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Radio Number for Strong Product $P_2 \boxtimes P_n$

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Abstract

A radio labeling of a graph G is a function f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(u) - f(v)| \geq diam(G) + 1 - d_G(u,v)$, where $diam(G)$ and $d_G(u,v)$ are diameter and distance between u and v in graph G respectively. The radio number $rn(G)$ of G is the smallest number k such that G has radio labeling with $\max\{f(v) : v \in V(G)\} = k$. We investigate radio number for strong product of P_2 and P_n.

Keywords: Interference, channel assignment, radio labeling, radio number, strong product.

2010 MSC: 05C15, 05C78.

1 Introduction

In 1980, Hale[5] initiated the problem to determine the minimum number of channels in a given network which is now popular as a channel assignment problem. He classified transmitter as *very close* and *close* transmitter according to the interference between them. He called *very close* transmitters if a pair of transmitters has major interference and called *close* transmitters if a pair of transmitters has minor interference. Hale[5] gave the graphical representation for the channel assignment problem wherein he represented transmitters by vertices and interference between a pair of transmitters by edges. Two transmitters are joined by an edge if major interference occurs between them and minor interference is taken as vertices at distance two in a graph.

In 1991, Roberts[10] suggested a solution for channel assignment problem and proposed that a pair of transmitters having minor interference must receive different channels and a pair of transmitters having major interference must receive channels that are at least two apart. Motivated through this Griggs and Yeh[4] introduced the distance two labeling which is defined as follows:

A distance two labeling (or $L(2,1)$-labeling) of a graph $G = (V(G), E(G))$ is a function f from vertex set $V(G)$ to the set of nonnegative integers such that the following conditions are satisfied:

(1) $|f(u) - f(v)| \geq 2$ if $d(u,v) = 1$.

(2) $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$.

The difference between the largest and the smallest label assigned by f is called the span of f and the minimum span over all $L(2,1)$-labeling of G is called the λ-number of G, denoted by $\lambda(G)$. The $L(2,1)$-labeling has been explored in past two decades by many researchers like Yeh[17, 18], Georges and Mauro[3], Sakai[11], Chang and Kuo[1], Wang[15], Vaidya and Bantva[12] and Vaidya et al.[13].

But as time passed, practically it has been observed that the interference among transmitters might go beyond two levels. Radio labeling extends the number of interference level considered in $L(2,1)$-labeling from two to the largest possible - the diameter of G. The diameter of G is denoted by $diam(G)$ or simply by d is the maximum distance among all pairs of vertices in G. Motivated through the problem of channel assignment of FM radio stations Chartrand et. al[2] introduced the concept of radio labeling of graph as follows.

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A radio labeling of a graph G is an injective function $f : V(G) \rightarrow \{0, 1, 2, ...\}$ such that the following is satisfied for all u, $v \in V(G)$:

$$|f(u) - f(v)| \geq diam(G) + 1 - d_G(u, v).$$

The radio number denoted by $rn(G)$ is the minimum span of a radio labeling for G. Note that when $diam(G)$ is two then radio labeling and distance two labeling are identical. The radio labeling is studied in the past decade by many researchers like Liu[6], Liu and Xie[7, 8], Liu and Zhu[9] and Vaidya and Vihol[14].

In this paper, we completely determine the radio number of strong product of P_2 with P_n. Through out this discussion, the order of $P_2 \boxtimes P_n$ is p and we consider $n \geq 3$ as $P_2 \boxtimes P_2$ is simply K_4 for which $L(2, 1)$-labeling and radio labeling coincide. Moreover terms not defined here are used in the sense of West[16].

2 Main results

The strong product $G \boxtimes H$ of G and H is the graph in which the vertex (u, v) is adjacent to the vertex (u', v') if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$, or $uu' \in E(G)$ and $vv' \in E(H)$.

For $P_2 \boxtimes P_{2k+1}$, let v_0 and v_0' be the centers. Let v_{L1}, v_{L2}, ... ,v_{Lk} be the vertices on the left side and v_{R1}, v_{R2}, ... ,v_{Rk} be the vertices on the right side with respect to center v_0 and v_{L1}', v_{L2}', ... ,v_{Lk}' be the vertices on the left side and v_{R1}', v_{R2}', ... ,v_{Rk}' be the vertices on the right side with respect to center v_0'.

