Solutions of fractional difference equations using S-transforms

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Abstract

In the present paper, we define the nabla discrete Sumudu transform (S-transform) and present some of its basic properties. We obtain the nabla discrete Sumudu transform of fractional sums and differences. We apply this transform to solve some fractional difference equations with initial value problems. Finally, using S-transforms, we prove that discrete Mittag-Leffler function is the eigen function of Caputo type fractional difference operator ∇^α.

Keywords: Difference equation, fractional difference, Caputo type, initial value problem, Sumudu transform.

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1 Introduction

In the literature there are numerous integral transforms that are widely used in physics, astronomy, as well as engineering. In order to solve the differential equations, the integral transforms were extensively used and thus there are several works on the theory and application of integral transforms such as the Laplace, Fourier, Mellin, and Hankel, to name but a few. In the sequence of these transforms in early 90s, Watugala [13] introduced a new integral transform named the Sumudu transform and further applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of the functions

$$A = \{f(t) : \exists M, \tau_1, \tau_2, |f(t)| < Me^{\frac{t}{\tau_j}}, \quad \text{if} \quad t \in (-1)^j \times [0, \infty)\}$$

by the following formula

$$G(u) = S[f(t); u] = \int_0^\infty f(ut)e^{-t}dt, \quad t \in (-\tau_1, \tau_2).$$

The existence and uniqueness and properties of the Sumudu transform and its derivatives were discussed in [2, 3, 4, 5, 8, 9, 11]. Although the Sumudu transform of a function has a deep connection to its Laplace transform, the main advantage of the Sumudu transform is the fact that it may be used to solve problems without resorting to a new frequency domain because it preserves scales and unit properties. By these properties, the Sumudu transform may be used to solve intricate problems in engineering and applied sciences that can hardly be solved when the Laplace transform is used. Moreover, some properties of the Sumudu transform make it more advantageous than the Laplace transform.

Fractional calculus has gained importance during the past three decades due to its applicability in diverse fields of science and engineering, such as, viscoelasticity, diffusion, neurology, control theory, and statistics. The analogous theory for discrete fractional calculus was initiated and properties of the theory of fractional sums and differences were established. Recently, a series of papers continuing this research has appeared in

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which G.V.S.R.Deekshitulu and J.Jagan Mohan discussed some basic inequalities, comparison theorems and qualitative properties of the solutions of fractional difference equations [6, 7].

Now, we introduce some basic definitions and results concerning nabla discrete fractional calculus. Throughout the article, for notations and terminology we refer [1]. Let $u(n) : \mathbb{N}_0^+ \to \mathbb{R}$ and $m - 1 < \alpha < m$ where $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}^+$.

Definition 1.1. *The fractional sum operator of order α is defined as*

$$\nabla^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j + \alpha - 1}{j} u(n - j) = \sum_{j=1}^{n} \binom{n - j + \alpha - 1}{n - j} u(j). \tag{1.1}$$

Definition 1.2. *The Caputo type fractional difference operator of order α is defined as*

$$\nabla^{\alpha} u(n) \quad = \quad \nabla^{\alpha - m}[\nabla^m u(n)] = \sum_{j=0}^{n-1} \binom{j - \alpha + m - 1}{j} \nabla^m u(n - j) \tag{1.2}$$

$$= \quad \sum_{j=1}^{n} \binom{n - j - \alpha - 1}{n - j} u(j) - \sum_{k=0}^{m-1} \binom{n + k - \alpha - 1}{n - 1} [\nabla^k u(j)]_{j=0}. \tag{1.3}$$

2   S-Transforms and Properties

Now we initiate the study of S-transforms in the present section. Let $u(n), v(n) : \mathbb{N}_0^+ \to \mathbb{R}$.

Definition 2.1. *The S-transform of $u(n)$ is defined as*

$$S[u(n)] = \frac{1}{z} \sum_{j=1}^{\infty} u(j)(1 - \frac{1}{z})^{j-1} \tag{2.1}$$

for each $z \in \mathbb{C} \setminus \{0\}$ for which the series converges.

