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Orthonormal series expansion and finite spherical Hankel transform of generalized functions

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Abstract

The finite spherical Hankel transformation is extended to generalized functions by using orthonormal series expansion of generalized functions. A complete orthonormal family of spherical Bessel functions is derived and certain spaces of testing functions and generalized functions are defined. The inversion and uniqueness theorems are obtained. The operational transform formula is derived and is applied to solve the problem of the propagation of heat released from a spherically symmetric point heat source.

Keywords: Finite spherical Hankel transform, orthonormal series expansion of generalized functions.

2010 MSC: 46F12, 44A15, 46F10, 41A58.

1 Introduction

Several authors treated the problem of expanding the elements of a distribution space using different orthonormal systems. Zemanian [2], [5] constructed the testing function space A for suitable complete orthonormal system $\{\Psi_n\}$ of eigenfunctions of the differential operator η. The elements of the dual space A' are generalized functions, each of which can be expanded into a series of eigenfunctions Ψ_n. As a special case of his general theory he defined the finite Fourier, Hermite, Jacobi and finite Hankel transformations of generalized functions where the inverse transformations are obtained by using orthonormal series expansions of generalized functions.

Bhosale and More[3] and Panchal and More[4] extended certain finite integral transformations to generalized functions by using the method of Zemanian. In this paper the variant of finite spherical Hankel transformation introduced by Chen I.I.H.[1] is extended it to certain space of generalized functions whose inverse is obtained in terms of Fourier-spherical Bessel series.

2 Preliminary Results, Notations and Terminology

Let I=$\{x/0 \leq x \leq a < \infty\}$ and $N_0 = \mathrm{N} \cup \{0\}$, where N is the set of natural numbers. Consider the self adjoint differential operator

$$L_0 = (x^{-1}D_x x^2 D_x x^{-1})$$

denoting the conventional or generalized partial differential operators, where $D_x = \frac{\partial}{\partial x}$. Let $J_{\frac{1}{2}}(x)$ and $j_0(x)$ be the Bessel function of the first kind of order 1/2, and spherical Bessel function of order zero respectively. Consider the eigenfunction system $\{\psi_n(x)\}_{n=1}^{\infty}$ corresponding to the differential operator L_0 where $\psi_n(x) = C_n x j_0(\lambda_n x)$, $C_n = \frac{2}{a[J'_{\frac{1}{2}}(\lambda_n a)]}\sqrt{\frac{\lambda_n}{\pi}}$, and the corresponding eigenvalues $\lambda_n, n = 1, 2, 3, ...$ are the positive roots of $j_0(\lambda z) = 0$ arranged in ascending order of magnitude.We see that,

$$L_0\psi_n(x) = -\lambda_n^2\psi_n(x). \tag{2.1}$$

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Let $L_2(I)$ be the linear space of functions that are absolutely square integrable on I and $< f, g >$ denote the inner product defined by,

$$< f, \overline{g} >= (f, g) = \int_I f(x)\overline{g}(x)dx. \tag{2.2}$$

Thus,

$$||f||_2^2 =< f, \overline{f} >= (f, f) = \int_I |f(x)|^2 dx \tag{2.3}$$

is the norm on $L_2(I)$. Hence

$$(\psi_m(x), \psi_n(x)) = \begin{cases} 1 & \text{m=n} \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

and $\int_I |\psi_n(x)|^2 dx = 1$ implies that $\psi_n(x) \in L_2(I)$ for every $n \in N_0$.

We define the finite spherical Hankel transform of $f(x) \in L_2(I)$ denoted by $SH[f(x)](n) = F_{SH}(n)$ as,

$$F_{SH}(n) = (f(x), \psi_n(x)) = \int_I f(x)\psi_n(x)dx. \tag{2.5}$$

The following theorem provides the inversion of the transformation defined in (2.5).

Theorem 2.1. *Every $f(x) \in L_2(I)$ admits the Fourier-spherical Bessel series expansion*

$$f(x) = \sum_{n=1}^{\infty} (f(x), \psi_n(x))\psi_n(x) \tag{2.6}$$

where the series converges point-wise on I.

3 Testing Function Space $S - H(I)$

For $n \in N_0$ we denote by $S - H(I)$ the space all complex valued smooth functions $\phi(x)$ defined on I such that for each non negative integers n *and* k.

i)

$$\eta^k(\phi) = \eta^0(L_0^k\phi) = \left\{ \int_I [L_0^k\phi(x)]^2 dx \right\}^{\frac{1}{2}} < \infty \tag{3.1}$$

ii)

$$(L_0^k\phi, \psi_n(x)) = (\phi, L_0^k\psi_n(x)) \tag{3.2}$$

Obviously $L_2(I) \subset S - H(I)$. The space $S - H(I)$ is a linear space and η^k is a seminorm on $S - H(I)$. Moreover η^0 is a norm on $S - H(I)$. Thus $\eta^k, k \in N_0$ is a countable multi-norm on $S - H(I)$. Also $S - H(I)$ is complete and hence a Frechet space. Thus $S - H(I)$ turns out to be a testing function space.