For $P_2 \boxtimes P_{2k}$, let v_{L0} and v_{R0}, v_{L0}' and v_{R0}' be the centers. Let v_{L1}, v_{L2}, ... ,$v_{L(k-1)}$ be the vertices on the left side and v_{R1}, v_{R2}, ... ,$v_{R(k-1)}$ be the vertices on the right side with respect to centers v_{L0} and v_{R0} and v_{L1}', v_{L2}', ... ,$v_{L(k-1)}'$ be the vertices on the left side and v_{R1}', v_{R2}', ... ,$v_{R(k-1)}'$ be the vertices on the right side with respect to centers v_{L0}' and v_{R0}'.

Let for $P_2 \boxtimes P_{2k+1}$, $V(P_2 \boxtimes P_{2k+1}) = V_L \cup V_R \cup V_L' \cup V_R'$

$V_L = \{v_0, v_{L1}, v_{L2}, ... ,v_{Lk}\}$

$V_R = \{v_0, v_{R1}, v_{R2}, ... ,v_{Rk}\}$

$V_L' = \{v_0', v_{L1}', v_{L2}', ... ,v_{Lk}'\}$

$V_R' = \{v_0', v_{R1}', v_{R2}', ... ,v_{Rk}'\}$

Let for $P_2 \boxtimes P_{2k}$, $V(P_2 \boxtimes P_{2k}) = V_L \cup V_R \cup V_L' \cup V_R'$

$V_L = \{v_{L0}, v_{L1}, v_{L2}, ... ,v_{L(k-1)}\}$

$V_R = \{v_{R0}, v_{R1}, v_{R2}, ... ,v_{R(k-1)}\}$

$V_L' = \{v_{L0}', v_{L1}', v_{L2}', ... ,v_{L(k-1)}'\}$

$V_R' = \{v_{R0}', v_{R1}', v_{R2}', ... ,v_{R(k-1)}'\}$

In $P_2 \boxtimes P_n$, we say two vertices u and v are on opposite side if $u \in V_L$ or V_L' and $v \in V_R$ or V_R'.

We define the level function on $V(P_2 \boxtimes P_n)$ to the set of whole numbers W from a center vertex w by

$L(u) = \{d(u, w) : w$ is a center vertex $\}$, for any $u \in V(P_2 \boxtimes P_n)$.

In $P_2 \boxtimes P_n$, the maximum level is k if $n = 2k + 1$ and $k - 1$ if $n = 2k$.

Observation 2.1. *For $P_2 \boxtimes P_n$,*

$$(1) \quad |V(P_2 \boxtimes P_n)| = \begin{cases} 4k + 2 & if \quad n = 2k + 1 \\ 4k & if \quad n = 2k \end{cases}$$

$$(2) \quad d(u,v) \quad \leq \quad \begin{cases} L(u)+L(v) & if \quad n=2k+1 \\ L(u)+L(v)+1 & if \quad n=2k \end{cases}$$

(3) If u_i, $u_{i+1} \in V(P_2 \boxtimes P_n)$, $1 \leq i \leq p-1$ are on opposite side and $d(u_i, u_{i+1}) = d(u_{i+1}, u_{i+2})$ or $d(u_i, u_{i+1}) = d(u_{i+1}, u_{i+2}) \pm 1$ then $d(u_i, u_{i+2}) = 1$.

Theorem 2.2. *Let $P_2 \boxtimes P_n$ be a strong product of P_2 and P_n and $k = \lfloor \frac{n}{2} \rfloor$ then*

$$rn(P_2 \boxtimes P_n) \quad \geq \quad \begin{cases} 2k(2k+1)+1 & if \quad n=2k+1 \\ 2k(2k-1)+1 & if \quad n=2k \end{cases}$$

Moreover, the equality holds if and only if there exist a radio labeling f with ordering $\{u_1, u_2, \ldots, u_p\}$ of vertices of $P_2 \boxtimes P_n$ such that $f(u_1) = 0 < f(u_2) < f(u_3) < \ldots < f(u_p)$, where all the following holds (for all $1 \leq i \leq p-1$):
(1) u_i and u_{i+1} are on opposite side,
(2) $\{u_1, u_p\} = \{w_1, w_2\}$ where w_1, w_2 are center vertex.