Definition 2.2. *A function $u(n)$ is of exponential order r, $r > 0$ if there exists a constant $A > 0$ such that $|u(n)| \le Ar^{-n}$ for sufficiently large n.*

The following lemma discusses the convergence of S-transform.

Lemma 2.1. *Suppose $u(n)$ is of exponential order r, $r > 0$. Then $S[u(n)]$ exists for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$.*

Now we derive some important properties of S-transforms.

Theorem 2.1. *(Linearity) For any constants a and b,*

$$S[au(n) + bv(n)] = aS[u(n)] + bS[v(n)]. \tag{2.2}$$

The following lemma relates the shifted S-transform to the original.

Lemma 2.2. *(Shifting Theorem) Let $k \in \mathbb{N}_0^+$ and let $u(n)$ and $v(n)$ are of exponential order r, $r > 0$. Then for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$,*

$$S[u(n - k)] = (1 - \frac{1}{z})^k S[u(n)]. \tag{2.3}$$

and

$$S[u(n + k)] = (1 - \frac{1}{z})^{-k} \frac{1}{z} \Big[zS[u(n)] - u(1) - (1 - \frac{1}{z})^1 u(2) - ..... - (1 - \frac{1}{z})^{k-1} u(k) \Big]. \tag{2.4}$$

Definition 2.3. *The convolution of $u(n)$ and $v(n)$ is defined as*

$$u(n) * v(n) = \sum_{m=1}^{n} u(m)v(n - m + 1). \tag{2.5}$$

Lemma 2.3. *(Convolution Theorem) Let $u(n)$ and $v(n)$ are of exponential order r, $r > 0$. Then for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$,*

$$S[u(n) * v(n)] = zS[u(n)]S[v(n)]. \tag{2.6}$$

Proof. Consider

$$
\begin{aligned}
S[u(n) * v(n)] &= \frac{1}{z} \sum_{j=1}^{\infty} [u(j) * v(j)] \, (1 - \frac{1}{z})^{j-1} \\
&= \frac{1}{z} \sum_{j=1}^{\infty} \Big[\sum_{m=1}^{j} u(m)v(j - m + 1) \Big] (1 - \frac{1}{z})^{j-1} \\
&= z \Big[\frac{1}{z} \sum_{m=1}^{\infty} u(m)(1 - \frac{1}{z})^{m-1} \Big] \Big[\frac{1}{z} \sum_{j=1}^{\infty} v(j - m + 1)(1 - \frac{1}{z}z)^{j-m} \Big].
\end{aligned}
$$

Take $j - m + 1 = i$ then i varies from 1 to ∞. Then

$$
S[u(n) * v(n)] = z \Big[\frac{1}{z} \sum_{m=1}^{\infty} u(m)(1 - z)^{m-1} \Big] \Big[\frac{1}{z} \sum_{i=1}^{\infty} v(i)(1 - z)^{i-1} \Big] = zS[u(n)]S[v(n)].
$$

$\square$

Henry L Gray and Nien fan Zhang [10] defined the following function, which is very useful in solving initial value problems

Definition 2.4. *For any complex numbers α and β, $(\alpha)_{\beta}$ be defined as follows.*

$$
(\alpha)_{\beta} = \begin{cases}
\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} & \text{when } \alpha \text{ and } \alpha + \beta \text{ are neither zero nor negative integers,} \\
1 & \text{when } \alpha = \beta = 0, \\
0 & \text{when } \alpha = 0, \ \beta \text{ is neither zero nor negative integer,} \\
undefined & \text{otherwise.}
\end{cases}
$$

Remark 2.1. *It is clear from the above definition that*

$$(\alpha)_{\beta} = \Gamma(\beta + 1) \binom{\alpha + \beta - 1}{\alpha - 1} = \Gamma(\beta + 1) \binom{\alpha + \beta - 1}{\beta}. \tag{2.7}$$

