

Transient solution of an $M^{[X]}/G/1$ queuing model with feedback, random breakdowns and Bernoulli schedule server vacation having general vacation time distribution

G. Ayyappan^{a,*} and S.Shyamala^b

^a *Department of Mathematics, Pondicherry Engineering College, Pondicherry, India.*

^b *Department of Mathematics, Arunai Engineering College, Thiruvannamalai, Tamil Nadu, India.*

Abstract

This paper analyze an $M^{[X]}/G/1$ queue with feedback, random server breakdowns and Bernoulli schedule server vacation with general(arbitrary) distribution. Customers arrive in batches with compound Poisson process and are served one by one with first come first served basis. Both the service time and vacation time follow general (arbitrary) distribution. After completion of a service the may go for a vacation with probability θ or continue staying in the system to serve a next customer, if any with probability $1 - \theta$. With probability p , the customer feedback to the tail of original queue for repeating the service until the service be successful. With probability $1 - p = q$, the customer departs the system if service be successful. The system may breakdown at random following Poisson process, whereas the repair time follows exponential distribution. We obtain the time dependent probability generating function in terms of their Laplace transforms and the corresponding steady state results explicitly. Also we derive the system performance measures like average number of customers in the queue and the average waiting time in closed form.

Keywords: $M^{[X]}/G/1$ queue, Poisson arrival, probability generating function, Bernoulli schedule, steady state, mean queue size, mean waiting time.

2010 MSC: 60K25, 60K30.

©2012 MJM. All rights reserved.

1 Introduction

Due to a lot of significance in the decision making process, the research on queuing theory has been extensively increased. Queuing theory has made a revolution in industry and logistics sector apart from its immense applications in many other areas like air traffic, bio-sciences, population studies, health sectors, manufacturing and production sections etc. According to the prevailing demands or situations in real life scenario, queuing models have been encountered enormously, in research perspective.

Most recently research studies on queues with server breakdown have been attracted, as an important area of queuing theory and have been studied extensively and successfully due to their various applications in production, communication systems. Mostly in the queuing literature, the server may be considered as a reliable one, such that service station never fails. But in real situations mostly the servers are unreliable, we often encounter the cases where service stations may fail which can be repaired. Similarly, many phenomena always occur in the area of computer communication networks and flexible manufacturing system etc. Since the performance of such a system may be heavily affected by server breakdowns, followed by a repair immediately, such systems with a repairable service stations are well worth investigating from the queuing theory point of view as well as reliability point of view.

*Corresponding author.

E-mail addresses: ayyappanpec@hotmail.com (G. Ayyappan) and subramaniyanshyamala@gmail.com (S.Shyamala).

Recently, there have been several contributions in considering non-Markovian single server queuing system, in which the server may experience with break downs and repairs, such system with repairable server has been studied as queuing models by many authors including Avi-Itzhak and Naor [2], Graver [6], Takine and Sengupta [21], Wang [24], Tang [22] Assani and Artalejo [1] etc.

Another feature in queuing theory is the study of queuing models with vacations. when the system is empty, the server becomes idle and this idle time may be utilized by the server for being engaged for other purposes. Thus, the non-availability of the server in the system is known as vacation. During the last three or four decades, queuing theorists are interested in the study of queuing models with vacations immensely, because of their applicability and theoretical structures in real life situations such as manufacturing and production systems, computer and communication systems, service and distribution systems, etc.

The most remarkable works have been done in recent past by some researchers on vacation models including Choudhary [3], Keilson and Servi [9], Krishna Kumar [13], Levy and Yechiali [10], Wang [24], Madan [15, 16, 17, 18, 19], Thangaraj [23] etc. The details about vacation queues can be found in the survey of Doshi [5].

Transient state measures, which are very important to track down the functioning of the system at any instant of time. In this paper we present an analysis of the transient state behavior of a queuing system where breakdowns may occur at random, and once the system breaks down, it enters a repair process and the customer whose service is interrupted goes back to the head of the queue. At the same time the server may go on vacation. The vacations follow a Bernoulli distribution, that is, after a service completion, the server may go for a vacation with probability p ($0 \leq p \leq 1$) or may continue to serve the next customer, if any, with probability $1 - p$. The service time and the vacation time are generally distributed, while the repair time is exponentially distributed. The customers arrive in batches to the system and served one by one on a "first come - first served" basis.

