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Nonlocal impulsive fractional semilinear differential equations with almost sectorial operators

M.C. Ranjini[a] and A. Anguraj[b,*]

[a,b] Department of Mathematics, P.S.G. College of Arts and Science, Coimbatore-641 014, Tamil Nadu, India.

Abstract

This paper is concerned with the existence and uniqueness of mild solutions for a class of impulsive fractional semilinear differential equations with nonlocal condition in a Banach space by using the concepts of almost sectorial operators. The results are established by the application of the Banach fixed point theorem and Krasnoselskii's fixed point theorem.

Keywords: Fractional differential equations, impulses, nonlocal condition, almost sectorial operator, semigroup of growth γ, mild solution

2010 MSC: 34A37, 34K37, 47A60.

1 Introduction

Sectorial operators, that is, linear operators A defined in Banach spaces, whose spectrum lies in a sector

$$S_w = \{\lambda \in \mathbb{C}/\{0\} \mid |arg\lambda| \le w\} \cup \{0\} \quad for\ some \ \ 0 \le w \le \frac{\pi}{2}$$

and whose resolvent satisfies an estimate

$$||(\lambda - A)^{-1}|| \quad \le \quad C|\lambda|^{-1}, \qquad \forall \quad \lambda \in \mathbb{C}\backslash S_w, \tag{1.1}$$

have been studied extensively during the last 40 years, both in abstract settings and for their applications to partial differential equations. Many important elliptic differential operators belong to the class of sectorial operators, especially when they are considered in the Lebesgue spaces or in spaces of continuous functions (see [1] and [[2], chapter 3]). However, if we look at spaces of more regular functions such as the spaces of Holder continuous functions, we find that these elliptic operators do no longer satisfy the estimate (1.1) and therefore are not sectorial as was pointed out by Von Wahl (see [[3], Ex.3.1.33], see [4]).

Neverthless, for these operators estimates such as

$$||(\lambda - A)^{-1}|| \le \frac{C}{|\lambda|^{1-\gamma}}, \qquad \lambda \in \sum_{w,v} = \{\lambda \in \mathbb{C} : |arg(\lambda - w)| < v\} \tag{1.2}$$

where $\gamma \in (0,1)$, $w \in \mathbb{R}$ and $v \in (\frac{\pi}{2}, \pi)$, can be obtained, (see[4]) which allows to define an associated "analytic semigroup" by means of the Dunford Integral

$$T(t) \quad = \quad \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda - A)^{-1} d\lambda, \qquad t > 0 \tag{1.3}$$

where $\Gamma_\theta = \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$.

*Corresponding author.

E-mail addresses: angurajpsg@yahoo.com (A.Anguraj) and ranjiniprasad@gmail.com (M.C.Ranjini).

In the literature, a linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ which satisfy the condition (1.2) is called almost sectorial and the operator family $\{T(t), T(0) = I, t \geq 0\}$ is said the "semigroup of growth γ" generated by A. The operator family $T(t)_{t\geq 0}$ has properties similar at those of analytic semigroup which allow to study some classes of partial differential equations via the usual methods of semigroup theory. Concerning almost sectorial operators, semigroups of growth γ and applications to partial differential equations, we refer the reader to [4, 5, 6, 7, 8] and the references there in.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc., involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. Though the concepts and the calculus of fractional derivative are few centuries old, it is realized only recently that these derivatives form an excellent framework for modeling real world problems.

In the consequence, fractional differential equations have been of great interest. For details, see the monographs of Kilbas et al.[9], Lakshimkantham et al.[10], Miller and Ross [11], Podlubny [12] and the papers in [13, 14, 15, 16] and the references therein.

On the otherhand, the theory of impulsive differential equations has undergone rapid development over the years and played a very imortant role in modern applied mathematical models of real processes arising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics. See, the monographs of Bainov and Simeonov [17], Benchohra et al.[18], Lakshmikantham et al.[19], Samoilenko and Perestyuk[20], A. Anguraj et al.[21, 22] and the references therein. However impulsive fractional differential equations have been studied by the authors, see for instance [23, 24, 25].

