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On degree of approximation of conjugate series of a Fourier series by product summability

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Abstract

In this paper a theorem on degree of approximation of a function $f \in Lip(\alpha, r)$ by product summability $(E,q)(\bar{N}, p_n)$ of conjugate series of Fourier series associated with f has been proved.

Keywords: Degree of Approximation, $Lip(\alpha, r)$ class of function, (E,q) mean, $(\bar{N}, p_n)$ mean, $(E,q)(\bar{N}, p_n)$ product mean, Fourier series, conjugate of the Fourier series, Lebesgue integral.

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1 Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \longrightarrow \infty, as\ n \longrightarrow \infty, (P_{-i} = p_{-i} = 0, i \geq 0). \tag{1.1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\upsilon=0}^{n} p_\upsilon s_\upsilon, \tag{1.2}$$

defines the sequence $\{t_n\}$ of the $(\bar{N}, p_n)$-mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \longrightarrow s,\ as\ n \longrightarrow \infty, \tag{1.3}$$

then the series $\sum a_n$ is said to be $(\bar{N}, p_n)$ summable to s.
The conditions for regularity of $(\bar{N}, p_n)$-summability are easily seen to be [1]

$$\begin{cases} (i) P_n \to \infty, \quad as \quad n \to \infty, \\ (ii) \sum_{i=0}^{n} p_i \leq C \mid P_n \mid, \quad as \quad n \to \infty. \end{cases} \tag{1.4}$$

The sequence-to-sequence transformation, [1]

$$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon, \tag{1.5}$$

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defines the sequence $\{T_n\}$ of the (E,q) mean of the sequence $\{s_n\}$. If

$$T_n \to s, \quad as \quad n \to \infty, \tag{1.6}$$

then the series Σa_n is said to be (E,q) summable to s. Clearly (E,q) method is regular. Further, the (E,q) transformation of the $(\bar{N},p_n)$ transform of $\{s_n\}$ is defined by

$$\begin{aligned}\tau_n &= \frac{1}{(1+q)^n}\Sigma_{k=0}^{n}\binom{n}{k}q^{n-k}T_k \\ &= \frac{1}{(1+q)^n}\Sigma_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\Sigma_{\upsilon=0}^{k}p_\upsilon s_\upsilon\right\}\end{aligned} \tag{1.7}$$

If

$$\tau_n \to s, \ as \ n \to \infty, \tag{1.8}$$

then $\sum a_n$ is said to be $(E,q)(\bar{N},p_n)$-summable to s.

Let $f(t)$ be a periodic function with period 2π and L-integrable over $(-\pi,\pi)$. The Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\ nx + b_n sin\ nx) \equiv \sum_{n=0}^{\infty} A_n(x), \tag{1.9}$$

and the conjugate series of the Fourier Series (1.9) is

$$\sum_{n=1}^{\infty}(b_n cos\ nx - a_n sin\ nx) \equiv \sum_{n=0}^{\infty} B_n(x). \tag{1.10}$$

Let $\bar{s}_n(f:x)$ be the n-th partial sum of (1.10). The L_∞-norm of a function $f: R \to R$ is defined by

$$\parallel f \parallel_\infty = sup\{|f(x)| : x \in R\} \tag{1.11}$$

and the L_υ-norm is defined by

$$\parallel f \parallel_\upsilon = \left(\int_0^{2\pi} |f(x)|^\upsilon\right)^{\frac{1}{\upsilon}}, \upsilon \geq 1. \tag{1.12}$$

The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\parallel \cdot \parallel_\infty$ is defined by [5]

$$\parallel P_n - f \parallel_\infty = sup\{|p_n(x) - f(x)| : x \in R\} \tag{1.13}$$

and the degree of approximation $E_n(f)$ a function $f \in L_\upsilon$ is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_\upsilon. \tag{1.14}$$