The rest of the paper has been organized as follows: in section 2, the mathematical description of our model has been found, in section 3, the transient solution of the system has been derived, in section 4, the steady state analysis has been discussed.

2 Mathematical description of the queuing model

To describe the required queuing model, we assume the following.

- Let $\lambda c_i dt; i = 1, 2, 3, \dots$ be the first order probability of arrival of 'i' customers in batches in the system during a short period of time $(t, t+dt)$ where $0 \leq c_i \leq 1, \sum_{i=1}^{\infty} c_i = 1, \lambda > 0$ is the mean arrival rate of batches.
- There is a single server which provides service following a general (arbitrary) distribution with distribution function $B(v)$ and density function $b(v)$. Let $\mu(x)dx$ be the conditional probability density function of service completion during the interval $(x, x+dx]$ given that the elapsed service time is x , so that

$$\mu(x) = \frac{b(x)}{1 - B(x)} \quad (2.1)$$

and therefore

$$b(v) = \mu(v)e^{-\int_0^v \mu(x) dx} \quad (2.2)$$

- After completion of service, if the customer is not satisfied with the service for certain reason or if customer received unsuccessful service, the customer may immediately join the tail of the original queue as a feedback customer for receiving another regular service with probability p ($0 < p < 1$). Otherwise the customer may depart forever from the system with probability $q (= 1 - p)$. The service discipline for feedback and newly customers are first come first served. Also service time for a feedback customer is independent of its previous service times.

- As soon as a service is completed, the server may take a vacation of random length with probability θ (or) he may stay in the system providing service with probability $1 - \theta$, where $0 \leq \theta \leq 1$.

- The vacation time of the server follows a general (arbitrary) with distribution function $V(s)$ and the density function $v(s)$. Let $\nu(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x so that

$$\nu(x) = \frac{v(x)}{1 - V(x)} \quad (2.3)$$

and therefore

$$v(s) = \nu(s)e^{-\int_0^s \nu(x)dx} \quad (2.4)$$

- The system may breakdown at random and the breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breakdown, the customer whose service is interrupted comes back to the head of queue.

- Once the system breaks down it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate $\beta > 0$.

- Various stochastic processes involved in the queuing system are assumed to be independent of each other.

3 Definitions and Equations governing the system

We let,

- $P_n(x, t)$ = Probability that at time 't' the server is active providing service and there are 'n' ($n \geq 0$) customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $p_n(t)$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in service irrespective of the value of x.
- $V_n(x, t)$ = Probability that at time 't', the server is on vacation with elapsed vacation time x, and there are 'n' ($n \geq 0$) customers waiting in the queue for service. Consequently $V_n(t)$ denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.
- $R_n(t)$ = Probability that at time t, the server is inactive due to breakdown and the system is under repair while there are 'n' ($n \geq 0$) customers in the queue.
- $Q(t)$ = Probability that at time 't' there are no customers in the system and the server is idle but available in the system.

The model is then, governed by the following set of differential-difference equations.

$$\frac{\partial}{\partial t} P_n(x, t) + \frac{\partial}{\partial x} P_n(x, t) + (\lambda + \mu(x) + \alpha)P_n(x, t) = \lambda \sum_{i=1}^{n-1} c_i P_{n-i}(x, t); n \geq 1 \quad (3.1)$$

$$\frac{\partial}{\partial t} P_0(x, t) + \frac{\partial}{\partial x} P_0(x, t) + (\lambda + \mu(x) + \alpha)P_0(x, t) = 0 \quad (3.2)$$

$$\frac{\partial}{\partial t} V_n(x, t) + \frac{\partial}{\partial x} V_n(x, t) + (\lambda + \nu(x))V_n(x, t) = \lambda \sum_{i=1}^{n-1} c_i V_{n-i}(x, t); n \geq 1 \quad (3.3)$$

$$\frac{\partial}{\partial t} V_0(x, t) + \frac{\partial}{\partial x} V_0(x, t) + (\lambda + \nu(x))V_0(x, t) = 0 \quad (3.4)$$

$$\frac{d}{dt} R_n(t) = -(\lambda + \beta)R_n(t) + \lambda \sum_{i=1}^{n-1} c_i R_{n-i}(t) + \alpha \int_0^\infty P_{n-1}(x, t)dx \quad (3.5)$$

$$\frac{d}{dt} R_0(t) = -(\lambda + \beta)R_0(t) \quad (3.6)$$

$$\frac{d}{dt} Q(t) = -\lambda Q(t) + \beta R_0(t) + \int_0^\infty V_0(x, t)\nu(x)dx + (1 - \theta)q \int_0^\infty P_0(x, t)\mu(x)dx \quad (3.7)$$

