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On a class of fractional q-Integral inequalities

Z. Dahmani[a,*] and A. Benzidane[b]

[a,b] Laboratory of Pure and Applied Mathematics, LPAM, Faculty SEI, UMAB University of Mostaganem, Algeria.

Abstract

In the present paper, we use the fractional q-calculus to generate some new integral inequalities for some monotonic functions. Other fractional q-integral results, using convex functions, are also presented.

Keywords: Convex function, fractional q-calculus, q-Integral inequalities.

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1 Introduction

The study of the q-integral inequalities play a fundamental role in the theory of differential equations. We refer the reader to [3, 8, 9, 14] for further information and applications. To motivate our work, we shall introduce some important results. The first one is given in [13], where Ngo et al. proved that for any positive continuous function f on $[0,1]$ satisfying $\int_x^1 f(\tau)d\tau \geq \int_x^1 \tau d\tau, x \in [0,1]$, and for $\delta > 0$, the inequalities

$$\int_0^1 f^{\delta+1}(\tau)d\tau \geq \int_0^1 \tau^{\delta} f(\tau)d\tau \tag{1.1}$$

and

$$\int_0^1 f^{\delta+1}(\tau)d\tau \geq \int_0^1 \tau f^{\delta}(\tau)d\tau \tag{1.2}$$

are valid.
In [11], W.J. Liu, G.S. Cheng and C.C. Li proved that

$$\int_a^b f^{\alpha+\beta}(\tau)d\tau \geq \int_a^b (\tau-a)^{\alpha} f^{\beta}(\tau)d\tau, \tag{1.3}$$

for any $\alpha > 0, \beta > 0$ and for any positive continuous function f on $[a,b]$, such that

$$\int_x^b f^{\gamma}(\tau)d\tau \geq \int_x^b (\tau-a)^{\gamma}d\tau;\ \gamma := min(1,\beta), x \in [a,b].$$

Recently, Liu et al. [12] proved another interesting form of integral result, and the following inequality

$$\frac{\int_a^b f^{\beta}(\tau)d\tau}{\int_a^b f^{\gamma}(\tau)d\tau} \geq \frac{\int_a^b (\tau-a)^{\delta} f^{\beta}(\tau)d\tau}{\int_a^b (\tau-a)^{\delta} f^{\gamma}(\tau)d\tau}, \beta \geq \gamma > 0, \delta > 0 \tag{1.4}$$

*Corresponding author.
E-mail addresses: zzdahmani@yahoo.fr (Z. Dahmani) and aekm27@yahoo.fr (A. Benzidane).

(where f is a positive continuous and decreasing function on $[a, b]$), was proved in this paper. Several interesting inequalities can be found in [12].

Many researchers have given considerable attention to (1),(3) and (4) and a number of extensions and generalizations appeared in the literature (e.g. [4, 5, 6, 7, 10, 11, 15, 16]).

The main purpose of this paper is to establish some new fractional q-integral inequalities on the specific time scales $T_{t_0} = \{t : t = t_0 q^n, n \in N\} \cup \{0\}$, where $t_0 \in R$, and $0 < q < 1$. Other fractional q-integral results, involving convex functions, are also presented. Our results have some relationships with those obtained in [12].

2 Notations and Preliminaries

In this section, we provide a summary of the mathematical notations and definitions used in this paper. For more details, one can consult [1, 2].
Let $t_0 \in R$. We define

$$T_{t_0} := \{t : t = t_0 q^n, n \in N\} \cup \{0\}, 0 < q < 1. \tag{2.5}$$

For a function $f : T_{t_0} \to R$, the ∇ q-derivative of f is:

$$\nabla_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} \tag{2.6}$$

for all $t \in T \setminus \{0\}$ and its ∇q-integral is defined by:

$$\int_0^t f(\tau)\nabla\tau = (1-q)t\sum_{i=0}^{\infty} q^i f(tq^i) \tag{2.7}$$

The fundamental theorem of calculus applies to the q-derivative and q-integral. In particular, we have:

$$\nabla_q \int_0^t f(\tau)\nabla\tau = f(t). \tag{2.8}$$

If f is continuous at 0, then

$$\int_0^t \nabla_q f(\tau)\nabla\tau = f(t) - f(0). \tag{2.9}$$

Let T_{t_1}, T_{t_2} denote two time scales. Let $f : T_{t_1} \to R$ be continuous let $g : T_{t_1} \to T_{t_2}$ be q-differentiable, strictly increasing, and $g(0) = 0$. Then for $b \in T_{t_1}$, we have:

$$\int_0^b f(t)\nabla_q g(t)\nabla t = \int_0^{g(b)} (f \circ g^{-1})(s)\nabla s. \tag{2.10}$$

