Disjunctive total domination in some tree networks

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Abstract. Networking has been an essential field of multidisciplinary study, including computational theory, mathematics, social sciences, computer science, and other theoretical and applied sciences. The vulnerability determines the network's resistance to interruption of information flow after the breakdown of particular stations or transmission connections. Recently, new vulnerability parameter namely the disjunctive total domination number has been defined by Henning and Naicker [14]. This measure finds the critical vertices with an important position in the graph. In this context, we consider and compute exact formulae for the disjunctive total domination number in some tree networks.

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Contents

1. Introduction

The connected graph can model the network, the vertex in the graph represents a network node, and the edge represents a contact connection between the two nodes [7, 16]. There are many parameters in graph theory for network analysis and to determine some properties of the network. Graph theory techniques facilitate representation and analysis during a vulnerability assessment of complex networks. The theory is based on a set of measurements that evaluate networks and include graph vulnerability parameters. The proposed solutions for the network's vulnerability were rooted in the graph theoretical principles, especially the concepts of domination [19].

Theory of domination is one of the most important branches of graph vulnerability, which has wide application in network designings. It has a wide variety of uses in many areas, such as computer science, communication networks, transportation networks, biological and social networks, operations research, chemistry, economics, engineering, and applied mathematics; the principle of domination has recently become the center of graph theory research activity. This is largely due to a variety of new parameters that can be developed from the basic definition of domination [10, 14, 15]. Disjunctive total domination is the new domination parameter defined recently. Henning and Naicker [14] defined the disjunctive total domination as a relaxation of total domination.

A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G and is denoted

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by $\gamma(G)$ [3, 10]. A *total dominating set*, abbreviated a TD-set, of a graph G, with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G [3, 15]. Let $S \subseteq V(G)$. The set S can be the *disjunctive total dominating set* of the graph G if and only if it satisfies any of the following properties. For every vertex $v \in V(G)$;

    (i) v is adjacent to a vertex of S,

    (ii) at least two vertices at a distance of 2 from the vertex v must be in the set S.

The *disjunctive total dominating set* of the graph G is briefly called DDT-set. The *disjunctive total domination number* of G is the minimum cardinality of a DTD-set of G and denoted by $\gamma_t^d(G)$. A DTD-set of cardinality $\gamma_t^d(G)$ is called a $\gamma_t^d(G)$-set. Clearly, every TD-set is a DTD-set, furthermore the result $\gamma_t^d(G) \leq \gamma_t(G)$ is obtained in [12–14]. This parameter is studied on grids, trees, permutation graphs, claw-free graphs, shadow distance graph of some special graphs and it is applied on some graph modifications such as bondage and subdivision [1, 2, 12–14].

We consider the disjunctive total domination number as a metric for network vulnerability. In this model, we find the critical vertices with an important position in the graph. Since disjunctive total domination number is considered to be a reasonable measure for the vulnerability of graphs, it is of particular interest to evaluate the disjunctive total domination number of different classes of graphs. Suppose one can break a more complex network into smaller networks, then under some conditions. In that case, the optimization problem's solutions on the smaller networks can be combined to solve the optimization problem on the larger network. Thus, calculation of the disjunctive total domination number for simple graph types is important.

For notation and graph theory terminology, we in general follow [10, 19]. Specifically, let $G = (V(G), E(G))$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The set of all adjacent vertices to vertex $v \in V(G)$ in G is called neighborhood and denoted by $N_G(v)$ or $N(v)$. The close neighborhood of this vertex is defined as $N_G(v) \cup \{v\}$ and denoted by $N_G[v]$ or $N[v]$. The other basic parameter for graphs is the degree of vertex $v \in V(G)$, which is defined as the number of vertices in $N_G(v)$ and denoted by $deg(v)$. Assume that the vertices u and v belong to the graph G. For these vertices, $d(u, v)$ is defined as the distance of the shortest path between these vertices.Furthermore, $diam(G)$ is defined as the *diameter* of G, and it is the highest distance value within the vertices of G. $\Delta(G) = max\{deg(v) | v \in V(G)\}$ and $\delta(G) = min\{deg(v) | v \in V(G)\}$ represent the maximum and minimum degree, respectively. The vertex with $deg(v) = 1$ is said to be a pendant vertex or leaf vertex. The vertex adjacent to the pendant vertex is called the support vertex.