The above equations are to be solved subject to the following boundary conditions

$$P_n(0, t) = (1 - \theta) \left[p \int_0^\infty P_n(x, t)\mu(x)dx + q \int_0^\infty P_{n+1}(x, t)\mu(x)dx \right] + \int_0^\infty V_{n+1}(x, t)\nu(x)dx + \beta R_{n+1}(t) + \lambda c_{n+1} Q(t); n \geq 0 \quad (3.8)$$

$$V_n(0, t) = \theta \int_0^\infty P_n(x, t) \mu(x) dx; n \geq 0 \quad (3.9)$$

Assuming there are no customers in the system initially so that the server is idle.

$$V_0(0) = 0; V_n(0) = 0; Q(0) = 1; P_n(0) = 0, n = 0, 1, 2, \dots \quad (3.10)$$

Generating functions of the queue length. The time dependent solution.

We define the probability generating functions

$$P_q(x, z, t) = \sum_{n=0}^{\infty} z^n P_n(x, t)$$

$$P_q(z, t) = \sum_{n=0}^{\infty} z^n P_n(t)$$

$$V_q(x, z, t) = \sum_{n=0}^{\infty} z^n V_n(x, t)$$

$$V_q(z, t) = \sum_{n=0}^{\infty} z^n V_n(t)$$

$$R_q(z, t) = \sum_{n=0}^{\infty} z^n R_n(t)$$

$$C(z) = \sum_{n=1}^{\infty} c_n z^n \quad (3.11)$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^\infty f(t) e^{-st} dt. \quad (3.12)$$

Taking Laplace transforms of equations (3.1) to (3.9) and using the probability generating function defined above.

$$\frac{\partial}{\partial x} \bar{P}_n(x, s) + (s + \lambda + \mu(x) + \alpha) \bar{P}_n(x, s) = \lambda \sum_{i=1}^{n-1} c_i \bar{P}_{n-i}(x, s) \quad (3.13)$$

$$\frac{\partial}{\partial x} \bar{P}_0(x, s) + (s + \lambda + \mu(x) + \alpha) \bar{P}_0(x, s) = 0 \quad (3.14)$$

$$\frac{\partial}{\partial x} \bar{V}_n(x, s) + (s + \lambda + \nu(x)) \bar{V}_n(x, s) = \lambda \sum_{i=1}^{n-1} c_i \bar{V}_{n-i}(x, s) \quad (3.15)$$

$$\frac{\partial}{\partial x} \bar{V}_0(x, s) + (s + \lambda + \nu(x)) \bar{V}_0(x, s) = 0 \quad (3.16)$$

$$(s + \lambda + \beta) \bar{R}_n(s) = \lambda \sum_{i=1}^{n-1} c_i \bar{R}_{n-i}(s) + \alpha \int_0^\infty \bar{P}_{n-1}(x, s) dx \quad (3.17)$$

$$(s + \lambda + \beta) \bar{R}_0(s) = 0 \quad (3.18)$$

$$(s + \lambda) \bar{Q}(s) = 1 + \beta \bar{R}_0(s) + \int_0^\infty \bar{V}_0(x, s) \nu(x) dx + (1 - \theta) q \int_0^\infty \bar{P}_0(x, s) \mu(x) dx \quad (3.19)$$

for boundary conditions,

$$\begin{aligned} \bar{P}_n(0, s) = (1 - \theta) \left[p \int_0^\infty \bar{P}_n(x, s) \mu(x) dx + q \int_0^\infty \bar{P}_{n+1}(x, s) \mu(x) dx \right] \\ + \int_0^\infty \bar{V}_{n+1}(x, s) \nu(x) dx + \beta \bar{R}_{n+1}(s) + \lambda c_{n+1} \bar{Q}(s); n \geq 0 \end{aligned} \quad (3.20)$$

$$\bar{V}_n(0, s) = \theta \int_0^\infty \bar{P}_n(x, s) \mu(x) dx; n \geq 0 \quad (3.21)$$

multiply equation (3.13) by z^n and add (3.14) implies

$$\frac{\partial}{\partial x} \bar{P}_q(x, z, s) + (s + \lambda - \lambda C(z) + \mu(x) + \alpha) \bar{P}_q(x, z, s) = 0 \quad (3.22)$$

performing similar operations to equations (3.15) to (3.18).