The q-factorial function is defined as follows:
If n is a positive integer, then

$$(t-s)^{\underline{(n)}} = (t-s)(t-qs)(t-q^2 s)...(t-q^{n-1}s). \tag{2.11}$$

If n is not a positive integer, then

$$(t-s)^{\underline{(n)}} = t^n \prod_{k=0}^{\infty} \frac{1-(\frac{s}{t})q^k}{1-(\frac{s}{t})q^{n+k}}. \tag{2.12}$$

The q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)^{\underline{(n)}} = \frac{1-q^n}{1-q}(t-s)^{\underline{(n-1)}}, \tag{2.13}$$

and the q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)^{\underline{(n)}} = -\frac{1-q^n}{1-q}(t-qs)^{\underline{(n-1)}}. \tag{2.14}$$

The q-exponential function is defined as

$$e_q(t) = \prod_{k=0}^{\infty}(1-q^k t), e_q(0) = 1 \tag{2.15}$$

The fractional q-integral operator of order $\alpha \geq 0$, for a function f is defined as

$$\nabla_q^{-\alpha} f(t) = \tfrac{1}{\Gamma_q(\alpha)}\int_0^t (t-q\tau)^{\underline{\alpha-1}} f(\tau)\nabla\tau; \quad \alpha > 0, t > 0, \tag{2.16}$$

where $\Gamma_q(\alpha) := \frac{1}{1-q}\int_0^1 (\frac{u}{1-q})^{\alpha-1} e_q(qu)\nabla u$.

3 Main Results

Theorem 3.1. *Let f and g be two positive and continuous functions on T_{t_0} such that f is decreasing and g is increasing on T_{t_0}. Then for all $\alpha > 0, \beta \geq \gamma > 0, \delta > 0,$ we have*

$$\frac{\nabla_q^{-\alpha}[f^\beta(t)]}{\nabla_q^{-\alpha}[f^\gamma(t)]} \geq \frac{\nabla_q^{-\alpha}[g^\delta f^\beta(t)]}{\nabla_q^{-\alpha}[g^\delta f^\gamma(t)]}, t > 0. \tag{3.17}$$

Proof. Let us consider

$$H(\tau,\rho) := \Big(g^\delta(\rho) - g^\delta(\tau)\Big)\Big(f^\beta(\tau)f^\gamma(\rho) - f^\gamma(\tau)f^\beta(\rho)\Big), \tau,\rho \in (0,t), t > 0. \tag{3.18}$$

We have

$$H(\tau,\rho) \geq 0. \tag{3.19}$$

Hence, we get

$$\begin{aligned}\int_0^t \frac{(t-q\tau)^{\underline{(\alpha-1)}}}{\Gamma_q(\alpha)} H(\tau,\rho)\nabla\tau &= g^\delta(\rho)f^\gamma(\rho)\nabla_q^{-\alpha}[f^\beta(t)] + f^\beta(\rho)\nabla_q^{-\alpha}[g^\delta(t)f^\gamma(t)] \\ &-f^\gamma(\rho)\nabla_q^{-\alpha}[g^\delta(t)f^\beta(t)] - g^\delta(\rho)f^\beta(\rho)\nabla_q^{-\alpha}[f^\gamma(t)] \geq 0.\end{aligned} \tag{3.20}$$

Consequently,

$$\begin{aligned}2^{-1}\int_0^t\int_0^t \frac{(t-q\rho)^{\underline{(\alpha-1)}}(t-q\tau)^{\underline{(\alpha-1)}}}{\Gamma_q^2(\alpha)} H(\tau,\rho)\nabla\tau\nabla\rho &= \nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\alpha}[g^\delta(t)f^\gamma(t)] \\ &-\nabla_q^{-\alpha}[f^\gamma(t)]\nabla_q^{-\alpha}[g^\delta(t)f^\beta(t)) \geq 0.\end{aligned} \tag{3.21}$$

Theorem 3.1 is thus proved.

□

Another result which generalizes Theorem 3.1 is described in the following theorem:

Theorem 3.2. *Suppose that f and g are two positive and continuous functions on T_{t_0}, such that f is decreasing and g is increasing on T_{t_0}. Then for all $\alpha > 0, \omega > 0, \beta \geq \gamma > 0, \delta > 0,$ we have*

$$\frac{\nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\omega}[g^\delta f^\gamma(t)] + \nabla_q^{-\omega}[f^\beta(t)]\nabla_q^{-\alpha}[g^\delta f^\gamma(t)]}{\nabla_q^{-\alpha}[f^\gamma(t)]\nabla_q^{-\omega}[g^\delta f^\beta(t)] + \nabla_q^{-\omega}[f^\gamma(t)]\nabla_q^{-\alpha}[g^\delta f^\beta(t)]} \geq 1; t > 0. \tag{3.22}$$