We have also seen articles dealing with nonlocal conditions. That is a classical initial condition $x(0) = x_0$ is extended to the following nonlocal condition $x(0) + g(x(.)) = x_0$, where x(.) is a solution and g is a mapping defined on some function space into $\mathbb{X}$. Such nonlocal conditions were first used by K. Deng, in [26]. In his paper, Deng indicated that the diffusion phenomenon of a small amount of gas in a transparent tube can give a better result than using the usual local condition. For the importance of nonlocal conditions in different fields, we refer the reader to [27, 28, 29, 30] and the references contained therein.

Very recently, Rong-Nian Wang et al.[31], studied the classical and mild solutions of abstract fractional cauchy problems using almost sectorial operators and in [32], A.N. Carvalho et al. established the existence of mild solutions for cauchy problem for non-autonomous evolution equation, in which the operator in the linear part depends on time t and for each t, it is almost sectorial.To the best of our knowledge, much less is known about the nonlocal impulsive fractional differential equations with almost sectorial operators. Using the concepts of the above mentioned papers, we proved the existence and uniqueness of mild solutions of the nonlocal impulsive fractional differential equations with almost sectorial operators.

Here, we consider the semilinear impulsive fractional differential equations with nonlocal conditions in the following form.

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in I = [0, T], \quad t \neq t_k \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad t = t_k, \quad k = 1, 2, ..., m. \\ x(0) + g(x) = x_0 \end{cases} \tag{1.4}$$

where ${}^cD^\alpha$ is the standard Caputo's fractional derivative of order α, $0 < \alpha < 1$ and $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is an almost sectorial operator on a Banach space $\mathbb{X}$. Here, $0 < t_1 < t_2 < ... < t_m = T$, $I_k \in C(\mathbb{X}, \mathbb{X})$, $k = 1, 2, ..., m$. Let $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively. The nonlocal condition

$$g(x) = \sum_{i=1}^{n} c_i x(s_i)$$

where c_i, i = 1,2,...n, are given constants and $0 < s_1 < s_2 ... < s_n \leq T$.

2 Preliminaries

In this section, we recall some notations, properties of $T(t)$ and the definition of a mild solution of (1.4) by investigating the Classical solutions of the system (1.4).

Proposition 2.1. *([5, 6]). Let A be the almost sectorial operator satisfying the conditions (1.2) and (1.3). Then the following properties are satisfied.*

(i) The operator A is closed, $T(t+s) = T(t)T(s)$ and $AT(t)x = T(t)Ax$, $\forall\ t, s \in [0, \infty)$ and each $x \in D(A)$.

(ii) $\frac{d}{dt}T(t) = AT(t)$.

(iii) There exists a constant $C_0 > 0$ such that $||A^n T(t)|| \leq C_n t^{-(n+\gamma)}$ $(t > 0)$.

Now, we state the necessary notions and facts on fractional calculus.

Definition 2.1. *([9]) The Riemann-Liouville fractional integral operator of order $q > o$ with the lower limit t_0 for a function f is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds, \quad t > t_0$$

provided the right-hand side is pointwise defined on $[t_0, \infty)$, where Γ is the gamma function.

Definition 2.2. *([9]) The Riemann-Liouville (R-L) derivative of order $q > 0$ with the lower limit t_0 for a function $f : [t_0, \infty) \to \mathbb{R}$ can be written as*

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{(n-q-1)} f(s) ds, \quad t > t_0, \quad n-1 < q < n.$$

Definition 2.3. *([9]). The Caputo fractional derivative of order $q > 0$ with the lower limit t_0 for a function $f : [t_0, \infty) \to \mathbb{R}$ can be written as*

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{(n-q-1)} f^{(n)}(s) ds = I^{(n-q)} f^{(n)}(t), \quad t > t_0, \quad n-1 < q < n.$$

Denote $E_{\alpha,\beta}$ the generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{\wp} \frac{\lambda^{\alpha-\beta} e^{\lambda}}{\lambda^{\alpha} - z} d\lambda, \quad \alpha, \beta > 0, \quad z \in \mathbb{C}$$

where $\wp$ is a contour which starts and ends at $-\infty$.

