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Global stability for reaction-diffusion SIR model with general incidence function

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Abstract. The aim of this work is to study the dynamics of a reaction-diffusion SIR epidemic model with a nonlinear general incidence function. The local stability of the disease-free equilibrium is obtained via characteristic equations. The global existence, positivity and boundedness of solutions for reaction-diffusion system with homogeneous Neumann boundary conditions are proved. We mainly use the technique of Lyapunov functional to establish the global stability of both disease-free and endemic equilibria. Numerical simulations are presented to illustrate our theoretical results by using a suitable discretization of the model.

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Contents

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1. Introduction

Reaction-diffusion equations model a variety of physical and biological phenomena. These equations describe how the concentration or density distributed in space varies under the influence of two processes: local interactions of species, and diffusion which causes the spread of species in space. In population dynamics, diffusion terms correspond to a random motion of individuals and reaction terms describe their reproduction.
Recently, reaction-diffusion equations have been used by many authors in epidemiology as well as virology. Wang and Wang [11] proposed a mathematical model to simulate the hepatitis B virus (HBV) infection with spatial dependance. They introduced the random mobility of viruses into the basic model proposed by Nowak et al [7] and they assume that the motion of virus follows a Fickian diffusion, that is to say, the population flux of the virus is proportional to the concentration gradient and the proportionality constantis taken to be negative. They also neglected the mobility of susceptible cells and infected cells. Wang et al [12] introduced into [11] an intracellular time delay between the infection of a cell and the production of new virus particles. They considered the initial conditions in a one-dimensional interval with Neumann boundary conditions. The authors neglected the diffusion by assuming that the space is homogeneous in order to establish the global stability of equilibrium solutions. When the space is inhomogeneous, the effects of diffusion and intracellular time delay are obtained by computer simulations. Xu and Ma [14] introduced the saturation response response to the model [12], and obtained sufficient conditions for the global stability of the infected steady rate. In [13], Shaoli et al proposed a diffused HBV model with CTL immune response and nonlinear incidence for the control of viral infections. They Showed that the free diffusion of the virus has no effect on the global stability of such HBV infection problem with Neumann homogeneous boundary conditions. In their work, Yang et al [16] considered the SIR epidemic model with time delay, nonlinear incidence rate was also presented and studied by Xu and Ma [15]. They introduced spacial diffusion in these models and assumed that the three diffusion coefficients are equal in order to prove the existence of traveling waves solutions for the models. They discussed the local stability of a disease-free steady state and an endemic steady state to these models under homogeneous Neumann boundary conditions. Hattal et al [6] take account of the term $e^{-\mu t}$ the probability of surviving from $t-\tau$ to time t in their diffusion model.
Our work is derived from the following SIR epidemic model with a general incidence function described by

$$\begin{cases} \dot{S} = B - \mu_1 S - f(S,I), \\ \dot{I} = f(S,I) - (\mu_2+\gamma)I, \\ \dot{R} = \gamma I - \mu_3 R, \end{cases} \tag{1.1}$$

where S, I and R are susceptible, infectious, and recovered classes, respectively. B is the recruitment rate of the population, μ_1 is the natural death rate of the population, μ_2 is the death rate due to disease, γ is the recovery rate of the infective individuals. $f(S,I)$ is the rate of transmission.

On the other hand, the spatial content of the environment has been ignored in the model (1.1). However, due to the large mobility of people within a country or even worldwide, spatially uniform models are not sufficient to give a realistic picture of disease diffusion. For this reason, the spatial effects cannot be neglected in studying the spread of epidemics.
This paper is organized as follows. The global existence, positivity and boundedness of solutions is described in Section 2. In Section 3, we give the Qualitative analysis of the spatial model (2.2) in which we determine the local and the global stability of the models. In addition, we present the numerical simulation to illustrate our results in Section 4. Finally, the conclusion of this paper is given in Section 5.