Proof. The relation (3.20) allows us to obtain

$$\begin{aligned}\int_0^t\int_0^t \frac{(t-q\rho)^{\underline{(\omega-1)}}(t-q\tau)^{\underline{(\alpha-1)}}}{\Gamma_q(\omega)\Gamma_q(\alpha)} H(\tau,\rho)\nabla\tau\nabla\rho &= \nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\omega}[g^\delta f^\gamma(t)] \\ +\nabla_q^{-\omega}[f^\beta(t)]\nabla_q^{-\alpha}[g^\delta f^\gamma(t)] - \nabla_q^{-\alpha}[f^\gamma(t)]\nabla_q^{-\omega}[g^\delta f^\beta(t)) &- \nabla_q^{-\omega}[f^\gamma(t)]\nabla_q^{-\alpha}[g^\delta f^\beta(t)] \geq 0,\end{aligned} \tag{3.23}$$

for any $\omega > 0$.
Hence, we have (3.22).

□

Remark 3.1. *It is clear that Theorem [3.1] would follow as a special case of Theorem [3.2] for* $\alpha = \omega$.

The third result is given by the following theorem:

Theorem 3.3. *Let* f *and* g *be two positive continuous functions on* T_{t_0}, *such that*

$$\Big(f^{\delta}(\tau)g^{\delta}(\rho) - f^{\delta}(\rho)g^{\delta}(\tau)\Big)\Big(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\Big) \geq 0; \tau, \rho \in (0,t), t > 0. \quad (3.24)$$

Then we have

$$\frac{\nabla_q^{-\alpha}[f^{\delta+\beta}(t)]}{\nabla_q^{-\alpha}[f^{\delta+\gamma}(t)]} \geq \frac{\nabla_q^{-\alpha}[g^{\delta}f^{\beta}(t)]}{\nabla_q^{-\alpha}[g^{\delta}f^{\gamma}(t)]}, \quad (3.25)$$

for any $\alpha > 0, \beta \geq \gamma > 0, \delta > 0$.

Proof. We consider the quantity:

$$K(\tau,\rho) := \Big(f^{\delta}(\tau)g^{\delta}(\rho) - f^{\delta}(\rho)g^{\delta}(\tau)\Big)\Big(f^{\gamma}(\rho)f^{\beta}(\tau) - f^{\gamma}(\tau)f^{\beta}(\rho)\Big); \tau, \rho \in (0,t), t > 0$$

and we use the same arguments as in the proof of Theorem [3.1]. □

Using two fractional parameters, we obtain the following generalization of Theorem [3.3]:

Theorem 3.4. *Let* f *and* g *be two positive continuous functions on* T_{t_0}, *such that*

$$\Big(f^{\delta}(\tau)g^{\delta}(\rho) - f^{\delta}(\rho)g^{\delta}(\tau)\Big)\Big(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\Big) \geq 0; \tau, \rho \in (0,t), t > 0. \quad (3.26)$$

Then for all $\alpha > 0, \omega > 0, \beta \geq \gamma > 0, \delta > 0$, *we have*

$$\frac{\nabla_q^{-\alpha}[f^{\delta+\beta}(t)]\nabla_q^{-\omega}[g^{\delta}f^{\gamma}(t)] + \nabla_q^{-\omega}[f^{\delta+\beta}(t)]\nabla_q^{-\alpha}[g^{\delta}f^{\gamma}(t)]}{\nabla_q^{-\alpha}[f^{\gamma+\delta}(t)]\nabla_q^{-\omega}[g^{\delta}f^{\beta}(t)] + \nabla_q^{-\omega}[f^{\gamma+\delta}(t)]\nabla_q^{-\alpha}[g^{\delta}f^{\beta}(t)]} \geq 1. \quad (3.27)$$

Remark 3.2. *Applying Theorem [3.4], for* $\alpha = \omega$, *we obtain Theorem [3.3].*

Involving convex functions, we have the following result:

Theorem 3.5. *Let* f *and* h *be two positive continuous functions on* T_{t_0} *and* $f \leq h$ *on* T_{t_0}. *If* $\frac{f}{h}$ *is decreasing and* f *is increasing on* $[0,\infty[$, *then for any convex function* $\phi; \phi(0) = 0$, *the inequality*

$$\frac{\nabla_q^{-\alpha}(f(t))}{\nabla_q^{-\alpha}(h(t))} \geq \frac{\nabla_q^{-\alpha}(\phi(f(t)))}{\nabla_q^{-\alpha}(\phi(h(t)))}, t > 0, \alpha > 0 \quad (3.28)$$

is valid.