Throughout this section we let A be an almost sectorial operator with semigroup of growth γ, where $0 < \gamma < 1$. In the sequel, we will define two families of operators based on the generalized Mittag-Leffler-type functions and the resolvent operators associated with A. They will be two families of linear and bounded operators.

Next, we consider the definition of mild solution of (1.4).

Consider, the following cauchy problem,

$$\begin{cases} {}^cD^{\alpha} x(t) = Ax(t) + f(t, x(t)), \quad 0 < \alpha < 1, \\ x(0) + g(x) = x_0 \in X \end{cases} \tag{2.5}$$

where f is an abstract function defined on [0, ∞) and with values in X, A is almost sectorial operator.

Using Mittag-Leffler function, the Classical solution of the system (2.5) is given by,

$$x(t) = [x_0 - g(x)] E_{\alpha,1}(At^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) f(s, x(s)) ds. \tag{2.6}$$

Denote the operators $P_{\alpha}(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha})$ and $S_{\alpha}(t) = E_{\alpha,1}(At^{\alpha})$. Then x(t) can be expressed as

$$x(t) = S_{\alpha}(t)[x_0 - g(x)] + \int_0^t P_{\alpha}(t-s) f(s, x(s)) ds. \tag{2.7}$$

where $S_\alpha(t)$ and $P_\alpha(t)$ can be expressed as

$$\begin{aligned} S_\alpha(t) &= \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda. \\ P_\alpha(t) &= \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda^\alpha - A)^{-1}d\lambda. \end{aligned}$$

where $\Gamma_\theta = \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$, is oriented counter-clockwise.

Lemma 2.1. *For each fixed $t > 0$, $S_\alpha(t)$ and $P_\alpha(t)$ are linear and bounded operators on $\mathbb{X}$. Moreover, there exist constants $C_s = C(\alpha,\gamma) > 0$, $C_p = C(\alpha,\gamma) > 0$ such that for all $t > 0$,*

$$||S_\alpha(t)|| \le C_s t^{-\alpha\gamma}, \qquad ||P_\alpha(t)|| = C_p t^{\alpha(1-\gamma)-1}, \qquad \text{where} \quad 0 < \gamma < 1.$$

Proof. Since, $t > 0$, $0 < \gamma < 1$, there exists a constant $C > 0$ such that

$$||(\lambda - A)^{-1}|| \le \frac{C}{|\lambda|^{1-\gamma}} \qquad ,\lambda \in \sum_{w,v}$$

From [32], observe that $\frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda - A)^{-1}d\lambda$ converge in the uniform operator topology for all $t > 0$ and by (1.3), we have that

$$\begin{aligned} ||S_\alpha(t)|| &\le \left\| \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda \right\| \\ &\le \frac{1}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\lambda|t}|\lambda|^{\alpha-1}||(\lambda^\alpha - A)^{-1}||d|\lambda| \\ &\le \frac{1}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\lambda|t}|\lambda|^{\alpha-1}\frac{C}{|\lambda|^{\alpha(1-\gamma)}}d|\lambda| \\ &\le \frac{C\ t^{-\alpha\gamma}}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\mu|}|\mu|^{\alpha\gamma-1}d\mu \\ &\le C_s\ t^{-\alpha\gamma} \end{aligned}$$

Also, we have

$$\begin{aligned} ||P_\alpha(t)|| &\le \left\| \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda^\alpha - A)^{-1}d\lambda \right\| \\ &\le \frac{1}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\lambda|t}||(\lambda^\alpha - A)^{-1}||d|\lambda| \\ &\le \frac{1}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\lambda|t}\frac{C}{|\lambda|^{\alpha(1-\gamma)}}d|\lambda| \\ &\le \frac{C\ t^{\alpha(1-\gamma)-1}}{2\pi}\int_{\Gamma_\theta} e^{-cos\theta|\mu|}|\mu|^{-\alpha(1-\gamma)}d\mu \\ &\le C_s\ t^{\alpha(1-\gamma)-1} \end{aligned}$$

□

Lemma 2.2. *([31]) For $t > 0$, $S_\alpha(t)$ and $P_\alpha(t)$ are continuous in the uniform operator topology. Moreover, for every $r > 0$, the continuity is uniform on $[r,\infty)$.*