Proof. Let f be an optimal radio labeling for $P_2 \boxtimes P_n$, where $f(u_1) = 0 < f(u_2) < f(u_3) < \ldots < f(u_p)$. Then $f(u_{i+1}) - f(u_i) \geq (d+1)$ - $d(u_i, u_{i+1})$, for all $1 \leq i \leq p-1$. Summing these $p-1$ inequalities we get

$$rn(P_2 \boxtimes P_n) = f(u_p) \geq (p-1)(d+1) - \sum_{i=1}^{p-1} d(u_i, u_{i+1}) \tag{2.1}$$

Case - 1 : n is odd.

For $P_2 \boxtimes P_{2k+1}$, we have

$$\sum_{i=1}^{p-1} d(u_i, u_{i+1}) \leq \sum_{i=1}^{p-1} [L(u_i) + L(u_{i+1})]$$

$$= 2 \sum_{u \in V(G)} L(u) - L(u_1) - L(u_p)$$

$$= 2 \sum_{u \in V(G)} L(u) \tag{2.2}$$

Substituting (2.2) in (2.1), we get

$$rn(P_2 \boxtimes P_n) = f(u_p) \geq (p-1)(d+1) - 2 \sum_{u \in V(G)} L(u)$$

For $P_2 \boxtimes P_{2k+1}$, $p = 4k+2$, $d = 2k$ and $\sum_{u \in V(G)} L(u) = 2k(k+1)$

$$rn(P_2 \boxtimes P_n) = f(u_p) \geq (4k+2-1)(2k+1) - 4(k(k+1))$$

$$= (4k+1)(2k+1) - 4k(k+1)$$

$$= 8k^2 + 4k + 2k + 1 - 4k^2 - 4k$$

$$= 4k^2 + 2k + 1$$

$$= 2k(2k+1) + 1$$

Case - 2 : n is even.

For $P_2 \boxtimes P_{2k}$, we have

$$\sum_{i=1}^{p-1} d(u_i, u_{i+1}) \leq \sum_{i=1}^{p-1} [L(u_i) + L(u_{i+1}) + 1]$$

$$= 2\sum_{u\in V(G)} L(u) \text{ - } L(u_1) \text{ - } L(u_p) + (p-1)$$

$$= 2\sum_{u\in V(G)} L(u) + (p-1) \tag{2.3}$$

Substituting (2.3) in (2.1), we get

$$rn(P_2 \boxtimes P_n) = f(u_p) \geq (p-1)(d+1) \text{ - } 2\sum_{u\in V(G)} L(u) \text{ - } (p-1)$$

For $P_2 \boxtimes P_{2k}$, $p = 4k$, $d = 2k-1$ and $\sum_{v\in V(G)} L(u) = 2k(k-1)$

$$rn(P_2 \boxtimes P_n) = f(u_p) \geq (4k-1)(2k-1+1) \text{ - } 4(k(k-1)) \text{ - } (4k-1)$$

$$= 8k^2 \text{ - } 2k \text{ - } 4k^2 + 1$$

$$= 4k^2 \text{ - } 2k + 1$$

$$= 2k(2k-1) + 1$$

Thus, from Case - 1 and Case - 2, we have

$$rn(P_2 \boxtimes P_n) \quad \geq \quad \begin{cases} 2k(2k+1)+1 & \text{if} \quad n = 2k+1 \\ 2k(2k-1)+1 & \text{if} \quad n = 2k \end{cases}$$

□

Theorem 2.3. *Let f be an assignment of distinct non-negative integers to $V(P_2 \boxtimes P_n)$ and $\{u_1, u_2, u_3, ..., u_p\}$ be the ordering of $V(P_2 \boxtimes P_n)$ such that $f(u_i) < f(u_{i+1})$ defined by $f(u_1) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - d(u_i, u_{i+1})$. Then f is a radio labeling if for any $1 \leq i \leq p-2$ and $k = \lfloor \frac{n}{2} \rfloor$ the following holds.*
(1) $d(u_i, u_{i+1}) \leq k+1$ if n is odd,
(2) $d(u_i, u_{i+1}) \leq k+1$ and $d(u_i, u_{i+1}) \neq d(u_{i+1}, u_{i+2})$ if n is even.