2. Presentation of the model

We consider the following SIR epidemic model with general incidence function and spatial diffusion:

$$\begin{cases} \dfrac{\partial S}{\partial t}(x,t) = d_S \Delta S(x,t) + B - \mu_1 S(x,t) - f(S(x,t), I(x,t)), \\ \dfrac{\partial I}{\partial t}(x,t) = d_I \Delta I(x,t) + f(S(x,t), I(x,t)) - (\mu_2 + \gamma) I(x,t), \\ \dfrac{\partial R}{\partial t}(x,t) = d_R \Delta R(x,t) + \gamma I(x,t) - \mu_3 R(x,t), \end{cases} \tag{2.1}$$

where $S(x,t)$, $I(x,t)$, and $R(x,t)$ represent the numbers of susceptible, infected, and removed individuals at location x and time t, respectively. The positive constants d_S, d_I, and d_R denote the corresponding diffusion rates for these three classes of individuals.
The aim of this work is to investigate the global dynamics of the reaction-diffusion system (2.1). Note that R does not appear in the first two equations; this allows us to study the system

$$\begin{cases} \dfrac{\partial S}{\partial t}(x,t) = d_S \Delta S(x,t) + B - \mu_1 S(x,t) - f(S(x,t), I(x,t)), \\ \dfrac{\partial I}{\partial t}(x,t) = d_I \Delta I(x,t) + f(S(x,t), I(x,t)) - (\mu_2 + \gamma) I(x,t), \end{cases} \tag{2.2}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, \quad \text{on} \quad \partial\Omega \times (0, +\infty), \tag{2.3}$$

and initial conditions

$$S(x,0) = \psi_1(x) \geq 0, \;\; I(x,0) = \psi_2(x) \geq 0, \quad x \in \bar{\Omega}. \tag{2.4}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. $\dfrac{\partial S}{\partial \nu}$ and $\dfrac{\partial S}{\partial \nu}$ are , respectively, the normal derivatives of S and I on $\partial\Omega$.

Remark 2.1. *The Neumann condition is used to ensure the mobility of people in bounded domain Ω.*

Let us put

$$f(S,I) = g(S,I)I$$

The incidence function $f(S,I)$ is assumed to be continuously differentiable in the interior of $\mathbb{R}^2_+$ and satisfies the following hypotheses:

H1: $f(0,I) = f(S,0) = 0$ for all $S \geq 0 \;\; I \geq 0$.

H2: $\dfrac{\partial f}{\partial S}(S,I) > 0$ for all $S > 0$ and $I > 0$.

H3: $\dfrac{\partial f}{\partial I}(S,I) > 0$ for all $S > 0$ and $I > 0$.

H4: $f(S,I) \geq f_2(S^0, 0)I$ for all $S > 0$ and $I > 0$.

Let us denote by f_1 and f_2 the partial derivatives of f with respect to the first and to the second variable.

3. Global existence, positivity and boundedness of solutions

n this section, we establish the global existence, positivity, and boundedness of solutions of problem (2.2)-(2.4). Hence, the population should remain nonnegative and bounded.

Proposition 3.1. *For any given data satisfying the condition (2.4), there exists a unique solution of problem (2.2)-(2.4) defined on* $[0, +\infty[$ *and this solution remains nonnegative and bounded for all* $t \geq 0$.

Proof. The system (2.2)-(2.4) can be written abstractly in the Banach space $X = C(\bar{\Omega}) \times C(\Omega)$ of the form:

$$\begin{aligned} u'(t) &= Au(t) + G(u(t)) \\ u(0) &= u_0 \in X, \end{aligned} \tag{3.1}$$

where $u = \begin{pmatrix} S \\ I \end{pmatrix}$, $u_0 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $Au(t) = \begin{pmatrix} d_S \Delta S \\ d_I \Delta I \end{pmatrix}$ and

$$G(u(t)) = \begin{pmatrix} B - \mu_1 S - f(S,I) \\ f(S,I) - (\mu_2 + \gamma) I \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}. \tag{3.2}$$

G is locally Lipchitz in X. From [8], we deduce that system (3.1) admits a unique local solution on $[0, T_{max}]$, where T_{max} is the maximal existence time for solution of system (3.1).
In addition, system (2.2) can be written in the form:

$$\begin{aligned} \frac{\partial S}{\partial t} - d_S \Delta S &= G_1(S,I) \\ \frac{\partial I}{\partial t} - d_I \Delta I &= G_2(S,I). \end{aligned} \tag{3.3}$$

The functions $G_1(S,I)$ and $G_2(S,I)$ are continuously differentiable and satisfy $G_1(0,I) = B \geq 0$ and $G_2(S,0) = f(S,0) \geq 0$ for all $S, I \geq 0$. Since initial data of system (2.2) are nonnegatives, we deduce the positivity of the local solution (see [10]).
Now, we show the boundedness of solutions. So from (2.2)-(2.4) we have

$$\begin{aligned} \frac{\partial S}{\partial t} - d_S \Delta S &\leq B - \mu_1 S, \\ \frac{\partial S}{\partial \nu} &= 0, \\ S(x,0) = \psi_1(x) \leq \|\psi_1\|_\infty &= \max_{x \in \bar{\Omega}} \psi_1(x). \end{aligned} \tag{3.4}$$