$$\frac{\partial}{\partial x} \bar{V}_q(x, z, s) + (s + \lambda - \lambda C(z) + \nu(x)) \bar{V}_q(x, z, s) = 0 \quad (3.23)$$

$$(s + \lambda - \lambda C(z) + \beta) \bar{R}_q(z, s) = \alpha z \int_0^\infty \bar{P}_q(x, z, s) dx \quad (3.24)$$

For the boundary conditions, we multiply equation(3.20)by z^{n+1} , sum over n from 0 to ∞ and use generating function defined above, we get

$$\begin{aligned} z\bar{P}_q(0, z, s) &= (1 - \theta)(pz + q) \int_0^\infty \bar{P}_q(x, z, s) \mu(x) dx \\ &+ \int_0^\infty \bar{V}_q(x, z, s) \nu(x) dx + \beta \bar{R}_q(z, s) + (1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s) \end{aligned} \quad (3.25)$$

Similarly multiply equation (3.21) by z^n and sum over n from 0 to ∞ and use generating function defined above

$$\bar{V}_q(0, z, s) = \theta \int_0^\infty \bar{P}_q(x, z, s) \mu(x) dx \quad (3.26)$$

Integrating equation(3.22) from 0 to x yields

$$\bar{P}_q(x, z, s) = \bar{P}_q(0, z, s) e^{-(s+\lambda-\lambda C(z)+\alpha)x - \int_0^x \mu(t) dt} \quad (3.27)$$

where $\bar{P}_q(0, z, s)$ is given by equation(3.25). Again integrating equation (3.27) by parts with respect to x yields

$$\bar{P}_q(z, s) = \bar{P}_q(0, z, s) \left[\frac{1 - \bar{B}(s + \lambda - \lambda C(z) + \alpha)}{(s + \lambda - \lambda C(z) + \alpha)} \right] \quad (3.28)$$

where

$$\bar{B}(s + \lambda - \lambda C(z) + \alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x} dB(x) \quad (3.29)$$

is Laplace - Stieltjes transform of the service time B(x). Now multiplying both sides of equation (3.27) by $\mu(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}_q(x, z, s) \mu(x) dx = \bar{P}_q(0, z, s) \bar{B}(s + \lambda - \lambda C(z) + \alpha) \quad (3.30)$$

Using equation (3.30) equation (3.26) becomes

$$\bar{V}_q(0, z, s) = \theta \bar{P}_q(0, z, s) \bar{B}(s + \lambda - \lambda C(z) + \alpha) \quad (3.31)$$

Similarly integrate equation (3.23) from 0 to x, we get

$$\bar{V}_q(x, z, s) = \bar{V}_q(0, z, s) e^{-(s+\lambda-\lambda C(z))x - \int_0^x \nu(t) dt} \quad (3.32)$$

substituting by the value of $\bar{V}_q(0, z, s)$ from (3.31), in equation (3.33) we get

$$\bar{V}_q(x, z, s) = \theta \bar{P}_q(0, z, s) \bar{B}(s + \lambda - \lambda(C(z)) + \alpha) e^{-(s+\lambda-\lambda z)x - \int_0^x \nu(t) dt} \quad (3.33)$$

Again integrating equation (3.33) by parts with respect to x

$$\bar{V}_q(z, s) = \theta \bar{P}_q(0, z, s) \bar{B}(s + \lambda - \lambda C(z) + \alpha) \left[\frac{1 - \bar{V}(s + \lambda - \lambda C(z))}{(s + \lambda - \lambda C(z))} \right] \quad (3.34)$$

where

$$\bar{V}(s + \lambda - \lambda C(z)) = \int_0^\infty e^{-(s + \lambda - \lambda C(z))x} dV(x) \quad (3.35)$$

is Laplace - Stieltjes transform of the vacation time $V(x)$. Now multiplying both sides of equation(3.33) by $\nu(x)$ and integrating over x , we get

$$\int_0^\infty \bar{V}_q(x, z, s) \nu(x) dx = \theta \bar{P}_q(0, z, s) \bar{B}(s + \lambda - \lambda C(z) + \alpha) \bar{V}(s + \lambda - \lambda C(z)) \quad (3.36)$$