Theorem 2.1. *If f satisfies the uniform Holder condition with exponent $\beta \in (0,1]$ and A is an almost sectorial operator, then any solution of the Cauchy problem (1.4) is a fixed point of the operator given below*

$$\Gamma x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)] + \int_0^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in [0,t_1]; \\ S_\alpha(t-t_1)\big(x(t_1^-) + I_1(x(t_1^-))\big) + \int_{t_1}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_1,t_2]; \\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) + \int_{t_m}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_m,T]. \end{cases}$$

In fact, from(2.7) it is easy to see that Theorem [2.1] holds, so the proof is omitted.

Now let us consider the set of functions $PC(I,\mathbb{X}) = \{x : I \to \mathbb{X} : x \in C((t_k, t_{k+1}], \mathbb{X}),\ \ k = 1, 2, ..., m$ and there exist $x(t_k^+)$ and $x(t_k^-)$, $\ k = 1, 2, ...m$ with $x(t_k^-) = x(t_k)\}$
endowed with the norm $||x||_{PC} = \sup_{t\in I} ||x(t)||$.
From Theorem[2.1], we can define the mild solution of the system (1.4) as follows:

Definition 2.4. *A function $x : I \to \mathbb{X}$ is called a mild solution of a system (1.4), if $x \in PC(I,\mathbb{X})$ and satisfies the following equation,*

$$x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)] + \int_0^t P_\alpha(t-s)\ f(s, x(s))ds, & t \in [0, t_1]; \\ S_\alpha(t-t_1)\big(x(t_1^-) + I_1(x(t_1^-))\big) + \int_{t_1}^t P_\alpha(t-s)\ f(s, x(s))ds, & t \in (t_1, t_2]; \\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) + \int_{t_m}^t P_\alpha(t-s)\ f(s, x(s))ds, & t \in (t_m, T]. \end{cases}$$

Remark 2.1. *It is easy to verify that a classical solution of (1.4) is a mild solution of the same system.*

3 Existence Results

In this section, we give the main results on the existence of mild solutions of the system (1.4).
To establish our results, we introduce the following hypotheses.

(H_1) $f : I \times \mathbb{X} \to \mathbb{X}$ is continuous and there exists a constant $M > 0$ such that

$$\begin{aligned} ||f(t,x) - f(t,y)|| &\le M||x-y||, \quad \forall\ t \in I, x, y \in \mathbb{X} \\ ||f(t,0)|| &\le k_1, \end{aligned}$$

where k_1 is a constant.

(H_2) $g : PC(I,\mathbb{X}) \to \mathbb{X}$ is continuous and there exists a constant b such that

$$\begin{aligned} ||g(x) - g(y)|| &\le b||x-y||_{PC}, \quad \forall\ t \in I, x, y \in PC(I,\mathbb{X}) \\ ||g(0)|| &\le k_2, \end{aligned}$$

where k_2 is a constant.

(H_3) for each $k = 1, 2, ..., m$, there exists $\rho_k > 0$ such that

$$\begin{aligned} ||I_k(x) - I_k(y)|| &\le \rho_k||x-y||, \quad \forall x, y \in \mathbb{X} \\ ||I_k(0)|| &\le k_3, \end{aligned}$$

where k_3 is a constant.

(H_4) For each $x_0 \in \mathbb{X}$, there exists a constant $r > 0$ such that
$r \ge \max_{1\le i\le m} \Big\{C_s T^{-\alpha\gamma}\big[||x_0|| + r(\rho_i + b + 1) + k_2 + k_3)\big] + C_p(Mr + k_1)\frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big\}$

Theorem 3.2. *Under the assumptions $(H_1) - (H_3)$, the system (1.4) has a unique mild solution $x \in PC(I,\mathbb{X})$ if*

$$N = \max_{1\le i\le m} \Big\{C_s T^{-\alpha\gamma}\big[b + 1 + \rho_i\big] + C_p M \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big\} < 1 \tag{3.8}$$