By the comparison principle [9], we have $S(x,t) \leq S_1(t)$,
where $S_1(t) = \psi_1(x) e^{-\mu_1 t} + \dfrac{B}{\mu_1}(1 - e^{-\mu_1 t})$ is the solution of the problem

$$\begin{aligned} \frac{dS_1}{dt} &= B - \mu_1 S_1, \\ S_1(0) &= \|\psi\|_\infty. \end{aligned} \tag{3.5}$$

Since $S_1(t) \leq \max\{\dfrac{B}{\mu_1}, \|\psi_1\|_\infty\}$ for $t \in [0, \infty)$, we have that

$$S(x,t) \leq \max\{\frac{B}{\mu_1}, \|\psi_1\|_\infty\}, \forall (x,t) \in \bar{\Omega} \times [0, T_{max}). \tag{3.6}$$

From Theorem 2.1 given by Alikakos in [1], to establish the L^∞ uniform boundedness of $I(x,t)$, it is sufficient to show the L^1 uniform boundedness of $I(x,t)$.
Since

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0$$

and

$$\frac{\partial}{\partial t}(S+I) - \Delta(d_S S + d_I I) \leq B - \mu_1(S+I),$$

we get

$$\frac{\partial}{\partial t}\Big(\int_\Omega (S+I)dx\Big) \leq mes(\Omega)B - \mu_1\Big(\int_\Omega (S+I)dx\Big). \tag{3.7}$$

Hence,

$$\int_\Omega (S+I)dx \leq mes(\Omega)\max\{\frac{B}{\mu_1}, \|\psi_1+\psi_2\|_\infty\}, \tag{3.8}$$

which implies that

$$\sup_{t\geq 0}\int_\Omega I(x,t)dx \leq K = mes(\Omega)\max\{\frac{B}{\mu_1}, \|\psi_1+\psi_2\|_\infty\}.$$

Using the Theorem 3.1 of [1], we deduce that there exists a positive constant K^* that depends on K and $\|\psi_1+\psi_2\|_\infty$ such that

$$\sup \|I(.,t)\|_\infty \leq K^*. \tag{3.9}$$

From the above, we have proved that $S(x,t)$ and $I(x,t)$ are L^∞ bounded on $\bar{\Omega}\times[0,T_{\max})$.
Therefore, it follows from the standard theory for semilinear parabolic systems(see [4]) that $T_{\max} = +\infty$. This completes the proof of the proposition. ■

4. Qualitative analysis of the spacial model

Considering (1.1), we get that the basic reproduction number of disease in the absence of spatial dependence is given by

$$R_0 = \frac{f_2(S^0,0)}{\mu_2+\gamma}. \tag{4.1}$$

This describes the average number of secondary infections produced by a single infectious individual during the entire infectious period. It is not hard to show that the system (2.2) is always a disease-free equlibrium of the form $E_0 = (\frac{B}{\mu_1}, 0)$ when $R_0 \leq 1$. Further, if $R_0 > 1$, the system (2.2) has an endemic stationary state $E^* = (S^*, I^*)$.

4.1. Local stability of the equilibria

The purpose of this part is to determine the local stability for reaction-diffusion equations (2.2)-(2.4) by applying the method of Hattaf presented in [5].
First, we linearize the dynamical system (2.2) around arbitrary spatially homogeneous fixed point $\bar{E}(\bar{S},\bar{I})$ for a small space and time dependent fluctuations and expand them in Fourier space. For this, let

$$\begin{aligned} S(\overrightarrow{x},t) &\sim \bar{S}e^{\lambda t}e^{i\overrightarrow{k}.\overrightarrow{x}}, \\ I(\overrightarrow{x},t) &\sim \bar{I}e^{\lambda t}e^{i\overrightarrow{k}.\overrightarrow{x}}, \end{aligned} \tag{4.2}$$

where $\overrightarrow{x} = (x, y) \in \mathbb{R}^2$ and $\overrightarrow{k}.\overrightarrow{k} :=< \overrightarrow{k}, \overrightarrow{k} >:= k^2$. $\overrightarrow{k}$ and λ are the wavenumber vector and frequency, respectively. Then, we can obtain the corresponding characteristic equation as follows:

$$\det(J - k^2 D - \lambda I_2) = 0, \tag{4.3}$$

where I_2 is the identity matrix, $D = \text{diag}(d_S, d_I)$ is the diffusion matrix, and J is the Jacobian matrix of (2.2) without diffusion ($d_S = d_I = 0$) at $\bar{E}$ which is given by

$$J = \begin{pmatrix} -\mu_1 - f_1(\bar{S}, \bar{I}) & -f_2(\bar{S}, \bar{I}) \\ f_1(\bar{S}, \bar{I}) & f_2(\bar{S}, \bar{I}) - (\mu_2 + \gamma) \end{pmatrix} \tag{4.4}$$

The characterization of the local stability of disease-free equilibrium E_0 is given by the following result.

