ & \leq \sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\alpha|^{\frac{1}{k}} |\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_3} + \frac{|\beta|^{\frac{1}{k}} |\Delta_s^r Y_k|^{\frac{1}{k}}}{\rho_3}, 0 \right) \right) \right]^{p_k} \\ & \leq \sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\alpha| |\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_3} + \frac{|\beta| |\Delta_s^r Y_k|^{\frac{1}{k}}}{\rho_3}, 0 \right) \right) \right]^{p_k} \\ & \leq \sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_1} + \frac{|\Delta_s^r Y_k|^{\frac{1}{k}}}{\rho_2}, 0 \right) \right) \right]^{p_k} \\ & \leq K \sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho_1}, 0 \right) \right) \right]^{p_k} + K \sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r Y_k|^{\frac{1}{k}}}{\rho_2}, 0 \right) \right) \right]^{p_k} \\ & \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

Hence $\sum_{k=1}^n \frac{1}{n} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r (\alpha X_k + \beta Y_k)|^{\frac{1}{k}}}{\rho_3}, 0 \right) \right) \right]^{p_k} \rightarrow 0$ as $k \rightarrow \infty$. Hence $\Gamma_{\mathcal{M}}(F, p, \Delta_s^r)$ is a linear space. This completes the proof. \square

Theorem 2.4. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the space $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ is complete with respect to the paranorm defined by

$$g(X) = \sup_{(k)} \left(\sum a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right)^{\frac{1}{H}},$$

where $H = \max \{1, \sup_k (p_k/H)\}$ and \bar{d} is translation metric.

Proof. Clearly, $g(0) = 0, g(-x) = g(x)$. It can also be seen easily that $g(x+y) \leq g(x) + g(y)$ for $X = (X_k), Y = (Y_k)$ in $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$, since \bar{d} is translation invariant. Now for any scalar λ , we have $|\lambda|^{\frac{p_k}{H}} < \max\{1, \sup |\lambda|\}$, so that $g(\lambda x) < \max\{1, \sup |\lambda|\}$, λ fixed implies $\lambda x \rightarrow 0$. Now, let $\lambda \rightarrow 0, X$ fixed for $\sup |\lambda| < 1$, we have

$$\left[\sum a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right]^{\frac{1}{H}} < \epsilon \text{ for } N > N(\epsilon).$$

Also for $1 \leq n \leq N$, since

$$\left[\sum a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right]^{\frac{1}{H}} < \epsilon,$$

there exists m such that

$$\left[\sum_{k=m}^{\infty} a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right]^{\frac{1}{H}} < \epsilon.$$

Taking λ small enough, we have

$$\left[\sum_{k=m}^{\infty} a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right]^{\frac{1}{H}} < 2\epsilon \text{ for all } k.$$

Since $g(\lambda X) \rightarrow 0$ as $\lambda \rightarrow 0$. Therefore g is a paranorm on $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$.

To show the completeness, let $(X^{(i)})$ be a Cauchy sequence in $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$. Then for a given $\epsilon > 0$ there is $r \in N$ such that

$$\left[\sum a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r (X^{(i)} - X^{(j)})|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right]^{\frac{1}{H}} < \epsilon \text{ for all } i, j > r. \quad (2.1)$$

Since \bar{d} is a translation, so equation (2.1) implies that

$$\left[\sum a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r (X_k^{(i)} - X_k^{(j)})|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right]^{\frac{1}{H}} < \epsilon \text{ for all } i, j > r \text{ and each } n. \quad (2.2)$$

Hence

$$\left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r (X_k^{(i)} - X_k^{(j)})|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right] < \epsilon \text{ for all } i, j > r.$$

Therefore $(X^{(i)})$ is a Cauchy sequence in $L(\mathbb{R})$. Since $L(\mathbb{R})$ is complete, $\lim_{j \rightarrow \infty} X_k^j = X_k$. Fixing $r_0 \geq r$ and letting $j \rightarrow \infty$, we obtain (2.2) that

$$\left[\sum a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r (X_k^{(i)} - X_k)|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right] \right] < \epsilon \text{ for all } r_0 > r, \quad (2.3)$$

since \bar{d} is a translation invariant. Hence

$$\left[\sum a_{nk} \left[\bar{d} \left(M_k \left(\frac{|\Delta_s^r (X^{(i)} - X)|^{\frac{1}{k}}}{\rho}, 0 \right) \right) \right]^{p_k} \right]^{\frac{1}{H}} < \epsilon$$

i.e. $X^{(i)} \rightarrow X$ in $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$. It is easy to see that $X \in \Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$. Hence $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ is complete. This completes the proof. \square