Using equation (3.28), equation (3.24) becomes

$$\bar{R}_q(z, s) = \frac{\alpha z \bar{P}_q(0, z, s) [1 - \bar{B}(s + \lambda - \lambda C(z) + \alpha)]}{[s + \lambda - \lambda C(z) + \beta][s + \lambda - \lambda C(z) + \alpha]} \quad (3.37)$$

Now using (3.30), (3.36) and (3.37) in equation (3.25) and solving for $\bar{P}_q(0, z, s)$ we get

$$\bar{P}_q(0, z, s) = \frac{f_1(z) f_2(z) [(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (3.38)$$

where $Dr = f_1(z) f_2(z) \{z - (1 - \theta)(pz + q)\bar{B}[f_1(z)] - \theta\bar{V}(s + \lambda - \lambda C(z))\bar{B}[f_1(z)]\} - \alpha\beta z \{1 - \bar{B}[f_1(z)]\}$

$$f_1(z) = s + \lambda - \lambda C(z) + \alpha$$

$$f_2(z) = s + \lambda - \lambda C(z) + \beta$$

substituting the value of $\bar{P}_q(0, z, s)$ from equation (3.38) in to equations (3.26), (3.34) and (3.37)

$$\bar{P}_q(z, s) = \frac{f_2(z) [1 - \bar{B}[f_1(z)]] [(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (3.39)$$

$$\bar{V}_q(z, s) = \frac{\theta f_1(z) f_2(z) \bar{B}[f_1(z)] [(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)] \left[\frac{1 - \bar{V}(s + \lambda - \lambda C(z))}{(s + \lambda - \lambda C(z))} \right]}{Dr} \quad (3.40)$$

$$\bar{R}_q(z, s) = \frac{\alpha z [1 - \bar{B}[f_1(z)]] [(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (3.41)$$

where Dr is given in the above.

4 The steady state analysis

In this section we shall derive the steady state probability distribution for our queuing model. To define the steady state probabilities, suppress the argument 't' where ever it appears in the time dependent analysis. By using well known Tauberian property,

$$Lt_{s \rightarrow 0} s \bar{f}(s) = Lt_{t \rightarrow \infty} f(t) \quad (4.1)$$

multiplying both sides of equation (3.39), (3.40), (3.41) and applying equation(4.1) and simplifying, we get

$$P_q(z) = \frac{f_2(z) (1 - \bar{B}[f_1(z)]) [\lambda(C(z) - 1)Q]}{Dr} \quad (4.2)$$

$$V_q(z) = \frac{p f_1(z) f_2(z) \bar{B}[f_1(z)] [\bar{V}(\lambda - \lambda C(z)) - 1] Q}{Dr} \quad (4.3)$$

$$R_q(z) = \frac{\theta\lambda\alpha z[1 - \bar{B}[f_1(z)]][(C((z)) - 1)Q]}{Dr} \quad (4.4)$$

where Dr and $f_1(z)$ and $f_2(z)$ are given by in previous section. Let $W_q(z)$ denotes the probability generating function of queue size irrespective of the state of the system. Then adding (4.2),(4.3) and (4.4), we get

$$W_q(z) = P_q(z) + V_q(z) + R_q(z) \quad (4.5)$$

$$W_q(z) = \frac{f_2(z)[1 - B[f_1(z)]][\lambda(C((z)) - 1)Q]}{Dr} + \frac{\theta f_1(z)f_2(z)B[f_1(z)][V(\lambda - \lambda C((z)) - 1)]Q}{Dr} + \frac{\lambda\alpha z[1 - B[f_1(z)]][(C((z)) - 1)Q]}{Dr} \quad (4.6)$$