Theorem 4.1. *The disease-free equilibrium E_0 is locally asymptotically stable if $R_0 \leq 1$ and it is unstable if $R_0 > 1$.*

Proof. Evaluating (4.3) at E_0, we have the following equation,

$$\begin{aligned} &(-\mu_1 - f_1(S^0, 0) - k^2 d_S - \lambda)(f_2(S^0, 0) - (\mu_2 + \gamma) - k^2 d_I - \lambda) \\ &+f_1(S^0, 0) f_2(S^0, 0) = 0. \end{aligned} \tag{4.5}$$

By developping (4.5), we get

$$\begin{aligned} &\lambda^2 + (\mu_1 + f_1(S^0, 0) - f_2(S^0, 0) + \mu_2 + \gamma + k^2 d_S + k^2 d_I)\lambda - \mu_1 f_2(S^0, 0) \\ &+\mu_1(\mu_2 + \gamma) + f_1(S^0, 0)(\mu_2 + \gamma) + \mu_1 k^2 d_I + f_1(S^0, 0) k^2 d_I + k^4 d_I d_S \\ &+k^2 d_S(\mu_2 + \gamma - f_2(S^0, 0)) = 0. \end{aligned} \tag{4.6}$$

Since $R_0 \leq 1$, we obtain

$$-f_2(S^0, 0) + \mu_2 + \gamma > 0.$$

Therefore, by the Routh-Hurwitz criterion all the roots of equation (4.5) have a negative real parts. This shows that equilibrium E_0 is locally asymptotically stable. ■

Next, we focus on the local stability of the endemic equilibrium E^*.

Theorem 4.2. *The endemic equilibrium E^* is locally asymptotically stable if $R_0 > 1$.*

Proof. Evaluating (4.3) at $E^*(S^*, I^*)$, we get

$$\begin{aligned} &\lambda^2 + (\mu_1 + f_1(S^*, I^*) - f_2(S^*, I^*) + \mu_2 + \gamma + k^2 d_S + k^2 d_I)\lambda - \mu_1 f_2(S^*, I^*) \\ &+\mu_1(\mu_2 + \gamma) + f_1(S^*, I^*)(\mu_2 + \gamma) + \mu_1 k^2 d_I + f_1(S^*, I^*) k^2 d_I + k^4 d_I d_S \\ &+k^2 d_S(\mu_2 + \gamma - f_2(S^*, I^*)) = 0. \end{aligned} \tag{4.7}$$

By using **H3**, we conclude that E^* is locally asymptotically stable. ■

4.2. Global stability of the equilibria

The purpose of this subsection is to determine the global stability for reaction-diffusion equations (2.2)-(2.4) by constructing Lyapunov functionals proposed in [2] and applying the method of Hattaf presented in [3].

Theorem 4.3. *If $R_0 \leq 1$, the disease-free equilibrium E_0 of (2.2)-(2.4) is globally asymptotically stable for all diffusion coefficients.*

We define

$$g(S,I)=\frac{f(S,I)}{I}.$$

Proof. Consider the following Lyapunov functional

$$V_1(t)=S(t)-S^0-\int_{S^0}^{S(t)}\frac{g(S^0,0)}{g(X,0)}dX+I,$$

where $S^0=\dfrac{B}{\mu_1}$. Calculating the time derivative of V along the positive solution of system (1.1), we get

$$\begin{aligned}\dot{V}_1(t)&=(1-\frac{g(S^0,0)}{g(S,0)})\dot{S}+\dot{I}\\&=(1-\frac{g(S^0,0)}{g(S,0)})(B-\mu_1 S)+\frac{g(S^0,0)}{g(S,0)})f(S,I)-(\mu_2+\gamma)I\\&=\mu_1 S^0(1-\frac{g(S^0,0)}{g(S,0)})(1-\frac{S}{S^0})+(\mu_2+\gamma)I\bigg(\frac{g(S^0,0)}{g(S,0)}g(S,I)\frac{1}{f_2(S^0,0)}R_0-1\bigg).\end{aligned}$$