Lemma 2.4. *Let $a \in \mathbb{R} \setminus \{....., -2, -1\}$ and $n \in \mathbb{N}_0^+$. Then for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$,*

$$S[(n)_a] = \Gamma(a + 1)z^a. \tag{2.8}$$

Proof. Consider

$$
\begin{aligned}
S[(n)_a] = S\Big[\frac{\Gamma(n + a)}{\Gamma(n)} \Big] &= \frac{1}{z} \sum_{j=1}^{\infty} \frac{\Gamma(j + a)}{\Gamma(j)} (1 - \frac{1}{z})^{j-1} \\
&= \frac{\Gamma(a + 1)}{z} \sum_{j=1}^{\infty} \binom{j + a - 1}{j - 1} (1 - \frac{1}{z})^{j-1} \\
&= \frac{\Gamma(a + 1)}{z} \Big[1 + (1 + a) + \frac{(1 + a)(2 + a)}{2!} (1 - \frac{1}{z})^1 + ... \Big] \\
&= \frac{\Gamma(a + 1)}{z} [1 - (1 - \frac{1}{z})]^{-a-1} = \Gamma(a + 1)z^a.
\end{aligned}
$$

$\square$

Remark 2.2. *From the above lemma, we get*

$$S\Big[\binom{n + a - 1}{n - 1} \Big] = z^a. \tag{2.9}$$

Lemma 2.5. *Suppose $u(n)$ is of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$. Then for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$,*

$$S\left[\nabla^{-\alpha} u(n)\right] = z^{\alpha} S[u(n)]. \tag{2.10}$$

Proof. Consider

$$
\begin{aligned}
S\left[\nabla^{-\alpha} u(n)\right] = S\left[\sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u(j)\right] &= \frac{1}{\Gamma(\alpha)} S\left[\sum_{j=1}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} u(j)\right] \\
&= \frac{1}{\Gamma(\alpha)} S\left[\sum_{j=1}^{n} (n-j+1)_{\alpha-1} u(j)\right] \\
&= \frac{1}{\Gamma(\alpha)} S\left[u(n) * (n)_{\alpha-1}\right] \\
&= \frac{z}{\Gamma(\alpha)} S[u(n)] S\left[(n)_{\alpha-1}\right] = z^{\alpha} S[u(n)].
\end{aligned}
$$

$\square$

Lemma 2.6. *Suppose $u(n)$ is of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}^{+}$ such that $m-1 < \alpha < m$. Then for all $z \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{z}| < r$,*

$$S\left[\nabla^{\alpha} u(n)\right] = z^{-\alpha}\left[S[u(n)] - \sum_{k=0}^{m-1} z^{k}[\nabla^{k} u(j)]_{j=0}\right]. \tag{2.11}$$

Proof. Consider

$$S\left[\nabla^{\alpha} u(n)\right] = S\left[\sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^{k} u(j)]_{j=0}\right] = S_1 + S_2$$

where

$$S_1 = S\left[\sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j)\right] = z^{-\alpha} S[u(n)]$$

and

$$S_2 = S\left[\sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^{k} u(j)]_{j=0}\right] = \sum_{k=0}^{m-1} S\left[\binom{n+k-\alpha-1}{n-1}\right] [\nabla^{k} u(j)]_{j=0}.$$

Now we consider

$$S\left[\binom{n+k-\alpha-1}{n-1}\right] = \frac{1}{\Gamma(k-\alpha+1)} S\left[(n)_{k-\alpha}\right] = z^{k-\alpha}.$$

Thus

$$S\left[\nabla^{\alpha} u(n)\right] = z^{-\alpha}\left[S[u(n)] - \sum_{k=0}^{m-1} z^{k}[\nabla^{k} u(j)]_{j=0}\right].$$

$\square$

3   Solutions of fractional difference equations using S-transforms

In this section, we will illustrate the possible use of the S-transform by applying it to solve some fractional order initial value problems.

In 2003, Atsushi Nagai [12] defined the discrete Mittag-Leffler function

$$F_{\alpha}(a, n) = \sum_{j=0}^{\infty} \left[a^{j} \binom{n+j\alpha-1}{n-j}\right] \tag{3.1}$$

which is a generalization of nabla exponential function on the time scale of integers. He also proved that $F_{\alpha}(a, n)$ is an eigen function of Caputo type fractional difference operator defined in (1.3), that is,

$$\nabla^{\alpha} F_{\alpha}(a, n) = a F_{\alpha}(a, n). \tag{3.2}$$

Now we prove the same using S-transforms.