Now, we make use of the following known theorems in our results.

Theorem 1.1. *[14] For $n \geq 3$, $\gamma_t^d(C_n) = \frac{2n}{5}$ if $n \equiv 0 (mod 5)$; and $\gamma_t^d(C_n) = \lceil \frac{2(n+1)}{5} \rceil$, otherwise.*

Theorem 1.2. *[14] For $n \geq 3$, $\gamma_t^d(P_n) = \lceil \frac{2(n+1)}{5} \rceil + 1$ if $n \equiv 1 (mod 5)$; and $\gamma_t^d(P_n) = \lceil \frac{2(n+1)}{5} \rceil$, otherwise.*

Lemma 1.3. *[14] If v is a support vertex in a graph G with exactly one neighbor w that is not a leaf, then there is a $\gamma_t^d(G)$-set that contains v. Further if $deg(w) = 2$, then there is a $\gamma_t^d(G)$-set that contains both v and w.*

2. Disjunctive Total Domination Numbers of Some Trees

In this section, the distinctive total dominance numbers of certain tree-type networks such as the double comet graph, the double star graph, the comet graph, the generalized caterpillars, the comb graph, the thorn graph P_n^*, the binomial tree and the complete k-ary tree are computed and exact formulae are presented.

Definition 2.1. *[8] The double star graph $S(x, y)$, where $x, y \geq 0$, is the graph consisting of the union of two star graphs $K_{1,x}$ and $K_{1,y}$ together with an edge joining their centers.*

Theorem 2.2. *If $G \cong S(x, y)$ of order $x + y$, where $x, y \geq 0$, then, $\gamma_t^d(G) = 2$.*

Proof. Note that $V(G) = V(K_{1,x}) \cup V(K_{1,y})$. Furthermore, let u_1 and u_2 be the central vertices of $K_{1,x}$ and $K_{1,y}$, respectively. It is easily seen that $deg(x_i) = deg(y_i) = 1$ for every vertices x_i and y_i, where $x_i \in V(K_{1,x})$ and $y_i \in V(K_{1,y})$, $deg(u_1) = x + 1$ and $deg(u_2) = y + 1$. If a DTD-set of G is considered S, then taking u_1 and u_2 to the set S yields $\gamma_t^d(G) \leq 2$. Furthermore, we have $\gamma_t^d(G) \geq 2$ for any graph G by the definition of disjunctive total domination number. So, $\gamma_t^d(G) \geq 2$. As a consequence, by combining the lower and upper bounds, we obtain $\gamma_t^d(G) = 2$. ∎

Definition 2.3. *[4] The comet graph $C(t,r)$ is the graph obtained by identifying one end of the path P_t with the center of the star graph $K_{1,r}$. This graph is illustrated in Figure 1.*
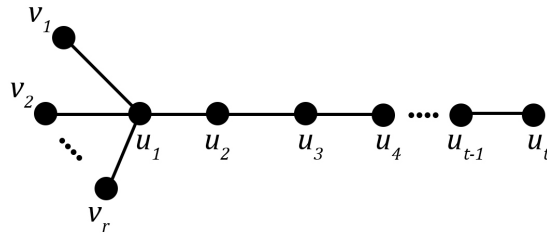

Figure 1: The comet graph $C(t,r)$.

