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Existence of positive periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale

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Abstract

Let $\mathbb{T}$ be a periodic time scale. The purpose of this paper is to use Krasnoselskii's fixed point theorem to prove the existence of positive periodic solutions on time scale of the nonlinear neutral dynamic equation with variable delay

$$(x(t) - g(t, x(t-\tau(t))))^{\triangle} = r(t)x(t) - f(t, x(t-\tau(t))).$$

We invert this equation to construct a sum of a contraction and a compact map which is suitable for applying the Krasnoselskii's theorem. The results obtained here extend the works of Raffoul [17] and Ardjouni and Djoudi [3].

Keywords: Positive periodic solutions, nonlinear neutral dynamic equations, fixed point theorem, time scales.

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1 Introduction

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of dynamic equations. Motivated by the papers [1]–[6], [9]–[17] and the references therein, we consider the following nonlinear neutral dynamic equation with variable delay

$$(x(t) - g(t, x(t-\tau(t))))^{\triangle} = r(t)x(t) - f(t, x(t-\tau(t))). \tag{1.1}$$

Throughout this paper we assume that $\tau : \mathbb{T} \to \mathbb{R}$ and that $id - \tau : \mathbb{T} \to \mathbb{T}$ is strictly increasing so that the function $x(t-\tau(t))$ is well defined over $\mathbb{T}$. Our purpose here is to use the Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions on time scales for equation (1.1). To reach our desired end we have to transform (1.1) into an integral equation written as a sum of two mapping; one is a contraction and the other is compact. After that, we use Krasnoselskii's fixed point theorem, to show the existence of a positive periodic solution for equation (1.1). In the special case $\mathbb{T} = \mathbb{R}$, in [3] we show that (1.1) has a positive periodic solution by using Krasnoselskii's fixed point theorem.

The organization of this paper is as follows. In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [18]. In Section 3, we present our main results on existence of positive periodic solutions of (1.1). The results presented in this paper extend the main results in [3, 17].

2 Preliminaries

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1], [2], [4]–[8], [14], [15] and

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papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [7] and [8] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Atici et al. [5] and Kaufmann and Raffoul [14]. The following two definitions are borrowed from [5] and [14].

Definition 2.1. *We say that a time scale $\mathbb{T}$ is periodic if there exist a $\omega > 0$ such that if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive ω is called the period of the time scale.*

Below are examples of periodic time scales taken from [14].

Example 2.1. *The following time scales are periodic.*
(1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih]$, $h > 0$ *has period* $\omega = 2h$.
(2) $\mathbb{T} = h\mathbb{Z}$ *has period* $\omega = h$.
(3) $\mathbb{T} = \mathbb{R}$.
(4) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ *where,* $0 < q < 1$ *has period* $\omega = 1$.

Remark 2.1 ([14]). *All periodic time scales are unbounded above and below.*

Definition 2.2. *Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scales with the period ω. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = n\omega$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.*

Remark 2.2 ([14]). *If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function μ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period ω.*

Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales ([7], Theorem 1.93).

Theorem 2.1 (Chain Rule). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our second theorem is the substitution rule ([7], Theorem 1.98).

Theorem 2.2 (Substitution). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t) \nu^{\triangle}(t) \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s) \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0$ for all $t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} Log(1 + \mu(z) p(z)) \triangle z\right). \tag{2.2}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t) y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.1 ([7]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t,s)$;

(iii) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1+\mu(t)p(t)}$;

(iv) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s) e_p(s,r) = e_p(t,r)$;

(vi) $e_p^{\triangle}(.,s) = p e_p(.,s)$ *and* $\left(\dfrac{1}{e_p(.,s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(.,s)}$.

Theorem 2.3 ([6], Theorem 2.1). *Let $\mathbb{T}$ be a periodic time scale with period $\omega > 0$. If $p \in C_{rd}(\mathbb{T})$ is a periodic function with the period $T = n\omega$, then*

$$\int_{a+T}^{b+T} p(u) \triangle u = \int_a^b p(u) \triangle u, \;\; e_p(b+T, a+T) = e_p(b,a) \;\; \text{if } p \in \mathcal{R},$$

and $e_p(t+T, t)$ is independent of $t \in \mathbb{T}$ whenever $p \in \mathcal{R}$.

Lemma 2.2 ([1]). *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u) \triangle u\right), \;\; \forall t \in \mathbb{T}.$$

Corollary 2.1 ([1]). *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u) \triangle u\right) < 1.$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1.1). For its proof we refer the reader to [18].

Theorem 2.4 (Krasnoselskii). *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{D}$, *implies* $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,

(ii) *$\mathcal{A}$ is compact and continuous,*

(iii) *$\mathcal{B}$ is a contraction mapping.*

Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3 Existence of positive periodic solutions

We will state and prove our main result in this section. After we provide an example to illustrate our results. Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and (a,b) are defined similarly.