Proof. Using the fact that on T_{t_0}, $\frac{\phi(f(.))}{f(.)}$ is an increasing function and $\frac{f}{h}$ is a decreasing function, we can write

$$L(\tau,\rho) \geq 0, \tau, \rho \in (0,t), t > 0, \quad (3.29)$$

where

$$L(\tau,\rho) := \frac{\phi(f(\tau))}{f(\tau)}f(\rho)h(\tau) + \frac{\phi(f(\rho))}{f(\rho)}f(\tau)h(\rho)$$
$$-\tfrac{\phi(f(\rho))}{f(\rho)}f(\rho)h(\tau) - \tfrac{\phi(f(\tau))}{f(\tau)}f(\tau)h(\rho), \tau, \rho \in (0,t), t > 0. \quad (3.30)$$

Multiplying both sides of (3.29) by $\frac{(t-q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)}$, then integrating the resulting inequality with respect to τ over $(0,t)$, yields

$$f(\rho)\nabla_q^{-\alpha}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] + \frac{\phi(f(\rho))}{f(\rho)}h(\rho)\nabla_q^{-\alpha}f(t)$$
$$-\tfrac{\phi(f(\rho))}{f(\rho)}f(\rho)\nabla_q^{-\alpha}h(t) - h(\rho)\nabla_q^{-\alpha}\Big[\tfrac{\phi(f(t))}{f(t)}f(t)\Big] \geq 0. \quad (3.31)$$

With the same arguments as before, we obtain

$$\nabla_q^{-\alpha} f(t)\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] - \nabla_q^{-\alpha} h(t)\nabla_q^{-\alpha}\Big[\frac{\phi(f(t))}{f(t)}f(t)\Big] \geq 0. \tag{3.32}$$

On the other hand, we have

$$\frac{\phi(f(\tau))}{f(\tau)} \leq \frac{\phi(h(\tau))}{h(\tau)}, \tau \in (0,t), t>0. \tag{3.33}$$

Therefore,

$$\frac{(t-q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(\tau)\frac{\phi(f(\tau))}{f(\tau)} \leq \frac{(t-q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(\tau)\frac{\phi(h(\tau))}{h(\tau)}, \tau \in (0,t), t>0. \tag{3.34}$$

The inequality (3.34) implies that

$$\nabla_q^{-\alpha}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] \leq \nabla_q^{-\alpha}\Big[\frac{\phi(h(t))}{h(t)}h(t)\Big]. \tag{3.35}$$

Combining (3.32) and (3.35), we obtain (3.28). □

To finish, we present to the reader the following result which generalizes the previous theorem:

Theorem 3.6. *Let f and h be two positive continuous functions on on T_{t_0} and $f \leq h$ on T_{t_0}. If $\frac{f}{h}$ is decreasing and f is increasing on T_{t_0}, then for any convex function $\phi; \phi(0) = 0$, we have*

$$\frac{\nabla_q^{-\alpha}(f(t))\nabla_q^{-\omega}(\phi(h(t))) + \nabla_q^{-\omega}(f(t))\nabla_q^{-\alpha}(\phi(h(t)))}{\nabla_q^{-\alpha}(h(t))\nabla_q^{-\omega}(\phi(f(t))) + \nabla_q^{-\omega}(h(t))\nabla_q^{-\alpha}(\phi(f(t)))} \geq 1, \alpha>0, \omega>0, t>0. \tag{3.36}$$

Proof. The relation (3.31) allows us to obtain

$$\begin{gathered}\nabla_q^{-\omega} f(t) J^{\alpha}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] + \nabla_q^{-\omega}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big]\nabla_q^{-\alpha} f(t) \\ -\nabla_q^{-\omega}\Big[\tfrac{\phi(f(t))}{f(t)}f(t)\Big]\nabla_q^{-\alpha}h(t) - \nabla_q^{-\omega}h(t)\nabla_q^{-\alpha}\Big[\tfrac{\phi(f(t))}{f(t)}f(t)\Big] \geq 0.\end{gathered} \tag{3.37}$$

On the other hand, we have:

$$\frac{(t-q\tau)^{(\omega-1)}}{\Gamma_q(\omega)}h(\tau)\frac{\phi(f(\tau))}{f(\tau)} \leq \frac{(t-q\tau)^{(\alpha-1)}}{\Gamma_q(\omega)}h(\tau)\frac{\phi(h(\tau))}{h(\tau)}, \tau \in [0,t], t>0. \tag{3.38}$$

Integrating both sides of (3.38) with respect to τ over $(0,t)$, yields

$$\nabla_q^{-\omega}\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] \leq \nabla_q^{-\omega}\Big[\frac{\phi(h(t))}{h(t)}h(t)\Big]. \tag{3.39}$$

By (3.35), (3.37) and (3.39), we get (3.36). □

Remark 3.3. *Applying Theorem [3.6], for $\alpha = \omega$, we obtain Theorem [3.5].*

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