By using $g(S,I)=\dfrac{f(S,I)}{I}$ which implies $g(S^0,0)=f_2(S^0,0)$, so we have

$$\dot{V}_1(t)\leq\mu_1 S^0(1-\frac{g(S^0,0)}{g(S,0)})(1-\frac{S}{S^0})+(\mu_2+\gamma)I\bigg(\frac{g(S,I)}{g(S,0)}R_0-1\bigg).$$

By **H4**, we get

$$\dot{V}_1(t)\leq\mu_1 S^0(1-\frac{g(S^0,0)}{g(S,0)})(1-\frac{S}{S^0})+(\mu_2+\gamma)I(R_0-1).$$

By using **H2** we obtain the following inequalities

$$1-\frac{g(S^0,0)}{g(S,0)}\geq 0\quad\text{for}\quad S\geq S^0,$$

$$1-\frac{g(S^0,0)}{g(S,0)}<0\quad\text{for}\quad S<S^0.$$

Thus, we have

$$(1-\frac{S}{S^0})(1-\frac{g(S^0,0)}{g(S,0)})\leq 0,$$

then,

$$\frac{dV_1}{dt}\leq 0$$

Now, From [3], we construct the Lyapunov functional for system (2.2) at E_0 as follows

$$W_1=\int_\Omega V_1(S(x,t),I(x,t))dx$$

Calculating the time derivative of W_1 along the solution of system (2.2)-(2.4), we have

$$\begin{aligned}\frac{dW_1}{dt} &= \int_\Omega \left\{ \mu_1 S^0 (1 - \frac{g(S^0,0)}{g(S,0)})(1 - \frac{S}{S^0}) \right. \\ &\left. + (\mu_2+\gamma) I \left(\frac{g(S^0,0)}{g(S,0)} \frac{f(S,I)}{I} \frac{1}{f_2(S^0,0)} R_0 - 1 \right) \right\} dx \\ &- d_S g(S^0,0) \int_\Omega \frac{g_1(S,0)}{(g(S,0))^2} \mid \nabla S \mid^2 dx \\ &\leq \int_\Omega \left\{ \mu_1 S^0 (1 - \frac{g(S^0,0)}{g(S,0)})(1 - \frac{S}{S^0}) + (\mu_2+\gamma) I (R_0 - 1) \right\} dx \\ &- d_S g(S^0,0) \int_\Omega \frac{g_1(S,0)}{(g(S,0))^2} \mid \nabla S \mid^2 dx.\end{aligned} \tag{4.8}$$

Since $R_0 \leq 1$, we have

$$\frac{dW_1}{dt} \leq 0.$$

Thus, the disease-free equilibrium E_0 is stable, and

$$\frac{dW_1}{dt} = 0,$$

if and only if $S = S_0$ and $I(R_0 - 1) = 0$. ■

Theorem 4.4. *If $R_0 > 1$, the endemic equilibrium E^* of (2.2)-(2.4) is globally asymptotically stable for all diffusion coefficients.*

Proof. Consider the following Lyapunov functional

$$V_2(t) = S(t) - S^* - \int_{S^*}^{S(t)} \frac{g(S^*,I^*)}{g(X,I^*)} dX + I^* \Phi(\frac{I(t)}{I^*}),$$

where $\Phi(x) = x - 1 - \ln x$, $x \in \mathbb{R}_+^*$. Obviously, $\Phi : \mathbb{R}_+^* \longrightarrow \mathbb{R}^+$ attains its global minimum at $x = 1$ and $\Phi(1) = 0$.

The function $\psi : x \longmapsto x - x^* - \int_{x^*}^{x} \frac{g(x^*,I^*)}{g(X,I^*)} dX$ has the global minimum at $x = x^*$ and $\psi(x^*) = 0$. Then, $\psi(x) \geq 0$ for any $x > 0$.

Hence, $V_2(t) \geq 0$ with equality holding if and only if $\frac{S(t)}{S^*} = \frac{I(t)}{I^*} = 1$ for all $t \geq 0$.

Finding the time derivative of $V_2(t)$ along the positive of system (2.1) gives

$$\begin{aligned}\frac{dV_2}{dt} &= (1 - \frac{g(S^*,I^*)}{g(S,I^*)})\dot{S} + (1 - \frac{I^*}{I})\dot{I} \\ &= (1 - \frac{g(S^*,I^*)}{g(S,I^*)})(B - \mu_1 S - f(S,I)) + (1 - \frac{I^*}{I})(f(S,I) - (\mu_2+\gamma)I)\end{aligned}$$

Note that $B = \mu_1 S^* + f(S^*,I^*)$ and $f(S^*,I^*) = (\mu_2+\gamma)I^*$.