A function f is said to satisfy Lipschitz condition (here after we write $f \in$ Lip α) if

$$|f(x+1) - f(x)| = O(|t|^\alpha), 0 < \alpha \leq 1. \tag{1.15}$$

and $f(x) \epsilon Lip(\alpha, r)$, for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = 0(|t|^\alpha), 0 < \alpha \leq 1, r \geq 1, t > 0. \tag{1.16}$$

For a given positive increasing function $\xi(t)$, the function $f(x) \in$ Lip $(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = 0(\xi(t)), r \geq 1, t > 0. \tag{1.17}$$

We use the following notation throughout this paper:

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}, \tag{1.18}$$

and

$$\overline{K}_n(t) = \frac{1}{\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}.$$

Further, the method $(E,q)(\overline{N},P_n)$ is assumed to be regular.

2 Known Theorems

Dealing with the degree of approximation by the product Misra et. al. [2] proved the following theorem using $(E,q)(\overline{N},p_n)$-mean of Conjugate Series of Fourier series:

Theorem 2.1. *If f is 2π-periodic function of class $Lip\alpha$, then degree of approximation by the product $(E,q)(\bar{N},p_n)$ summability mean of the conjugate series (1.10) of the Fourier Series (1.9) is given by $\|\tau_n - f|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right)$, $0<\alpha<1$, where τ_n is as defined in (1.7).*

Very recently Paikray et. al [3] established a theorem on degree of approximation by the product mean $(E,q)(\bar{N},p_n)$ of the Conjugate Series of fourier Series of a function of class $Lip(\alpha,r)$. They proved:

Theorem 2.2. *If f is a 2π-Periodic function of class $Lip(\alpha,r)$, then degree of approximation by the product $(E,q)(\bar{N},p_n)$ summability means on on he Conjugate Series (1.10) of the Fourier series (1.9) is given by $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{r}}}\right), 0<\alpha<1,\ r\geq 1$, where τ_n is as defined in (1.7).*

3 Main Theorem

In this paper, we have proved a theorem on degree of approximation by the product mean $(E,q)(\bar{N},p_n)$ of the conjugate series of the Fourier series of a function of class $Lip(\xi(t),r)$. We prove:

Theorem 3.3. *Let $\xi(t)$ be a positive increasing function and f a 2π- periodic function of the class $Lip(\xi(t),r), r\geq 1, t>0$. Then degree of approximation by the product $(E,q)(\bar{N},p_n)$ summability means on the Conjugate Series (1.10) of the Fourier series (1.9) is given by $\|\ \tau_n - f\ \|_\infty = O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right), r\geq 1$, where τ_n is as defined in (1.7).*

4 Required Lemmas

We require the following Lemmas to prove the theorem.

Lemma 4.1.

$$|\bar{K}_n(t)| = O(n), 0\leq t\leq \frac{1}{n+1}.$$

Proof. For $0\leq t\leq\frac{1}{n+1}$, we have $sin\ nt \leq n\ sin\ t$ then

$$\begin{aligned}
|\bar{K}_n(t)| &= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\upsilon t\cdot\cos\frac{t}{2}+\sin\upsilon t\cdot\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(\frac{cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)}{\sin\frac{t}{2}}+\sin\upsilon t\right)\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(O\left(2\sin\upsilon\frac{t}{2}\sin\upsilon\frac{t}{2}\right)+\upsilon\sin t\right)\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon(O(\upsilon)+O(\upsilon))\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{O(k)}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right| \\
&= O(n).
\end{aligned}$$

This proves the lemma. □

Lemma 4.2.

$$|\bar{K}_n(t)| = O\left(\frac{1}{t}\right), for \frac{1}{n+1} \leq t \leq \pi.$$

Proof. For $\frac{1}{n+1} \leq t \leq \pi$, by Jordan's lemma, we have $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$.Then

$$\begin{aligned}
|\bar{K}_n(t)| &= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\upsilon\frac{t}{2}\cdot\cos\frac{t}{2}+\sin\upsilon\frac{t}{2}\cdot\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}\frac{\pi}{2t}p_\upsilon\left(\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)+\sin\upsilon\frac{t}{2}\cdot\sin\frac{t}{2}\right)\right\}\right| \\
&\leq \frac{\pi}{2\pi(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right\}\right| = \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right\}\right| \\
&= \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right| \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