In order to obtain Q , we use the normalization condition, as follows

$$W_q(1) + Q = 1 \quad (4.7)$$

We see that at $z=1$, $W_q(z)$ is indeterminate of the form $0/0$. We apply L'Hospital rule in equation (4.6)

$$W_q(1) = \frac{\lambda QE(I)(\alpha + \beta)[1 - \bar{B}(\alpha)] + \theta\alpha\beta B(\alpha)E[V]}{(q + p\theta)\alpha\beta\bar{B}(\alpha) - \lambda(\alpha + \beta)(1 - \bar{B}(\alpha))E(I) - \theta\lambda\alpha\beta\bar{B}(\alpha)E(I)E[V]} \quad (4.8)$$

where $\bar{B}(0) = 1$, $\bar{V}(0) = 1$, $-V'(0) = E[V]$ the mean vacation time. Using equation (4.8) in equation (4.7)

$$Q = 1 - \frac{\lambda E(I)}{q + p\theta} \left\{ \frac{1}{\beta\bar{B}(\alpha)} + \frac{1}{\alpha\bar{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right\} \quad (4.9)$$

and the the utilization factor ρ of the system is given by

$$\rho = \frac{\lambda E(I)}{q + p\theta} \left\{ \frac{1}{\beta\bar{B}(\alpha)} + \frac{1}{\alpha\bar{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + \theta E(V) \right\} \quad (4.10)$$

where $\rho < 1$ is the stability condition under which the steady state exists, equation(4.9) gives the probability that the server is idle. Substitute Q from equation (4.9) in equation (4.6) $W_q(z)$ have been completely and explicitly determined which is the the probability generating function of the queue size.

The average queue size and average waiting time

Let L_q denote the mean number of customers in the queue under the steady state, then $L_q = \frac{d}{dz} W_q(z) |_{z=1}$, since this formula gives $0/0$ form, then we write $W_q(z) = \frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of equation (4.5) respectively, then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2} \quad (4.11)$$

where primes and double primes in equation (4.11) denote first and second derivation at $z=1$ respectively. Carrying out the derivatives at $z=1$, we have

$$N'(1) = \lambda E(I)Q[(\alpha + \beta) - \bar{B}(\alpha)(\theta\alpha\beta E(V) - \alpha - \beta)] \quad (4.12)$$

$$N''(1) = 2Q[\lambda E(I)]^2 \left\{ \left(\frac{\alpha}{\lambda E(I)} - 1 \right) + \bar{B}(\alpha) \left[1 - \frac{\alpha}{\lambda E(I)} - \theta\alpha E(V) - \theta\beta E(V) \right] + \frac{1}{2}\theta\alpha\beta E(V^2) + \bar{B}'(\alpha)(\alpha + \beta - \theta\alpha\beta E(V)) \right\} + \lambda QE(I(I - 1)) \left\{ (\alpha + \beta) + \bar{B}(\alpha)(\theta\alpha\beta E(V) - \alpha - \beta) \right\} \quad (4.13)$$

$$D'(1) = -\lambda E(I)(\alpha + \beta) + \bar{B}(\alpha) \{ \alpha\beta(q + p\theta) + \lambda E(I)(\alpha + \beta) - \theta\alpha\beta E(V) \} \quad (4.14)$$

$$D''(1) = 2[\lambda E(I)]^2 \left\{ \left(1 - \frac{\alpha + \beta}{\lambda E(I)} \right) + \bar{B}(\alpha) \left[-(q + p\theta) + \theta\alpha E(V) + \theta\beta E(V) - \frac{1}{2}\alpha\beta\theta E(V^2) \right] + \bar{B}'(\alpha) \left[-(q + p\theta)(\alpha + \beta) - \frac{\alpha\beta}{\lambda E(I)} + \alpha\beta\theta E(V) \right] \right\} + \lambda E(I(I - 1)) \left\{ -(\alpha + \beta) + \bar{B}(\alpha)(\alpha + \beta - \theta\alpha\beta E(V)) \right\} \quad (4.15)$$

where $E(V^2)$ is the second moment of the vacation time and Q has been found in equation (4.9). Then if we substitute the values of $N'(1)$, $N''(1)$, $D'(1)$ and $D''(1)$ from equations (4.12), (4.13), (4.14) and (4.15) in to equation (4.11), we obtain L_q in a closed form.

Mean waiting time of a customer could be found, as follows

$$W_q = \frac{L_q}{\lambda} \quad (4.16)$$

by using Little's formula.