Hence;

$$\begin{aligned}\frac{dV_2}{dt} &= \mu_1 S^*(1-\frac{g(S^*,I^*)}{g(S,I^*)})(1-\frac{S}{S^*})\\ &+ f(S^*,I^*)\Big[(1-\frac{f(S^*,I^*)}{f(S,I^*)})((1-\frac{f(S,I)}{f(S^*,I^*)})+(1-\frac{I^*}{I})(\frac{f(S,I)}{f(S^*,I^*)}-\frac{I}{I^*})\Big]\\ &= \mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)})+f(S^*,I^*)\Big[2-\frac{g(S^*,I^*)}{g(S,I^*)}+\frac{I}{I^*}\frac{g(S,I)}{g(S,I^*)}\\ &- \frac{I}{I^*}-\frac{g(S,I)}{g(S^*,I^*)}\Big]\\ &= \mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)})+f(S^*,I^*)\Big[3-\frac{g(S^*,I^*)}{g(S,I^*)}-\frac{g(S,I)}{g(S^*,I^*)}\\ &- \frac{g(S,I^*)}{g(S,I)}+\Big(\frac{I}{I^*}\times\frac{g(S,I)}{g(S,I^*)}-\frac{I}{I^*}-1+\frac{g(S,I^*)}{g(S,I)}\Big)\Big]\end{aligned}$$

By adding and substracting $\ln\dfrac{g(S^*,I^*)}{g(S,I^*)}+\ln\dfrac{g(S,I)}{g(S^*,I^*)}+\ln\dfrac{g(S,I^*)}{g(S,I)}$,
we get,

$$\begin{aligned}\frac{dV_2}{dt} &= \mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)})\\ &+ f(S^*,I^*)\Big[-\Phi\Big(\frac{g(S^*,I^*)}{g(S,I^*)}\Big)-\Phi\Big(\frac{g(S,I)}{g(S^*,I^*)}\Big)-\Phi\Big(\frac{g(S,I^*)}{g(S,I)}\Big)\\ &+ \Big(\frac{I}{I^*}\times\frac{g(S,I)}{g(S,I^*)}-\frac{I}{I^*}-1+\frac{g(S,I^*)}{g(S,I)}\Big)\Big]\\ &= \mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)})\\ &+ f(S^*,I^*)\Big[-\Phi\Big(\frac{g(S^*,I^*)}{g(S,I^*)}\Big)-\Phi\Big(\frac{g(S,I)}{g(S^*,I^*)}\Big)-\Phi\Big(\frac{g(S,I^*)}{g(S,I)}\Big)\\ &+ \frac{1}{I^*g(S,I^*)g(S,I)}\Big(g(S,I)-g(S,I^*)\Big)\Big(g(S,I)I-g(S,I^*)I^*\Big)\Big]\end{aligned}$$

By using **H2**, we have the following trivial inequalities

$$1-\frac{g(S^*,I^*)}{g(S,I^*)}\geq 0 \quad \text{for} \quad S\geq S^*,$$

$$1-\frac{g(S^*,I^*)}{g(S,I^*)}< 0 \quad \text{for} \quad S< S^*.$$

Thus, we have

$$(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)})\leq 0$$

By using **H4**, we have $g(S,I)$ is monotonically decreasing for I and by **H3** $g(S,I)I$ is monotocally increasing for I, and so

$$\Big(g(S,I)-g(S,I^*)\Big)\Big(g(S,I)I-g(S,I^*)I^*\Big)\leq 0.$$

On the other hand, the function Φ is always positive.

Now, we construct the Lyapunov functional for system (2.2) at E^* as follows

$$W_2 = \int_\Omega V_2(S(x,t), I(x,t))dx$$

Calculating the time derivative of W_2 along the solution of system (2.2)-(2.4), we have

$$\begin{aligned}\frac{dW_2}{dt} = \int_\Omega \Big\{ & \mu_1 S^*(1-\frac{S}{S^*})(1-\frac{g(S^*,I^*)}{g(S,I^*)}) \\ & + f(S^*,I^*)\Big[-\Phi\Big(\frac{g(S^*,I^*)}{g(S,I^*)}\Big) - \Phi\Big(\frac{g(S,I)}{g(S^*,I^*)}\Big) - \Phi\Big(\frac{g(S,I^*)}{g(S,I)}\Big) \\ & + \frac{1}{I^* g(S,I^*) g(S,I)}\Big(g(S,I)-g(S,I^*)\Big)\Big(g(S,I)I - g(S,I^*)I^*\Big)\Big]\Big\}dx \\ & - d_S g(S^*,I^*)\int_\Omega \frac{g_1(S,I^*)}{(g(S,I^*))^2}|\nabla S|^2 dx\end{aligned}$$