Theorem 2.5. Let $A = (a_{nk})$ ($n, k = 1, 2, 3, \dots$) be an infinite matrix with complex entries. Then $A \in \Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ if and only if given $\epsilon > 0$ there exists $N = N(\epsilon) > 0$ such that $|a_{nk}| < \epsilon^n N^k$ ($n, k = 1, 2, 3, \dots$).

Proof. Let $X = (X_k) \in \Gamma$ and let $Y_n = \left(\sum_{k=1}^{\infty} a_{nk} \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right)^{p_k} \right)$, ($n = 1, 2, 3, \dots$). Then $(Y_n) \in \Gamma$ if and only if given any $\epsilon > 0$ there exists $N = N(\epsilon) > 0$ such that $|a_{nk}| < \epsilon^n N^k$ by using Theorem 4 of [3]. Thus $A \in \Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ if and only if the condition holds. \square

Theorem 2.6. If $A = (a_{nk})$ transforms Γ into $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ then $\lim_{n \rightarrow \infty} (a_{nk})q^n = 0$ for all integers $q > 0$ and each fixed $k = 1, 2, 3, \dots$, where $X = (X_k)$ be a sequence of fuzzy numbers and \bar{d} is translation invariant.

Proof. Let $Y_n = \left[\sum_{k=1}^{\infty} a_{nk} \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right)^{p_k} \right]$ ($n = 1, 2, 3, \dots$). Let $(X_k) \in \Gamma$ and $(Y_n) \in \Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$.

Take $(X_k) = \delta^k = (0, 0, 0, \dots, 1, 0, 0, \dots)$, 1 in the k^{th} place and zero's elsewhere, then $(X_k) \in \Gamma$. Hence $\sum_{k=1}^{\infty} |a_{nk}|q^n < \infty$ for every positive q . In particular $\lim_{n \rightarrow \infty} (a_{nk})q^n = 0$ for all positive integers q and each fixed $k = 1, 2, 3, \dots$. This completes the proof. \square

Theorem 2.7. If $A = (a_{nk})$ transforms $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ into Γ , then $\lim_{n \rightarrow \infty} (a_{nk})q^n = 0$ for all integers $q > 0$ and each fixed $k = 1, 2, 3, \dots$, where $X = (X_k)$ be a sequence of fuzzy numbers and \bar{d} is translation invariant.

Proof. Let

$$t_n = \left[\sum_{k=1}^{\infty} \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right)^{p_k} \right] \in \Gamma.$$

Let

$$s_n = \left[\sum_{k=1}^{\infty} \bar{d} \left(M_k \left(\frac{|0|^{\frac{1}{k}}}{\rho}, 0 \right) \right)^{p_k} \right] \in \Gamma.$$

Then $Y_n = (t_n - s_n) = \left[\sum_{k=1}^{\infty} a_{nk} \bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right)^{p_k} \right]$ and $\bar{d} \left(M_k \left(\frac{|\Delta_s^r X_k|^{\frac{1}{k}}}{\rho}, 0 \right) \right)^{p_k} \in \Gamma$. Hence $(Y_n) \in \Gamma$.

Therefore $(a_{nk})q^n \rightarrow 0$ as $n \rightarrow \infty \forall k$. This completes the proof. \square

Theorem 2.8. *If $A = (a_{nk})$ transforms $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$ into $\Gamma_{\mathcal{M}}(F, A, p, \Delta_s^r)$, then $\lim_{n \rightarrow \infty} (a_{nk})q^n = 0$ for all integers $q > 0$ and each fixed $k = 1, 2, 3, \dots$, where $X = (X_k)$ be a sequence of fuzzy numbers and \bar{d} is translation invariant.*

Proof. The proof of the Theorem follows from Theorem 2.6 and Theorem 2.7. \square

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