Example 3.1. *Let $u(n)$ is of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Then the solution of*

$$\nabla^\alpha u(n) = au(n), \tag{3.3}$$
$$u(0) = a_0 \tag{3.4}$$

is $F_\alpha(a, n)$, where a and a_0 are constants.

Solution: Taking S-transforms on both sides of (3.3), we have

$$S[\nabla^\alpha u(n)] = aS[u(n)]$$
$$\text{or } z^{-\alpha}\Big[S[u(n)] - u(0)\Big] = aS[u(n)]$$
$$\text{or } S[u(n)] = a_0\Big[\frac{z^{-\alpha}}{z^{-\alpha} - a}\Big]$$
$$\text{or } S[u(n)] = a_0\Big[1 + az^\alpha + a^2 z^{2\alpha} + \text{...............}\Big].$$

Applying inverse S-transforms on both sides, we get

$$u(n) = a_0 S^{-1}\Big[1 + az^\alpha + a^2 z^{2\alpha} + \text{...............}\Big]$$
$$= a_0\Big[S^{-1}\big(1\big) + aS^{-1}\big(z^\alpha\big) + a^2 S^{-1}\big(z^{2\alpha}\big) + \text{...............}\Big]$$
$$= a_0\Big[1 + a\binom{n + \alpha - 1}{n - 1} + a^2\binom{n + 2\alpha - 1}{n - 2} + \text{...............}\Big]$$
$$\text{or } u(n) = a_0\sum_{j=0}^{\infty}\Big[a^j\binom{n + j\alpha - 1}{n - j}\Big] = a_0 F_\alpha(a, n).$$

Thus the solution of (3.3) is the discrete Mittag-Leffler function defined in (3.1).

Remark 3.3. *It is clear from the above example that*

$$S\Big[F_\alpha(a, n)\Big] = \frac{z^{-\alpha}}{z^{-\alpha} - a}. \tag{3.5}$$

Example 3.2. *Let $u(n)$ and $v(n)$ are of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Find the solution of*

$$\nabla^\alpha u(n) = av(n), \tag{3.6}$$
$$u(0) = a_0 \tag{3.7}$$

where a and a_0 are constants.

Solution: Taking S-transforms on both sides of (3.6), we have

$$z^{-\alpha}\Big[S[u(n)] - u(0)\Big] = aS[v(n)]$$
$$\text{or } S[u(n)] = a_0 + a\Big[S[v(n)] \times z^\alpha\Big].$$

Applying inverse S-transforms on both sides and applying convolution theorem, we get

$$u(n) = a_0 + S^{-1}\Big[z \times S[v(n)] \times z^{\alpha-1}\Big]$$
$$= a_0 + \Big[v(n) * \binom{n + \alpha - 2}{n - 1}\Big]$$
$$\text{or } u(n) = a_0 + \sum_{j=1}^{n}\Big[v(j)\binom{n - j + \alpha - 1}{n - j}\Big].$$

Example 3.3. *Let $u(n)$ and $v(n)$ are of exponential order r, $r > 0$ and let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Find the solution of*

$$\nabla^\alpha u(n) = au(n) + bv(n), \tag{3.8}$$

$$u(0) = a_0 \tag{3.9}$$

where a, b and a_0 are constants.

Solution: Taking S-transforms on both sides of (3.8), we have

$$z^{-\alpha}\Big[S[u(n)] - u(0) \Big] = aS[u(n)] + bS[v(n)]$$

$$or \ S[u(n)] = a_0 \Big[\frac{z^{-\alpha}}{z^{-\alpha} - a}\Big] + b\Big[S[v(n)] \times z^\alpha\Big].$$

Applying inverse S-transforms on both sides and applying convolution theorem, we get

$$u(n) = a_0 S^{-1}\Big[\frac{z^{-\alpha}}{z^{-\alpha} - a}\Big] + bS^{-1}\Big[z \times S[v(n)] \times z^{\alpha-1}\Big]$$

$$= a_0 F_\alpha(a, n) + b\sum_{j=1}^{n}\Big[v(j)\binom{n - j + \alpha - 1}{n - j}\Big].$$

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