Lemma 3.1. *Every $\psi_n(x)$ is a member of $S - H(I)$.*

Proof. For each $k \in N_0$, from equations (2.1) and (3.1) we have

$$\begin{aligned} |\eta^k[\psi_n(x)]|^2 &\le \int_I |L_0^k\psi_n(x)|^2 dx \\ &\le |\lambda_n|^{2k} \int_I |\psi_n(x)|^2 dx \\ &= |\lambda_n|^{2k} < \infty. \end{aligned}$$

Next since λ_n are real then for $m \neq n$, we have

$$\begin{aligned} (L_0^k\psi_n(x), \psi_m(x)) &= \lambda_n^k(\psi_n(x), \psi_m(x)) \\ &= 0 = \lambda_m^k(\psi_n(x), \psi_m(x)) = (\psi_n(x), \lambda_m^k\psi_m(x)) \\ &= (\psi_n(x), L_0^k\psi_m(x)) \end{aligned}$$

and for $m = n$

$$(L_0^k\psi_n(x), \psi_n(x)) = (\lambda_n^k\psi_n(x), \psi_n(x)) = (\psi_n(x), \lambda_n^k\psi_n(x)) = (\psi_n(x), L_0^k\psi_n(x)).$$

Hence $\psi_n(x) \in$ S-H(I) for all $n \in N_0$. ☐

Lemma 3.2. *Every $\phi(x) \in S - H(I)$ can be expanded into the series*

$$\phi(x) = \sum_{n=0}^{\infty} (\phi(x), \psi_n(x))\psi_n(x) \tag{3.3}$$

where the series converges in $S - H(I)$.

Proof. Let $\phi(x) \in S - H(I)$, then $L_0^k\phi(x) \in L_2(I)$ and from theorem (2.1), we have

$$\begin{aligned} Ł_0^k\phi(x) &= \sum_{n=0}^{\infty} (L_0^k\phi(x), \psi_n(x))\psi_n(x) \\ &= \sum_{n=0}^{\infty} (\phi, L_0^k\psi_n(x))\psi_n(x) \\ &= \sum_{n=0}^{\infty} (\phi, \psi_n(x))\lambda_n^k\psi_n(x) \\ &= \sum_{n=0}^{\infty} (\phi, \psi_n(x))L_0^k\psi_n(x) \end{aligned}$$

which implies that $\eta^k[\phi(x) - \sum_{n=0}^{N}(\phi(x), \psi_n(x))\psi_n(x)] \to 0$ as $N \to \infty$ independently. Thus the series in (3.3) converges to $\phi(x)$ in $S - H(I)$. □

4 The Generalized Function Space $S - H'(I)$

The space of all continuous linear functions on $S - H(I)$, denoted by $S - H'(I)$ is called the dual of $S - H(I)$ and members of $S - H'(I)$ are called generalized functions on I. The number that $f \in S - H'(I)$ assigns to $\phi \in S - H(I)$ is denoted by $< f, \phi >$. Since the testing function space $S - H(I)$ is complete, so also is $S - H'(I)$[5]. Let $f(x)$ be a real valued continuous function locally integrable on I such that

$$\int_I |f(x)|^2 dx < \infty,$$

then f generates a member of $S - H'(I)$ through the definition

$$< f, \phi >= \int_I f(x)\phi(x)dx. \tag{4.1}$$

Clearly (4.1) defines a linear function f on $S - H(I)$ and the continuity of f can be verified by using Schwarz's inequality. Such members of $S - H'(I)$ are called regular generalized functions in $S - H'(I)$. All other generalized functions in $S - H'(I)$ are called singular generalized functions. Now we define a generalized differential operator L_0 on $S - H'(I)$ through the relationship

$$(f, L_0\phi) =< f, \overline{L_0\phi} >=< \overline{L_0}' f, \overline{\phi} >= (\overline{L_0}' f, \phi) \tag{4.2}$$

where $\overline{L_0}'$ is obtained from L_0 by reversing the order in which the differentiation and multiplication by smooth functions occurring in L_0, replacing each D_x by $-D_x$ and then taking the complex conjugate of the result. But this is precisely the same expression for L_0 [[5], sec 9.2, eq 4]. Thus $L_0 = \overline{L_0}'$ is defined as the generalized differential operator on $S - H'(I)$ through the equation

$$< L_0 f, \phi >=< f, L_0\phi >, \tag{4.3}$$

where $f \in S - H'(I), \phi \in S - H(I)$.