Since the function Φ is monotone on each side of 1 and is minimized at 1 and
$\Big(g(S,I)-g(S,I^*)\Big)\Big(g(S,I)I - g(S,I^*)I^*\Big) \leq 0,$
then

$$\frac{dW_2}{dt} \leq 0.$$

Thus, the endemic equilibrium E^* is globally asymptotically stable. ■

5. Numerical simulations

In this section, we present the numerical simulations to illustrate our theoretical results. To simplify, we consider system (2.2) under Neumann boundary conditions.

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, \quad t > 0, \quad x = 0, 1, \tag{5.1}$$

and initial conditions

$$S(x,0) = \begin{cases} 1.1x, & 0 \leq x < 0.5, \\ \\ 1.1(1-x), 0,5 \leq x \leq 1, \end{cases} \qquad I(x,0) = \begin{cases} 0.5x, & 0 \leq x < 0.5, \\ \\ 0.5(1-x), 0,5 \leq x \leq 1, \end{cases} \tag{5.2}$$

We choose the following data set of system (2.2): $d_S = 0.1$, $\gamma = 0.5$, $\mu_1 = 0.1$, $\beta = 0.2$, $d_I = 0.1$, $\mu_2 = 0.6$, $B = 0.5$. By calculation, we have $R_0 < 1$. In this case, system (2.2) has a disease-free equilibrium E_0. Hence, by Theorem 3.3, E_0 is globally asymptotically stable. Numerical simulation illustrates our result (see Figure 1).
In Figure 2, we choose $\beta = 0.8$ and do not change the other parameters values. By calculation, we have $R_0 > 1$ which satisfy Theorem 3.4; the system (2.2) has a unique endemic equilibrium E^*. Therefore, by Theorem 3.3, E^* is globally assymtotically stable. Numerical simulation illustrates well this result (see Figure 2).

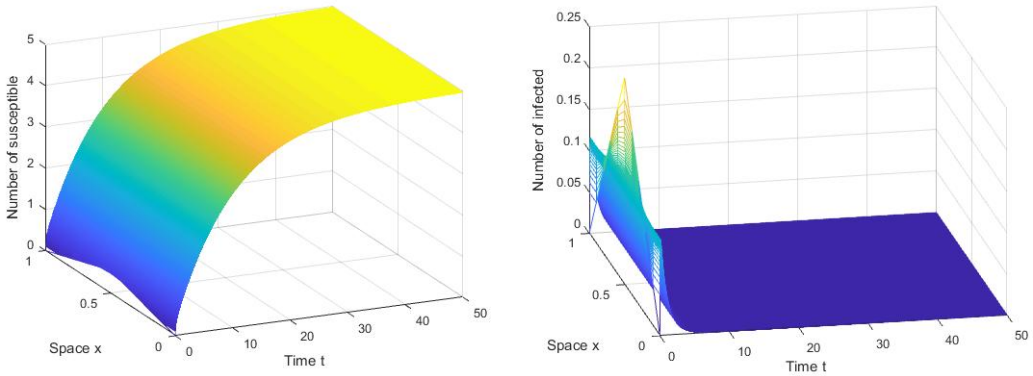

Figure 1: The temporal solution found by numerical of problem (2.2) with the Neumann boundary conditions (5.1) and initial conditions (5.2) when $R_0 \leq 1$.