Proof. Define $\Gamma : PC(I,\mathbb{X}) \to PC(I,\mathbb{X})$ by

$$\Gamma x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)] + \int_0^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in [0,t_1]; \\ S_\alpha(t-t_1)\big(x(t_1^-) + I_1(x(t_1^-))\big) + \int_{t_1}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_1,t_2]; \\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) + \int_{t_m}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_m,T]. \end{cases}$$

Clearly, the fixed points of the operator Γ are the solutions of the problem (1.4). We shall use the Banach contracton principle to prove that Γ has a fixed point.
We shall show that Γ is a contraction.
Let $x, y \in PC(I,\mathbb{X})$. Then for each $t \in [0,t_1]$ and by the lemma (2.1), we have

$$\begin{aligned} ||\Gamma x(t) - \Gamma y(t)|| &\leq ||S_\alpha(t)||\ ||g(x) - g(y)|| + \int_0^t ||P_\alpha(t-s)||\ ||f(s,x(s)) - f(s,y(s))||ds \\ &\leq C_s t^{-\alpha\gamma}\ b\ ||x-y|| + C_p \int_0^t (t-s)^{\alpha(1-\gamma)-1}\ M||x(s) - y(s)||ds \\ &\leq \Big[C_s T^{-\alpha\gamma}\ b + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big]\ ||x-y||_{PC} \end{aligned}$$

For $t \in (t_1,t_2]$,

$$\begin{aligned} ||\Gamma x(t) - \Gamma y(t)|| &\leq ||S_\alpha(t-t_1)||\big[||x(t_1^-) - y(t_1^-)|| + ||I_1(x(t_1^-)) - I_1(y(t_1^-))||\big] \\ &\quad + \int_{t_1}^t ||P_\alpha(t-s)||\ ||f(s,x(s)) - f(s,y(s)||ds \\ &\leq C_s(t-t_1)^{-\alpha\gamma}\big[||x(t_1^-) - y(t_1^-)|| + \rho_1||x(t_1^-) - y(t_1^-)||\big] \\ &\quad + C_p \int_{t_1}^t (t-s)^{\alpha(1-\gamma)-1}\ M||x(s) - y(s)||ds \\ &\leq \Big[C_s\ T^{-\alpha\gamma}(\rho_1+1) + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big]\ ||x-y||_{PC} \end{aligned}$$

Similarly, for all $t \in (t_i + t_{i+1}]$,

$$||\Gamma x(t) - \Gamma y(t)|| \leq \Big[C_s\ T^{-\alpha\gamma}(\rho_i+1) + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big]\ ||x-y||_{PC}$$

and for $t \in (t_m, T]$,

$$||\Gamma x(t) - \Gamma y(t)|| \leq \Big[C_s\ T^{-\alpha\gamma}(\rho_m+1) + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big]\ ||x-y||_{PC}$$

Thus, for all $t \in [0,T]$,

$$\begin{aligned} ||\Gamma x(t) - \Gamma y(t)|| &\leq \max_{1\leq i\leq m} \Big\{C_s\ T^{-\alpha\gamma}(b+\rho_i+1) + M\ C_p\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big\}\ ||x-y||_{PC} \\ &\leq N\ ||x-y||_{PC} \end{aligned}$$

Thus, by the equation (3.8), Γ is a contraction mapping. As a consequence of Banach fixed point theorem, we deduce that Γ has a unique fixed point $x_0 \in PC(I,\mathbb{X})$ which is a solution of the problem (1.4). □

Our next result is based on Krasnoselskii's fixed point theorem.

Lemma 3.3. *(Krasnoselskii's Fixed point theorem)([14]). Let $\mathbb{X}$ be a Banach space, let E be a bounded closed convex subset of $\mathbb{X}$ and let Γ_1, Γ_2 be maps of E into $\mathbb{X}$ such that $\Gamma_1 x + \Gamma_2 y \in E$ for every pair $x, y \in E$. If Γ_1 is a contraction and Γ_2 is completely continuous, then the equation $\Gamma_1 x + \Gamma_2 x = x$ has a solution on E.*

Theorem 3.3. *Assume that the hypothesis $(H_1)-(H_4)$ are satisfied, then the system has atleast one mild solution on I.*