Some properties of $S - H(I)$ and $S - H'(I)$

I) $\mathscr{D}(I) \subset S - H(I) \subset \mathscr{E}(I)$ and since $\mathscr{D}(I)$ is dense in $\mathscr{E}(I), S - H(I)$ is also dense in $\mathscr{E}(I)$. It follows $\mathscr{E}'(I)$ is a subspace of $S - H'(I)$. The convergence of a sequence in $\mathscr{D}(I)$ implies its convergence in $S - H(I)$. The restriction of any $f \in S - H'(I)$ to $\mathscr{D}(I)$ is in $\mathscr{D}'(I)$. Moreover the convergence in $S - H'(I)$ implies convergence in $\mathscr{D}'(I)$.

II) For each $f \in S - H'(I)$ there exists a non negative integer r and a positive constant C such that

$$| < f, \phi > | \leq C \max_{0 \leq k \leq r} \eta^k(\phi)$$

for every $\phi \in S - H(I)$. Here r and C depends on f but not on ϕ.

III) The mapping $\phi \to L_0\phi$ is continuous linear mapping of $S - H(I)$ into itself. It follows that $f \to L_0 f$ is also a continuous linear mapping of $S - H'(I)$ whenever f is a regular generalized function in $S - H'(I)$.

5 Finite Spherical Hankel transformation of generalized functions

We define the finite spherical Hankel transform of generalized function $f \in L - H'(I)$, denoted by $\mathcal{SH}[f] = \mathcal{F}_{SH}(n)$ as,

$$\mathcal{SH}[f(x)](n) = \mathcal{F}_{SH}(n) = (f(x), \psi_n(x)) \tag{5.1}$$

where $\psi_n(x) \in S - H(I)$ for $n \in N_0$. We see that $\mathcal{SH}$ is a linear and continuous mapping on S-H'(I), which maps $f \in S - H'(I)$ into a function $\mathcal{F}_{SH}(n)$ defined on N_0. The following theorem provides the inversion of the transformation defined in (5.1).

Theorem 5.1. *Let $f \in$ S-H'(I), then the series*

$$\sum_{n=0}^{\infty}(f(x), \psi_n(x))\psi_n(x) \tag{5.2}$$

converges to f in $S - H'(I)$.

Proof. From lemma 3.2 we have for every $\phi \in S - H(I)$, the series $\sum_{n=0}^{\infty}(\phi, \psi_n(x))\psi_n(x)$ converges to ϕ in $S - H(I)$, then for $f \in S - H'(I)$, we write

$$\begin{aligned}(f, \phi) &= (f, \sum_{n=0}^{\infty}(\phi, \psi_n(x)), \psi_n(x)) \\ &= \sum_{n=0}^{\infty} \overline{(\phi, \psi_n(x))}(f, \psi_n(x)) \\ &= \sum_{n=0}^{\infty}(f, \psi_n(x))(\psi_n(x), \phi(x)) \\ &= \sum_{n=0}^{\infty}((f, \psi_n(x))\psi_n(x), \phi(x)) \\ &= \left(\sum_{n=0}^{\infty}(f, \psi_n(x))\psi_n(x), \phi(x)\right).\end{aligned}$$

Thus the series $\sum_{n=0}^{\infty}(f, \psi_n(x))\psi_n(x)$ converges weakly to f in S-H'(I).

The above theorem lead to define the inverse of the finite spherical Hankel transformation of $f \in S - H'(I)$, denoted by $\mathcal{SH}^{-1}\mathcal{F}_{SH}(n) = f(x)$, as

$$\begin{aligned}\mathcal{SH}^{-1}\mathcal{F}_{SH}(n) = f(x) &= \sum_{n=0}^{\infty}\mathcal{F}_{SH}(n)\psi_n(x) \\ &= \sum_{n=0}^{\infty}(f(x), \psi_n(x))\psi_n(x).\end{aligned} \tag{5.3}$$

□

Theorem 5.2. *(Uniqueness Theorem): Let $f, g \in$ S-H'(I) are such that $\mathcal{SH}[f](n) = \mathcal{F}_{SH}(n) = \mathcal{G}_{SH}(n) = \mathcal{SH}[g](n)$ for every $n \in N_0$, then $f = g$ in the sense of equality in $S - H'(I)$.*