References

- [1] A. Aissani, and J.R. Artalejo, On the Single server retrial queue subject to breakdown, *Queueing System*, 30(1998), 309-321.
- [2] B. Avi-Itzhak, and P. Naor, One queuing problems with the service station subject to breakdown, *Operations Research*, 11(1963), 303-320.
- [3] G. Choudhury, A batch arrival queue with a vacation time under single vacation policy, *Computers and Operations Research*, 29(14)(2002), 1941-1955.
- [4] G. Choudhury, Some aspects of an $M/G/1$ queueing system with optional second service, *TOP*, 11(1)(2003), 141-150.
- [5] B.T. Doshi, Queueing systems with vacations- a survey, *Queueing Systems*, 1(1986), 29-66.
- [6] D.P. Graver, A waiting line with interrupted service including priorities, *Journal of Royal Stat. Society B*, 24(1960), 73-80.
- [7] D. Gross and C. Harris, *Fundamentals of Queueing Theory*, Third Edition, John Wiley and Sons, Inc., New York,(1998).
- [8] J. C. Ke, Modified T vacation policy for an $M/G/1$ queueing system with an un-reliable server and startup, *Mathematical and Computer Modelling*, 41(2005), 1267-1277.
- [9] J. Keilson and L.D.Servi, Oscillating random walk models for $G1/G/1$ vacation systems with Bernoulli schedules, *Journal of Applied Probability*, 23(1986), 790-802.
- [10] Y. Levy, and U. Yechiali, Utilization of idle time in an $M/G/1$ queueing system, *Management Science*, 22(1975), 202-211.
- [11] J. Keilson, and L.D. Servi, Dynamic of the $M/G/1$ vacation model, *Operation Research*, 35(4)(1987), July-August.
- [12] B. Krishna Kumar and D. Arivudainambi, Transient solution of an $M/M/1$ queue with catastrophes, *Computers and Mathematics with Applications*, 40(2000), 1233-1240.
- [13] B. Krishnakumar, and D. Arivudainambi, An $M/G/1/1$ feedback queue with regular and optional services, *Int. J. Inform. Manage. Sci*, 12(1)(2001), 67-73.
- [14] Y. Levi, and U. Yechilai, An $M/M/s$ queue with server vacations, *INFOR*, 14(2)(1976), 153-163.
- [15] K.C. Madan, An $M/G/1$ queue with second optional service, *Queueing Systems*, 34(2000), 37-46.
- [16] K.C. Madan, and A. Baklizi, An $M/G/1$ queue with additional second stage and optional service, *International Journal of Information and Management Sciences*, 13(1)(2002), 13-31.
- [17] K.C. Madan, and A.Z. Abu Al-Rub, On a single server queue with optional phase type server vacations based on exhaustive deterministic service and a single vacation policy, *Applied Mathematics and Computation*, 149(2004), 723-734.

- [18] K.C. Madan, W. Abu-Deyyeh, and M. Gharaibeh, On two parallel servers with random breakdowns, *Soochow Journal of Mathematics*, 29(4)(2003), 413-423.
- [19] F.A. Maraghi, K.C. Madan, and K. Darby-Dowman, Batch arrival queuing system with random breakdowns and Bernoulli schedule server vacations having general vacation time distribution, *International Journal of Information and Management Sciences*, 20(1)(2009), 55-70.
- [20] P.R. Parthasarathy, and R. Sudhesh, Transient solution of a multi server Poisson queue with N-policy, *Computer and Mathematics with Applications*, 55(2008), 550-562.
- [21] T. Takine, and B. Sengupta, A single server queue with service interruptions, *Queuing Systems*, 26(1997), 285-300.
- [22] Y.H. Tang, A single-server $M/G/1$ queuing system subject to breakdowns-some reliability and queuing problem, *Microelectronics and Reliability*, 37(2)(1997), 315-321.
- [23] V. Thangaraj and S. Vanitha, $M/G/1$ Queue with Two-Stage Heterogeneous Service Compulsory Server Vacation and Random Breakdowns, *Int. J. Contemp. Math. Sciences*, 5(7)(2010), 307 - 322.
- [24] K.H. Wang, Infinite source $M/M/1$ queue with breakdown, *Journal of the Chinese Institute of Industrial Engineers*, 7(1990), 47-55.

Received: March 2, 2013; *Accepted:* April 17, 2013

UNIVERSITY PRESS