Proof. Define operator $\Gamma : PC(I,\mathbb{X}) \to PC(I,\mathbb{X})$, as in Theorem [3.2] by

$$\Gamma x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)] + \int_0^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in [0,t_1]; \\ S_\alpha(t-t_1)\big(x(t_1^-) + I_1(x(t_1^-))\big) + \int_{t_1}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_1,t_2]; \\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) + \int_{t_m}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_m,T]. \end{cases}$$

Define B_r as $B_r = \Big\{x \in PC(I,\mathbb{X}) : ||x||_{PC} \leq r\Big\}$. Then, B_r is a closed, bounded and convex subset of $PC(I,\mathbb{X})$. On B_r, we define the operators Γ_1 and Γ_2 as follows.

$$\Gamma_1 x(t) = \begin{cases} S_\alpha(t)[x_0 - g(x)], & t \in [0,t_1]; \\ S_\alpha(t-t_1)\big(x(t_1^-) + I_1(x(t_1^-))\big), & t \in (t_1,t_2]; \\ . \\ . \\ . \\ S_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big), & t \in (t_m,T]. \end{cases}$$

and

$$\Gamma_2 x(t) = \begin{cases} \int_0^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in [0,t_1]; \\ \int_{t_1}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_1,t_2]; \\ . \\ . \\ . \\ \int_{t_m}^t P_\alpha(t-s)\ f(s,x(s))ds, & t \in (t_m,T]. \end{cases}$$

Now, we show that $\Gamma_1 + \Gamma_2$ has a fixed point in B_r. The proof is divided into three steps.
Step 1: $\Gamma_1 x + \Gamma_2 y \in B_r$, for every pair $x, y \in B_r$.
Consider for any $x, y \in B_r$ and for $t \in [0,t_1]$, we have

$$\begin{aligned} ||\Gamma_1 x(t) + \Gamma_2 y(t)|| &\leq ||S_\alpha(t)||\ \big[||x_0|| + ||g(x) - g(0)|| + ||g(0)||\big] \\ &\quad + \int_0^t ||P_\alpha(t-s)||\ [||f(s,y(s)) - f(s,0)|| + ||f(s,0)||]ds \\ &\leq C_s\ t^{-\alpha\gamma}\ \big[||x_0|| + b||x|| + k_2\big] + C_p \int_0^t (t-s)^{\alpha(1-\gamma)-1}\ (M||y|| + k_1)ds \\ &\leq C_s\ T^{-\alpha\gamma}\ \big[||x_0|| + br + k_2\big] + C_p\ (Mr + k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)} \end{aligned}$$

For $t \in (t_1,t_2]$, we have

$$\begin{aligned} ||\Gamma_1 x(t) + \Gamma_2 y(t)|| &\leq ||S_\alpha(t-t_1)||\big[||x(t_1^-)|| + ||I_1(x(t_1^-))||\big] + \int_{t_1}^t ||P_\alpha(t-s)||\ ||f(s,y(s))||ds \\ &\leq C_s\ (t-t_1)^{-\alpha\gamma}\ \big[r + (\rho_1 r + k_3)\big] + C_p \int_{t_1}^t (t-s)^{\alpha(1-\gamma)-1}\ (Mr + k_1)ds \\ &\leq C_s\ T^{-\alpha\gamma}\ \big[r(1+\rho_1) + k_3\big] + C_p\ (Mr + k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)} \end{aligned}$$

Similarly, we have

$$||\Gamma_1 x(t)+\Gamma_2 y(t)|| \leq C_s\ T^{-\alpha\gamma}\ \big[r(1+\rho_i)+k_3\big]+C_p\ (Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)},\ \ \forall\ \ t\in(t_i,t_{i+1}]$$

and

$$||\Gamma_1 x(t)+\Gamma_2 y(t)|| \leq C_s\ T^{-\alpha\gamma}\ \big[r(1+\rho_m)+k_3\big]+C_p\ (Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)},\ \ \forall\ \ t\in(t_m,T]$$

Thus, for all $t\in[0,T]$ and by (H_4), we have

$$\begin{aligned} ||\Gamma_1 x(t)+\Gamma_2 y(t)|| &\leq \max_{1\leq i\leq m}\Big\{C_s\ T^{-\alpha\gamma}\ \big[||x_0||+r(1+\rho_i+b)+k_2+k_3\big]+C_p\ (Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}\Big\} \\ &\leq r \end{aligned}$$

which means that $\Gamma_1 x+\Gamma_2 y\in B_r$ for any $x,y\in B_r$.