This proves the lemma. □

5 Proof of Theorem 3.1

Using Riemann-Lebesgue theorem, we have for the n-th partial sum $\bar{s}_n(f:x)$ of the conjugate Fourier series (1.10) of $f(x)$, following Titchmarch [4]

$$\bar{s}_n(f:x) - f(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\bar{K}_n dt,$$

the (N, p_n) transform of $\bar{s}_n(f:x)$ using (1.2) is given by

$$t_n - f(x) = \frac{2}{\pi P_n}\int_0^\pi \psi(t)\sum_{k=0}^{n}p_k\frac{\cos\frac{t}{2}-\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt,$$

denoting the $(E,q)(N,p_n)$ transform of $\bar{s}_n(f:x)$ by τ_n, we have

$$\begin{aligned}
\|\tau_n - f\| &= \frac{1}{\pi(1+q)^n}\int_0^\pi \psi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\sin\left(\upsilon+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right\}dt \\
&= \int_0^\pi \psi(t)\bar{K}_n(t)dt \\
&= \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right\}\psi(t)\bar{K}_n(t)dt \\
&= I_1 + I_2, \quad \text{say.} \qquad (5.1)
\end{aligned}$$

Now

$$
\begin{aligned}
|I_1| &= \frac{2}{\pi(1+q)^n}\left|\int_0^{\frac{1}{n+1}}\psi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right\}dt\right| \\
&= \left|\int_0^{\frac{1}{n+1}}\psi(t)\bar{K}_n(t)dt\right| \\
&= \left(\int_0^{\frac{1}{n+1}}\left(\frac{\psi(t)}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{n+1}}(\xi(t)\bar{K}_n(t))^s dt\right)^{\frac{1}{s}}, \quad \text{using Holder's inequality} \\
&= O(1)\left(\int_0^{\frac{1}{n+1}}\xi(t)n^s dt\right)^{\frac{1}{s}} \\
&= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left(\frac{n^s}{n+1}\right)^{\frac{1}{s}} \\
&= O\left(\xi\left(\frac{1}{n+1}\right)\frac{1}{(n+1)^{\frac{1}{s}-1}}\right) \\
&= O\left(\xi\left(\frac{1}{n+1}\right)\frac{1}{(n+1)^{-\frac{1}{r}}}\right) \\
&= O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right). \qquad (5.2)
\end{aligned}
$$

Next

$$
\begin{aligned}
|I_2| &\le \left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\phi(t)}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{1}{n+1}}^{\pi}(\xi(t)\bar{K}_n(t))^s dt\right)^{\frac{1}{s}}, \text{ using Holder's inequality} \\
&= O(1)\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t}\right)^s dt\right)^{\frac{1}{s}}, \quad \text{using Lemma 4.1} \\
&= O(1)\left(\int_{\frac{1}{\pi}}^{n+1}\left(\frac{\xi\left(\frac{1}{y}\right)}{\frac{1}{y}}\right)^s\frac{dy}{y^2}\right)^{\frac{1}{s}}. \qquad (5.3)
\end{aligned}
$$

Since $\xi(t)$ is a positive increasing function, so is $\xi(1/y)/(1/y)$. Using second mean value theorem we get

$$
= O\left((n+1)\xi\left(\frac{1}{n+1}\right)\right)\left(\int_\delta^{n+1}\frac{dy}{y^2}\right)^{\frac{1}{s}}, \quad \text{for some} \quad \frac{1}{\pi}\le\delta\le n+1
$$

$$
= O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right)
$$

Then from (5.2) and (5.3), we have

$$
|\ \tau_n - f(x)\ | = O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right), for \quad r\ge 1.
$$

$$
\|\ \tau_n - f(x)\ \|_\infty = \sup_{-\pi<x<\pi}|\tau_n - f(x)| = O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right), r\ge 1.
$$

This completes the proof of the theorem.

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