Define $P_T = \{\varphi : \mathbb{T} \to \mathbb{R} \mid \varphi \in C \text{ and } \varphi(t+T) = \varphi(t)\}$ where C is the space of continuous real-valued functions on $\mathbb{T}$. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|\varphi\| = \sup_{t \in \mathbb{T}} |\varphi(t)| = \sup_{t \in [0,T]} |\varphi(t)|.$$

We will need the following lemma whose proof can be found in [14].

Lemma 3.3. *Let $x \in C_T$. Then $\|x^{\sigma}\| = \|x \circ \sigma\|$ exists and $\|x^{\sigma}\| = \|x\|$.*

In this paper we assume that $r \in \mathcal{R}^+$ is continuous and for all $t \in \mathbb{T}$,

$$r(t+T) = r(t), \;\; (id - \tau)(t+T) = (id - \tau)(t), \tag{3.3}$$

where id is the identity function on $\mathbb{T}$. Also, we assume

$$\int_0^T r(s) \triangle s > 0. \tag{3.4}$$

We also assume that the functions $g(t,x)$ and $f(t,x)$ are continuous in their respective arguments and periodic in t with period T, that is,

$$g(t+T,x) = g(t,x), \ f(t+T,x) = f(t,x). \tag{3.5}$$

The following lemma is fundamental to our results.

Lemma 3.4. *Suppose (3.3)–(3.5) hold. If $x \in P_T$, then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) &= g(t, x(t-\tau(t))) \\ &\quad + \int_t^{t+T} G(t,s) \left[f(s, x(s-\tau(s))) - r(s) g(s, x(s-\tau(s))) \right] \triangle s, \end{aligned} \tag{3.6}$$

where

$$G(t,s) = \frac{e_r(t, \sigma(s))}{1 - e_{\ominus r}(t+T,t)}. \tag{3.7}$$

Proof. Let $x \in P_T$ be a solution of (1.1). First we write this equation as

$$\begin{aligned} &(x(t) - g(t, x(t-\tau(t))))^{\triangle} - r(t)(x(t) - g(t, x(t-\tau(t)))) \\ &= -f(t, x(t-\tau(t))) + r(t) g(t, x(t-\tau(t))). \end{aligned}$$

Multiply both sides of the above equation by $e_{\ominus r}(\sigma(t), 0)$ we get

$$\begin{aligned} &\left\{ (x(t) - g(t, x(t-\tau(t))))^{\triangle} - r(t)(x(t) - g(t, x(t-\tau(t)))) \right\} e_{\ominus r}(\sigma(t), 0) \\ &= \{-f(t, x(t-\tau(t))) + r(t) g(t, x(t-\tau(t)))\} e_{\ominus r}(\sigma(t), 0). \end{aligned}$$

Since $e_{\ominus r}(t,0)^{\triangle} = -r(t) e_{\ominus r}(\sigma(t), 0)$ we find

$$\begin{aligned} &[(x(t) - g(t, x(t-\tau(t)))) e_{\ominus r}(t,0)]^{\triangle} \\ &= \{-f(t, x(t-\tau(t))) + r(t) g(t, x(t-\tau(t)))\} e_{\ominus r}(\sigma(t), 0). \end{aligned}$$

Taking the integral from t to $t+T$, we obtain

$$\begin{aligned} &\int_t^{t+T} [(x(s) - g(s, x(s-\tau(s)))) e_{\ominus r}(s,0)]^{\triangle} \triangle s \\ &= \int_t^{t+T} \{-f(s, x(s-\tau(s))) + r(s) g(s, x(s-\tau(s)))\} e_{\ominus r}(\sigma(s), 0) \triangle s. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &(x(t+T) - g(t+T, x(t+T-\tau(t+T)))) e_{\ominus r}(t+T, 0) \\ &- (x(t) - g(t, x(t-\tau(t)))) e_{\ominus r}(t,0) \\ &= \int_t^{t+T} \{-f(s, x(s-\tau(s))) + r(s) g(s, x(s-\tau(s)))\} e_{\ominus r}(\sigma(s), 0) \triangle s. \end{aligned}$$

Dividing both sides of the above equation by $e_{\ominus r}(t,0)$ and using the fact that $x(t+T) = x(t)$, (3.3), (3.5) and

$$\frac{e_{\ominus r}(t+T,0)}{e_{\ominus r}(t,0)} = e_{\ominus r}(t+T,t), \ \frac{e_{\ominus r}(\sigma(s),0)}{e_{\ominus r}(t,0)} = e_r(t, \sigma(s)),$$

we obtain

$$\begin{aligned} &x(t) - g(t, x(t-\tau(t))) \\ &= \int_t^{t+T} \frac{e_r(t, \sigma(s))}{1 - e_{\ominus r}(t+T,t)} \{f(s, x(s-\tau(s))) - r(s) g(s, x(s-\tau(s)))\} \triangle s. \end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. □