Step 2: Γ_1 is contraction on B_r.
Let $x,y\in B_r$. By (H_2) and (H_3) , for each $t\in[0,t_1]$,

$$\begin{aligned} ||\Gamma_1 x(t)-\Gamma_1 y(t)|| &\leq ||S_\alpha(t)||\ ||g(x)-g(y)|| \\ &\leq C_s t^{-\alpha\gamma}\ b\ ||x-y|| \\ &\leq b\ C_s T^{-\alpha\gamma}\ ||x-y|| \end{aligned}$$

For $t\in(t_1,t_2]$,

$$\begin{aligned} ||\Gamma_1 x(t)-\Gamma_1 y(t)|| &\leq ||S_\alpha(t-t_1)||\big[||x(t_1^-)-y(t_1^-)||+||I_1(x(t_1^-))-I_1(y(t_1^-))||\big] \\ &\leq C_s T^{-\alpha\gamma}\ [1+\rho_1]||x-y|| \end{aligned}$$

Similarly, for all $t\in(t_i,t_{i+1}]$,

$$||\Gamma_1 x(t)-\Gamma_1 y(t)|| \leq C_s\ T^{-\alpha\gamma}(\rho_i+1)\ ||x-y||$$

and therefore for all $t\in(t_m,T]$,

$$||\Gamma_1 x(t)-\Gamma_1 y(t)|| \leq C_s\ T^{-\alpha\gamma}(\rho_m+1)\ ||x-y||$$

Thus, for all $t\in[0,T]$,

$$\begin{aligned} ||\Gamma_1 x(t)-\Gamma_1 y(t)|| &\leq \max_{1\leq i\leq m}\Big\{C_s\ T^{-\alpha\gamma}(b+\rho_i+1)\Big\}\ ||x-y|| \\ &\leq N||x-y|| \end{aligned}$$

Thus, from equation (3.8), Γ_1 is contraction on B_r.
Step 3: Now, we show that Γ_2 is a completely continuous operator.
For that consider, for any $t\in[0,t_1]$, we have

$$\begin{aligned} ||\Gamma_2 x(t)|| &\leq \int_0^t ||P_\alpha(t-s)||\ ||f(s,x(s))||ds \\ &\leq C_p\int_0^t (t-s)^{\alpha(1-\gamma)-1}\ (M||x||+k_1)ds \\ &\leq C_p(Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)} \end{aligned}$$

Similarly, for all $t\in(t_i,t_{i+1}]$,

$$||\Gamma_2 x(t)|| \leq C_p(Mr+k_1)\ \frac{T^{\alpha(1-\gamma)}}{\alpha(1-\gamma)}$$

Thus, from the above inequalities, $\Big\{\Gamma_2 x : x \in B_r\Big\}$ is uniformly bounded for every $t \in [0,T]$.

Next, we will prove that $\Big\{\Gamma_2 x : x \in B_r\Big\}$ is equicontinuous.
Let, $s_1, s_2 \in [0,t_1]$, with $s_1 < s_2$, then $\forall s_1, s_2$, we have

$$\begin{aligned}
||(\Gamma_2 x)(s_2) - (\Gamma_2 x)(s_1)|| &\leq \int_0^{s_2} ||P_\alpha(s_2 - s)||\; ||f(s,x(s))||ds - \int_0^{s_1} ||P_\alpha(s_1 - s)||\; ||f(s,x(s))||ds \\
&\leq C_p\Big[\int_0^{s_1} \big[(s_2 - s)^{\alpha(1-\gamma)-1} - (s_1 - s)^{\alpha(1-\gamma)-1}\big]\; ||f(s,x(s))||ds \\
&\quad + \int_{s_1}^{s_2} (s_2 - s)^{\alpha(1-\gamma)-1}\; ||f(s,x(s))||ds\Big] \\
&\leq \frac{C_p\;(Mr + k_1)}{\alpha(1-\gamma)}\Big[s_2^{\alpha(1-\gamma)} - s_1^{\alpha(1-\gamma)}\Big]
\end{aligned}$$