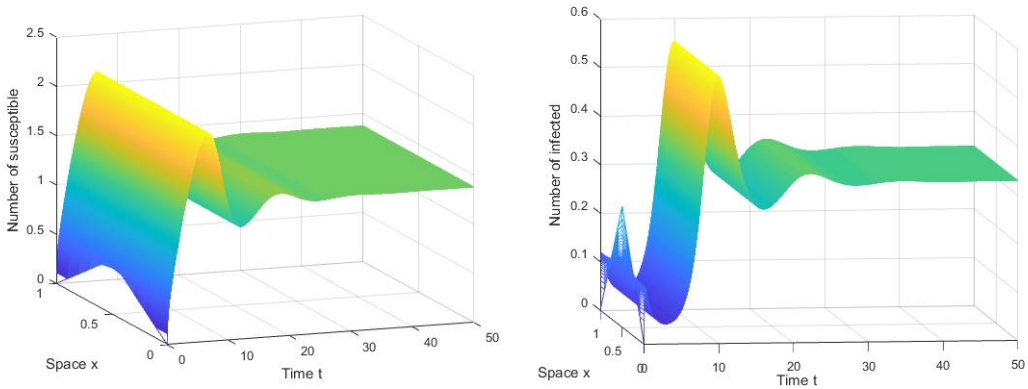

Figure 2: The temporal solution found by numerical of problem (2.2) with the Neumann boundary conditions (5.1) and initial conditions (5.2) when $R_0 > 1$.

6. Conclusion

In this paper, we investigated the dynamics of a reaction-diffusion epidemic model with general incidence function. The global dynamics of the model are completely determined by the basic reproduction number R_0. We proved that the disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$, which leads to the eradication of disease from population. When $R_0 > 1$ then disease-free equilibrium becomes unstable and a unique endemic equilibrium exists and is globally asymptotically stable, which means that the disease persists in the population.

From our theoretical and numerical results, we conclude that the spatial diffusion has no effect on the stability behavior of equilibria in the case of Neumann conditions and spatially constant coefficients.

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References

[1] N. D. ALIKAKOS, An application of the invariance principle to reaction-diffusion equations, *Journal of Differential Equations,* vol. **33**(2)(1979), 201-225.

[2] K. HATTAF, A.A. LASHARI, Y. LOUARTASSI, AND N. YOUSFI, A delayed SIR epidemic model with general incidence rate, *Electronic Journal of Qualitative Theory of Differential Equations,* **3**(2013), 1-9.

[3] K. HATTAF AND N. YOUSFI, Global stability for reaction-diffusion equations in biology, *Computers and Mathematics with applications*, **66**(2013), 1488-1497.

[4] D. HENRY, Geometric Theory of Semilinear Parabolic Equations ,vol. **840** of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1993.

[5] EL MEHDI LOTFI, MEHDI MAZIANE, KHALID HATTAF, AND NOURA YOUSFI, Partial Differential Equations of an Epidemic Model with Spatial Diffusion , *Int. Journal of Partial Differential Equations,* 2014.

[6] EL MEHDI LOTFI, MEHDI MAZIANE, MAROUANE MAHROUF, KHALID HATTAF, AND NOURA YOUSFI, Global stability of a diffused SIR epidemic model with general incidence rate and time delay, *Int. Journal of Mathematical analysis,* **10**(17)(2016), 807-816.

[7] M.A. NOWAK, S. BONHOEFFER, A.M.HILL, R. BOEHME, H.C. THOMAS, H.C. THOMAS, H. MCDADE, Viral dynamics in hepatitis B virus infection, *Proc. Natl. Acad. Sci. USA* **93**(1996), 4398-4402.

[8] A. PAZY, Semigroups of Linear Operators and Applications to Partial Differentiel Equations,Vol. **44** of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.

[9] M. H. PROTTER AND H. F. WEINBERGER, Maximum Principles in Differential Equations, *Prentice Hall, Englewood Cliffs, NJ, USA,* 1967.

[10] J. SMOLLER, Shock Waves and Reaction-Diffusion Equations, Springer, *Berlin, Germany,* 1983.

[11] K. WANG, W. WANG, Propagation of HBV with spatial dependance, *Math. Biosci.,* **210**(2007), 78-95.

[12] K. WANG, W. WANG, S. SONG Dynamics of an HBV model with diffusion and delay, *J. Theoret. Biol.,* **253**(2008), 36-44.

[13] S.L. WANG, X.L. FENG, Y.N. HE, Global asymptotical properties for a diffused HBV infection model with CTL immune response and nonlinear incidence, *Acta Math. Sci.,* **5**(2011), 1959-1967.

[14] R. XU, Z.E. MA, An HBV model with diffusion and time delay, *J. Theoret. Biol*., **257**(2009), 499-509.

[15] R. XU, Z.E. MA, Global stability of a SIR epidemic model with nonlinear incdence rate and time delay, *Nonlinear Anal. RWA.,* **10**(2009), 3175-3189.

[16] J. YANG, S. LIANG, Y. ZHANG, Travelling waves of a delayed SIR epidemic model with nonlinear incidence rate and spatial diffusion, *PLoS ONE*, **6**(2011), e21128.