To simplify notation, we let

$$m=\frac{\exp\left(-\int_0^{2T}|r(u)|\triangle u\right)}{1-e_{\ominus r}(T,0)},\ M=\frac{\exp\left(\int_0^{2T}|r(u)|\triangle u\right)}{1-e_{\ominus r}(T,0)}.$$

It is easy to see that for all $(t,s)\in[0,2T]\times[0,2T]$,

$$m\leq G(t,s)\leq M, \tag{3.8}$$

and from Lemma 2.1 and Theorem 2.3, we have for all $t,s\in\mathbb{R}$,

$$G(t+T,s+T)=G(t,s). \tag{3.9}$$

To apply Theorem 2.4, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is compact. So, we let $(\mathbb{B},\|\cdot\|)=(P_T,\|\cdot\|)$ and $\mathbb{D}=\{\varphi\in\mathbb{B}:L\leq\varphi\leq K\}$, where L is non-negative constant and K is positive constant. We express equation (3.6) as

$$\varphi(t)=(\mathcal{B}\varphi)(t)+(\mathcal{A}\varphi)(t):=(H\varphi)(t),$$

where $\mathcal{A},\mathcal{B}:\mathbb{D}\to\mathbb{B}$ are defined by

$$(\mathcal{A}\varphi)(t)=\int_t^{t+T}G(t,s)\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\}\triangle s, \tag{3.10}$$

and

$$(\mathcal{B}\varphi)(t)=g(t,\varphi(t-\tau(t))). \tag{3.11}$$

In this section, we obtain the existence of a positive periodic solution of (1.1) by considering the two cases; $g(t,x)\geq 0$ and $g(t,x)\leq 0$ for all $t\in\mathbb{R}$, $x\in\mathbb{D}$. We assume that function $g(t,x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant k such that

$$|g(t,x)-g(t,y)|\leq k\|x-y\|,\ \text{for all } t\in[0,T],\ x,y\in\mathbb{D}. \tag{3.12}$$

In the case $g(t,x)\geq 0$, we assume that there exist a non-negative constant k_1 and positive constant k_2 such that

$$k_1x\leq g(t,x)\leq k_2x,\ \text{for all } t\in[0,T],\ x\in\mathbb{D}, \tag{3.13}$$

$$k_2<1, \tag{3.14}$$

and for all $t\in[0,T]$, $x\in\mathbb{D}$

$$\frac{L(1-k_1)}{mT}\leq f(t,x)-r(t)g(t,x)\leq\frac{K(1-k_2)}{MT}. \tag{3.15}$$

Lemma 3.5. *For $\mathcal{A}$ defined in (3.10), Suppose that the conditions (3.3)–(3.5) and (3.13)–(3.15) hold. Then $\mathcal{A}:\mathbb{D}\to\mathbb{B}$ is compact.*

Proof. We first show that $\mathcal{A}:\mathbb{D}\to\mathbb{B}$. Clearly, if φ is continuous, then $\mathcal{A}\varphi$ is. Evaluating (3.10) at $t+T$ gives

$$(\mathcal{A}\varphi)(t+T)=\int_{t+T}^{t+2T}G(t+T,s)\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\}\triangle s.$$

Use Theorem 2.2 with $u=s-T$ to get

$$\begin{aligned}(\mathcal{A}\varphi)(t+T)=\int_t^{t+T}&G(t+T,u+T)\{f(u+T,\varphi(u+T-\tau(u+T)))\\&-r(u+T)g(u+T,\varphi(u+T-\tau(u+T)))\}\triangle u.\end{aligned}$$

From (3.3), (3.4) and (3.9), we obtain

$$\begin{aligned}(\mathcal{A}\varphi)(t+T)&=\int_t^{t+T}G(t,u)\{f(u,\varphi(u-\tau(u)))-r(u)g(u,\varphi(u-\tau(u)))\}\triangle s\\&=(\mathcal{A}\varphi)(t).\end{aligned}$$

That is, $\mathcal{A}:\mathbb{D}\to\mathbb{B}$.

We show that $\mathcal{A}(\mathbb{D})$ is uniformly bounded. For $t\in[0,T]$ and for $\varphi\in\mathbb{D}$, we have

$$\begin{aligned}|(\mathcal{A}\varphi)(t)| &\le \left|\int_t^{t+T} G(t,s)\left[f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\right]\triangle s\right|\\ &\le MT\frac{K(1-k_2)}{MT}=K(1-k_2).\end{aligned}$$

by (3.8) and (3.15). Thus from the estimation of $|(\mathcal{A}\varphi)(t)|$ we arrive

$$\|\mathcal{A}\varphi\|\le K(1-k_2).$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded.