Proof. Let $f(u_1) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - d(u_i, u_{i+1})$, for any $1 \leq i \leq p-1$ and $k = \lfloor \frac{n}{2} \rfloor$.

For each $i = 1, 2, ..., p-1$, let $f_i = f(u_{i+1}) - f(u_i)$. Now we want to prove that f is a radio labeling if (1) and (2) holds. *i.e.* for any $i \neq j$, $|f(u_j) - f(u_i)| \geq d + 1 - d(u_i, u_j)$

Case - 1 : n is odd.

If $n = 2k+1$ then $d = 2k$ and let (1) holds.

Let $j > i$ then $f(u_j) - f(u_i) = f_i + f_{i+1} + ... + f_{j-1}$

$$= (j-i)(d+1) \text{ - } d(u_i, u_{i+1}) \text{ - } d(u_{i+1}, u_{i+2}) \text{ - } ... \text{ - } d(u_{j-1}, u_j)$$

$$\geq (j-i)(d+1) - (j-i)(k+1) \text{ as } d(u_i, u_{i+1}) \leq k+1$$

$$= (j-i)(2k+2) - (j-i)(k+1)$$

$$= (j-i)(2k+2-k-1)$$

$$= (j-i)(k+1)$$

$$\geq d + 1 - d(u_i, u_j).$$

Case - 2 : n is even.

If $n = 2k$ then $d = 2k - 1$ and let (2) holds.

Let $j > i$ then $f(u_j) - f(u_i) = f_i + f_{i+1} + ... + f_{j-1}$

$= (j - i)(d + 1)$ - $d(u_i, u_{i+1})$ - $d(u_{i+1}, u_{i+2})$ - ... - $d(u_{j-1}, u_j)$

If $j - i =$ even then

$\geq (j - i)(d + 1) - \frac{j-i}{2}(k + 1) - \frac{j-i}{2}(k)$

$= (j - i)(2k) - (j - i)(k) - \frac{j-i}{2}$

$= (j - i)(k) - \frac{j-i}{2}$

$\geq d + 1 - d(u_i, u_j)$

If $j - i =$ odd then

$\geq (j - i)(d + 1) - \frac{j-i+1}{2}(k + 1) - \frac{j-i-1}{2}(k)$

$\geq d + 1 - d(u_i, u_j)$

Thus, in both the cases f is a radio labeling and hence the result. □

Theorem 2.4. *Let $P_2 \boxtimes P_n$ be a strong product of P_2 and P_n and $k = \lfloor \frac{n}{2} \rfloor$ then*

$$rn(P_2 \boxtimes P_n) \quad \leq \quad \begin{cases} 2k(2k+1)+1 & if \quad n = 2k+1 \\ 2k(2k-1)+1 & if \quad n = 2k \end{cases}$$

Proof. Here we consider following two cases.

Case - 1 : n is odd.

For $P_2 \boxtimes P_{2k+1}$, define f : V$(P_2 \boxtimes P_{2k+1}) \to \{0,1,2, ... ,2k(2k+1)+1\}$ by $f(u_{i+1}) = f(u_i) + d + 1$ - $L(u_i)$ - $L(u_{i+1})$ as per ordering of vertices shown in Table 1:

Table 1

$$v_0 \xrightarrow{k} v_{Rk} \xrightarrow{k+1} v_{L1} \xrightarrow{k+1} v'_{Rk} \xrightarrow{k+1} v'_{L1} \xrightarrow{k} v_{R(k-1)}$$

$$\xrightarrow{k+1} v_{L2} \xrightarrow{k+1} v'_{R(k-1)} \xrightarrow{k+1} v'_{L2} \xrightarrow{k} \dots \xrightarrow{k} v_{R1}$$

$$\xrightarrow{k+1} v_{Lk} \xrightarrow{k+1} v'_{R1} \xrightarrow{k+1} v'_{Lk} \xrightarrow{k} v'_0$$

Case - 2 : n is even.