Proof. Let $\phi \in S-H(I)$, and $f, g \in S-H'(I)$ then

$$\begin{aligned}< f,\phi > - < g,\phi > &=< \sum_{n=0}^{\infty}(f,\psi_n(x))\psi_n(x),\phi(x) > \\ &- < \sum_{n=0}^{\infty}(g,\psi_n(x))\psi_n(x),\phi(x) > \\ &=< \sum_{n=0}^{\infty}\mathcal{F}_{SH}(n)\psi_n(x),\phi(x) > \\ &- < \sum_{n=0}^{\infty}\mathcal{G}_{SH}(n)\psi_n(x),\phi(x) > \\ &=< \sum_{n=0}^{\infty}[\mathcal{F}_{SH}(n)-\mathcal{G}_{SH}(n)]\psi_n(x),\phi(x) > \\ &= 0\end{aligned}$$

for all n $\in N_0$. Hence $f = g$ in $S-H'(I)$. □

6 An Operational Calculus

Let $f(x) \in S-H'(I), \psi_n(x) \in S-H(I)$ and since the differential operator L_0 is a continuous linear mapping of $S-H'(I)$ into itself, then from equation (4.3), we have

$$\begin{aligned}\mathcal{SH}[L_0^k f](n) =< L_0^k f, \psi_n(x) > &=< f, L_0^k\psi_n(x) > \\ &=< f, -\lambda_n^{2k}\psi_n(x) > \\ &= -\lambda_n^{2k} < f,\psi_n(x) > \\ &= -\lambda_n^{2k}\mathcal{SH}[f](n) \\ &= -\lambda_n^{2k}\mathcal{F}_{SH}(n).\end{aligned} \tag{6.1}$$

We can use this fact to solve the distributional differential equations of the form

$$P(L_0)u = g \tag{6.2}$$

where P is a polynomial and the given g and unknown u are the generalized functions in S-H'(I). Applying the finite spherical Hankel transformation defined in (5.1) to the differential equation (6.2), we get

$$P(-\lambda_n^2)\mathcal{SH}[u](n) = \mathcal{SH}[g](n), \;\; n \in N_0. \tag{6.3}$$

If $P(-\lambda_n^2) \neq 0$ for all $n \in N_0$, we divide (6.3) by $P(-\lambda_n^2)$ and apply inverse finite spherical Hankel transform defined in (5.3), and get

$$u(x) = \sum_{n=0}^{\infty}\frac{\mathcal{SH}[g](n)}{P(-\lambda_n^2)}\psi_n(x) \tag{6.4}$$

where the series converges in $S-H'(I)$. In view of Theorem (5.1) and (5.2) the solution $u(x)$ in $S-H'(I)$ exists and is unique.

7 Application of finite spherical Hankel transform

The propagation of heat released from a spherically symmetric point heat source is governed by the heat conduction equation of the form

$$x^{-1}\frac{\partial^2(xu)}{\partial x^2} = k^{-1}\frac{\partial u}{\partial t} \tag{7.1}$$

where $k = K/\rho C_\nu$ is the thermal diffusivity for conductivity K, ρ is density, and C_ν is the heat capacity, respectively. We consider the folllowing initial and boundary conditions:

$$u(x,t) = f(x) \ \ when \quad t = 0 \quad at \ \ x = 0;$$
$$u(x,t) = 0 \ \ at \ \ x = a, \quad t > 0. \tag{7.2}$$

We now find the generalized solution $u(x,t)$ of this problem in the space $S - H'(I)$. Multiplying equation (7.1) by x^2, substituting $u = x^{-1}v(x,t)$ and then multiplying by x^{-1} we get

$$x^{-1}(x^2\frac{\partial^2}{\partial x^2} + 2x\frac{\partial}{\partial t})(x^{-1}v) = (1/k)\frac{\partial v}{\partial t} \tag{7.3}$$

Now applying the finite spherical Hankel transform defined in (5.1) to (7.3) we get

$$\frac{d\mathcal{V}_{SH}}{dt} + \lambda_n^2 k\mathcal{V}_{SH} = 0, \tag{7.4}$$

where $\mathcal{V}_{SH}$ is a finite spherical Hankel transform of $v(x,t)$. The solution of this equation is given by

$$\mathcal{V}_{SH}(\lambda_n, t) = Cexp(-\lambda_n^2 kt) \tag{7.5}$$

where the constant C can determined from the inial and boundary conditions given in (7.2) . Hence we have

$$\mathcal{V}_{SH}(\lambda_n, t) = \mathcal{F}_{SH}(n)exp(-\lambda_n^2 kt) \tag{7.6}$$

where $\mathcal{F}_{SH}(n)$ is the finite spherical Hankel transform of $f(t)$. Applying inverse finite spherical Hankel transform defined in (5.3), we get

$$v(x) = \sum_{n=0}^{\infty} \mathcal{F}_{SH}(n)exp(-\lambda_n^2 kt)\psi_n(x) \tag{7.7}$$

where the series converges in $S - H'(I)$. In view of Theorem (5.1) and (5.2) the solution $v(x)$ in $S - H'(I)$ exists and is unique. Thus $u(x,t) = x^{-1}v(x,t)$ is the required solution.

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