Similarly, $\forall s_1, s_2 \in (t_i, t_{i+1}]$, with $s_1 < s_2$, $i = 1, 2, ..., m$, we have

$$||(\Gamma_2 x)(s_2) - (\Gamma_2 x)(s_1)|| \leq \frac{C_p\;(Mr + k_1)}{\alpha(1-\gamma)}\Big[(s_2 - t_i)^{\alpha(1-\gamma)} - (s_1 - t_i)^{\alpha(1-\gamma)}\Big]$$

Thus, from the above inequalities, we have $\lim_{s_2 \to s_1} ||(\Gamma_2 x)(s_2) - (\Gamma_2 x)(s_1)|| = 0$. So, Γ_2 is equicontinuous. Moreover, it is clear that from the lemma (2.2), Γ_2 is continuous. So, Γ_2 is a completely continuous operator.

Therefore, Krasnoselskii's fixed point theorem shows that $\Gamma = \Gamma_1 + \Gamma_2$ has a fixed point on B_r and hence the system (1.4) has a solution on I. □

4 Example

Let $\hat{A} = (-i\Delta + \sigma)^{\frac{1}{2}}$, $D(\hat{A}) = W^{1,3}(\mathbb{R}^2)$ (a sobolev space)
be as in example 6.3([31]), in which the authors demonstrate that $\hat{A}$ is an almost sectorial operator for some $0 < w < \frac{\pi}{2}$ and $\gamma = \frac{1}{6}$. We denote the semigroup associated with $\hat{A}$ by T(t) and $||T(t)|| \leq C_0 t^{-\frac{1}{6}}$, where C_0 is a constant.
Let $\mathbb{X} = L^3(\mathbb{R}^2)$, we consider the following problem.

$$\begin{cases}
{}^cD^{\frac{1}{2}}x(t) = \hat{A}x(t) + \frac{\cos t}{(t+6)^2}\;\frac{|x(t)|}{1+|x(t)|}, & t \in I = [0,1], \quad t \neq \frac{1}{2} \\
\Delta x(\frac{1}{2}) = \frac{|x(\frac{1}{2}^-)|}{15+|x(\frac{1}{2}^-)|}, & t = \frac{1}{2} \\
x(0) + \frac{1}{2}x(\frac{1}{5}) = x(1)
\end{cases}$$

where

$$\begin{aligned}
f(t,x(t)) &= \frac{\cos t}{(t+6)^2}\;\frac{|x(t)|}{1+|x(t)|}, \\
I_1(x) &= \frac{|x|}{15+|x|}, \\
g(x) &= \frac{1}{2}x(\frac{1}{5}).
\end{aligned}$$

By direct computations, we see that

$$\begin{aligned}
\|f(t,x(t)) - f(t,y(t))\| &= \left|\frac{\cos t}{(t+6)^2}\right|\;\left\|\frac{|x(t)|}{1+|x(t)|} - \frac{|y(t)|}{1+|y(t)|}\right\| \leq \frac{1}{36}\|x(t) - y(t)\| \\
\|I_1(x) - I_1(y)\| &\leq \frac{1}{15}\|x - y\| \\
\|g(x) - g(y)\| &\leq \frac{1}{2}\|x - y\|
\end{aligned}$$

So,it is clear that the functions f, g and I_k satisfy the assumptions (H_1), (H_2) and (H_3) with M $= \frac{1}{36}$ b $= \frac{1}{2}$, and $\rho_1 = \frac{1}{15}$. Then, choosing for instance $\alpha = \frac{1}{2}$ and T = 1, we have from the equation (3.8),

$$N = C_s\big[\frac{7}{4} + \frac{1}{15}\big] + C_p\frac{1}{36}\frac{12}{5} < 1$$

for the suitable values of the constants C_s and C_p. Moreover the assumption (H_4) is also satisfied. Thus, all the assumptions of Theorem [3.2] and Theorem [3.3] are satisfied and hence by the conclusion of the Theorems [3.2] and [3.3], the nonlocal impulsive fractional problem (1.4) has a unique solution on [0,1].

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