For $P_2 \boxtimes P_{2k}$, define $f : \mathrm{V}(P_2 \boxtimes P_{2k}) \rightarrow \{0,1,2, \ldots ,2k(2k-1)+1\}$ by $f(u_{i+1}) = f(u_i) + d - L(u_i) - L(u_{i+1})$ as per ordering of vertices shown in Table 2:

Table 2

$$v_{L0} \xrightarrow{k} v_{R(k-1)} \xrightarrow{k+1} v_{L1} \xrightarrow{k} v'_{R(k-2)} \xrightarrow{k+1} v'_{L2} \xrightarrow{k} v_{R(k-3)}$$

$$\xrightarrow{k+1} \ldots \xrightarrow{k+1} v_{L(k-1)} \xrightarrow{k} v_{R0} \xrightarrow{1} v'_{L0} \xrightarrow{k} v'_{R(k-1)}$$

$$\xrightarrow{k+1} v'_{L1} \xrightarrow{k} v'_{R(k-2)} \xrightarrow{k+1} v'_{L2} \xrightarrow{k} v'_{R(k-3)} \xrightarrow{k+1} \ldots$$

$$\xrightarrow{k+1} v'_{L(k-1)} \xrightarrow{k} v'_{R0}$$

Thus in Case - 1 and Case - 2, it is possible to assign labeling to the vertices of $P_2 \boxtimes P_n$ with span equal to the lower bound satisfying the condition of Theorem 2.3. Hence f is a radio labeling. □

Theorem 2.5. *Let $P_2 \boxtimes P_n$ be a strong product of P_2 and P_n and $k = \lfloor \frac{n}{2} \rfloor$ then*

$$rn(P_2 \boxtimes P_n) \quad = \quad \begin{cases} 2k(2k+1)+1 & \text{if} \quad n = 2k+1 \\ 2k(2k-1)+1 & \text{if} \quad n = 2k \end{cases}$$

Proof. The proof follows from Theorem 2.2 and Theorem 2.4. □

Example 2.1. *In Figure 1, ordering of the vertices and optimal radio labeling of $P_2 \boxtimes P_9$ is shown.*

$v_0 \rightarrow v_{R4} \rightarrow v_{L1} \rightarrow v'_{R4} \rightarrow v'_{L1} \rightarrow v_{R3} \rightarrow v_{L2} \rightarrow v'_{R3} \rightarrow v'_{L2} \rightarrow v_{R2} \rightarrow v_{L3} \rightarrow$

$v'_{R2} \rightarrow v'_{L3} \rightarrow v_{R1} \rightarrow v_{L4} \rightarrow v'_{R1} \rightarrow v'_{L4} \rightarrow v'_0 = rn(P_2 \boxtimes P_9)$

Example 2.2. *In Figure 2, ordering of the vertices and optimal radio labeling of $P_2 \boxtimes P_{10}$ is shown.*

$v_{L0} \rightarrow v_{R4} \rightarrow v_{L1} \rightarrow v_{R3} \rightarrow v_{L2} \rightarrow v_{R2} \rightarrow v_{L3} \rightarrow v_{R1} \rightarrow v_{L4} \rightarrow v_{R0} \rightarrow v'_{L0} \rightarrow$

$v'_{R4} \rightarrow v'_{L1} \rightarrow v'_{R3} \rightarrow v'_{L2} \rightarrow v'_{R2} \rightarrow v'_{L3} \rightarrow v'_{R1} \rightarrow v'_{L4} \rightarrow v'_{R0} = rn(P_2 \boxtimes P_{10})$