It remains to show that $\mathcal{A}(\mathbb{D})$ is equicontinuous. Let $\varphi_n\in\mathbb{D}$, where n is a positive integer. Next we calculate $(\mathcal{A}\varphi_n)^{\triangle}(t)$ and show that it is uniformly bounded. By making use of (3.3) and (3.5) we obtain by taking the derivative in (3.3) that

$$\begin{aligned}(\mathcal{A}\varphi_n)^{\triangle}(t) &= [G(t,t+T)-G(t,t)]\{f(t,\varphi_n(t-\tau(t)))-r(t)g(t,\varphi_n(t-\tau(t)))\}\\ &\quad + r(t)(\mathcal{A}\varphi_n)^{\sigma}(t).\end{aligned}$$

Consequently, by invoking (3.8), (3.15) and Lemma 3.3, we obtain

$$\left|(\mathcal{A}\varphi_n)^{\triangle}(t)\right|\le\frac{K(1-k_2)}{MT}+\|r\|K(1-k_2)\le D,$$

for some positive constant D. Hence the sequence $(\mathcal{A}\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}\varphi_{n_k})$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. □

Lemma 3.6. *Suppose that (3.12) holds. If $\mathcal{B}$ is given by (3.11) with*

$$k<1, \tag{3.16}$$

then $\mathcal{B}:\mathbb{D}\to\mathbb{B}$ is a contraction.

Proof. Let $\mathcal{B}$ be defined by (3.4). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t+T)=(\mathcal{B}\varphi)(t)$. So, for any $\varphi,\psi\in\mathbb{D}$, we have

$$\begin{aligned}|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)| &\le |g(t,\varphi(t-\tau(t)))-g(t,\psi(t-\tau(t)))|\\ &\le k\|\varphi-\psi\|.\end{aligned}$$

Then $\|\mathcal{B}\varphi-\mathcal{B}\psi\|\le k\|\varphi-\psi\|$. Thus $\mathcal{B}:\mathbb{D}\to\mathbb{B}$ is a contraction by (3.16). □

Theorem 3.5. *Suppose (3.3)–(3.5) and (3.12)–(3.16) hold. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

Proof. By Lemma 3.5, the operator $\mathcal{A}:\mathbb{D}\to\mathbb{B}$ is compact and continuous. Also, from Lemma 3.6, the operator $\mathcal{B}:\mathbb{D}\to\mathbb{B}$ is a contraction. Moreover, if $\varphi,\psi\in\mathbb{D}$, we see that

$$\begin{aligned}&(\mathcal{B}\psi)(t)+(\mathcal{A}\varphi)(t)\\ &= g(t,\psi(t-\tau(t)))\\ &+\int_t^{t+T} G(t,s)\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\}\triangle s\\ &\le k_2K+M\int_t^{t+T}\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\}\triangle s\\ &\le k_2K+MT\frac{K(1-k_2)}{MT}=K.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\mathcal{B}\psi)(t)+(\mathcal{A}\varphi)(t)\\
&=g(t,\psi(t-\tau(t)))\\
&+\int_t^{t+T} G(t,s)\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\}\triangle s\\
&\geq k_1L+m\int_t^{t+T}\{f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\}\triangle s\\
&\geq k_1L+mT\frac{L(1-k_1)}{mT}=L.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x\in\mathbb{D}$ such that $x=\mathcal{A}x+\mathcal{B}x$. By Lemma 3.4 this fixed point is a solution of (1.1) and the proof is complete. □

Remark 3.3. *When $\mathbb{T}=\mathbb{R}$, Theorem 3.5 reduces to Theorem 3.1 of [3].*

In the case $g(t,x)\leq 0$, we substitute conditions (3.13)–(3.15) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3x\leq g(t,x)\leq k_4x,\ \text{for all } t\in[0,T],\ x\in\mathbb{D}, \tag{3.17}$$

$$-k_3<1, \tag{3.18}$$

and for all $t\in[0,T]$, $x\in\mathbb{D}$

$$\frac{L-k_3K}{mT}\leq f(t,x)-r(t)g(t,x)\leq\frac{K-k_4L}{MT}. \tag{3.19}$$

Theorem 3.6. *Suppose (3.3)–(3.5), (3.12) and (3.16)–(3.19) hold. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

The proof follows along the lines of Theorem 3.5, and hence we omit it.

Remark 3.4. *When $\mathbb{T}=\mathbb{R}$, Theorem 3.6 reduces to Theorem 3.2 of [3].*

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