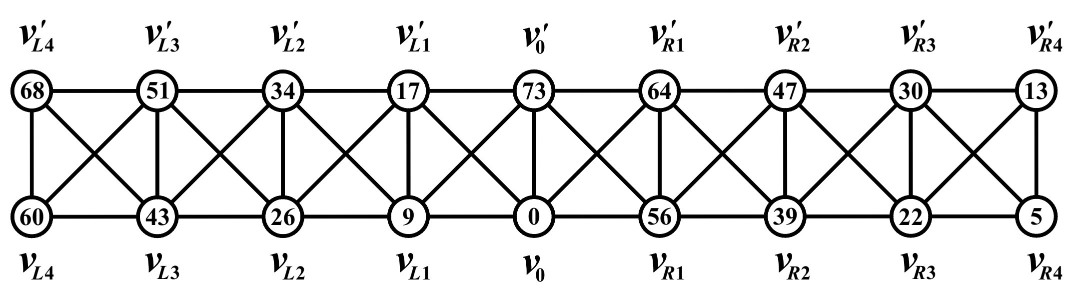

Figure 1. $rn(P_2 \boxtimes P_9)$ **= 73**

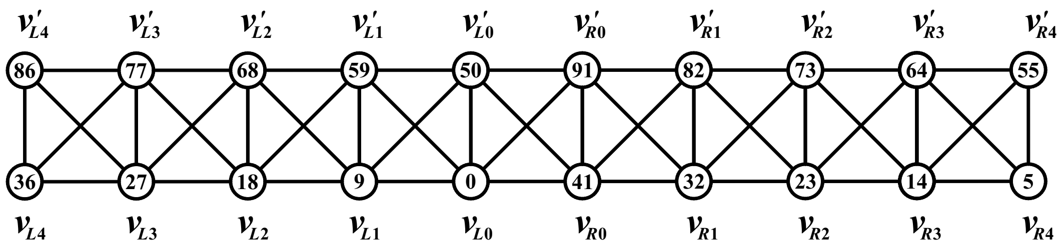

Figure 2. $rn(P_2 \boxtimes P_{10}) = \mathbf{91}$

3 Concluding Remarks

The assignment of channels is of great importance for the establishment of transmitter network which is free of interference. The radio labeling is an intelligent move in this direction because the level of interference is maximum at diametrical distance. We take up this problem in the context of strong product of P_2 and P_n and determine radio number for the same. To derive similar results for other graph families is an open area of research.

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References

[1] G. J. Chang and D. Kuo, The L(2,1)-labeling problem on graphs, *SIAM J. Discrete Math.*, 9(2)(1996), 309-316.

[2] G. Chartrand, D. Erwin, F. Harary and P. Zhang, Radio labeling of graphs, *Bull. Inst. Combin. Appl.*, 33(2001), 77-85.

[3] J. P. Georges and D. W. Mauro, Labeling trees with a condition at distsnce two, *Discrete Math.*, 269(2003), 127-148.

[4] J. R. Griggs and R. K. Yeh, Labeling graphs with condition at distance 2, *SIAM J. Discrete Math.*, 5(4)(1992), 586-595.

[5] W. K. Hale, Frequency assignment: Theory and applications, *Proc. IEEE*, 68(12)(1980), 1497-1514.

[6] D. Liu, Radio number for trees, *Discrete Mathematics*, 308(2008), 1153-1164.

[7] D. Liu, M. Xie, Radio number of square cycles, *Congr.Numer.*, 169(2004), 105-125.

[8] D. Liu, M. Xie, Radio number of square paths, *Ars Combin.*, 90(2009), 307-319.

[9] D. Liu and X. Zhu, Multilevel distance labelings for paths and cycles, *SIAM J. Discrete Math.,* 19(3)(2005), 610-621.

[10] F. S. Roberts, T-coloring of graphs: recent results and open problems, *Discrete Math.,* 93(1991), 229-245.

[11] D. Sakai, Labeling Chordal Graphs: Distance Two Condition, *SIAM J. Discrete Math.,* 7(1)(1994), 133-140.

[12] S. K. Vaidya and D. D. Bantva, Labeling cacti with a condition at distance two, *Le Matematiche*, 66(2011), 29-36.

[13] S. K. Vaidya, P. L. Vihol, N. A. Dani and D. D. Bantva, $L(2,1)$-labeling in the context of some graph operations, *Journal of Mathematics Research*, 2(3)(2010), 109-119.

[14] S. K. Vaidya and P. L. Vihol, Radio labeling for some cycle related graphs, *International Journal of Mathematics and Soft Computing*, 2(2)(2012), 11-24.

[15] W. Wang, The L(2,1)-labeling of trees, *Discrete Applied Math.,* 154(2006), 598-603.

[16] D. B. West, *Introduction to Graph Theory*, Prentice -Hall of India, 2001.

[17] R. K. Yeh, A survey on labeling graphs with a condition at distance two, *Discrete Math.,* 306(2006), 1217-1231.

[18] R. K. Yeh, *Labeling Graphs with a Condition at Distance Two*, Ph.D.Thesis, Dept.of Math., University of South Carolina, Columbia, SC, 1990.

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