Theorem 2.4. *If $G \cong C(t,r)$ of order $t + r$, then* $\gamma_t^d(G) = \begin{cases} \left\lceil \frac{2(t+1)}{5} \right\rceil + 1 \text{ , if } t \equiv 0,1,4(mod5); \\[2ex] \left\lceil \frac{2(t+1)}{5} \right\rceil \quad \text{ , otherwise.} \end{cases}$

Proof. Note that $V(G) = V(P_{t-1}) \cup V(K_{1,r})$. Furthermore, $V(P_{t-1}) = \{u_2, u_3, ..., u_t\}$ and $V(K_{1,r}) = \{u_1, v_1, ..., v_r\}$, where u_1 is the center vertex. Suppose S is a DTD-set in G. By Lemma 1.3, u_1 must be in S. Thus, all vertices v_i and u_2 are disjunctively totally dominated by the vertex u_1. The disjunctive total undominated vertices by S are the vertices of the path graph with $(t-1)$ vertices. So, the rest of the proof has to be made similar to the proof of Theorem 1.2. In this case, if $t \equiv 0,1,4(mod5)$, then $\gamma_t^d(G) = |S| = \lceil 2(t+1)/5 \rceil + 1$, while if $t \equiv 2,3(mod5)$,then $\gamma_t^d(G) = |S| = \lceil 2(t+1)/5 \rceil$ are obtained. ∎

Definition 2.5. *[5] The graph obtained by adding x and y vertices, which are pendant, to the end vertices of the path graph with $n - x - y$ vertices is called double comet $DC(n,x,y)$. For $x,y \geq 1$ and $n \geq x + y + 2$ the double comet $DC(n,x,y)$ is one of the tree graphs. $DC(n,x,y)$ is a graph with n vertices, $x + y$ of which is leaves. This graph is illustrated in Figure 2.*

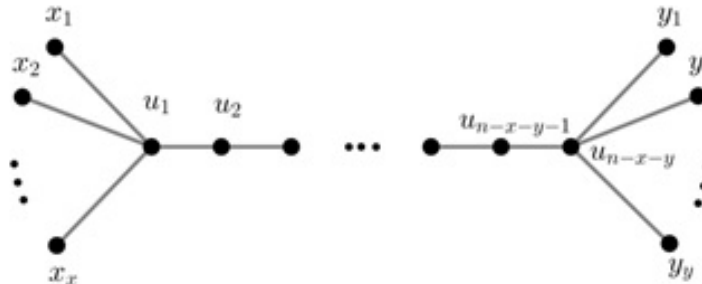Figure 2: The double comet graph $DC(n,x,y)$.

Theorem 2.6. *If $G \cong DC(n, x, y)$ with $x, y \geq 2$ and $n \geq x + y + 3$, then*

$$
\gamma_t^d(G) = \begin{cases} \left\lceil \dfrac{2(n-x-y+1)}{5} \right\rceil + 2 \,, \textit{if } (n-x-y) \equiv 4(mod 5); \\[2em] \left\lceil \dfrac{2(n-x-y+1)}{5} \right\rceil \qquad , \textit{if } (n-x-y) \equiv 2(mod 5); \\[2em] \left\lceil \dfrac{2(n-x-y+1)}{5} \right\rceil + 1 \,, \textit{otherwise.} \end{cases}
$$

Proof. Note that $V(G) = V(P_{n-x-y-2}) \cup V(K_{1,x}) \cup V(K_{1,y})$ in which $V(P_{n-x-y-2}) = \{u_2, u_3, ..., u_{n-x-y-1}\}$, $V(K_{1,x}) = \{u_1, x_1, x_2..., x_x\}$ and $V(K_{1,y}) = \{u_{n-x-y}, y_1, y_2..., y_y\}$. Suppose S is a DTD-set in G. By Lemma 1.3, the vertices u_1 and u_{n-x-y} must be taken to the set S. Thus the vertices not disjunctively totally dominated by the set S form the path graph. As in the proof of Theorem 1.2, the construction of the set S is continued. In this case, if $(n-x-y) \equiv 4(mod 5)$, then $\gamma_t^d(G) = |S| = \lceil (2(n-x-y+1)/5) \rceil + 2$; if $(n-x-y) \equiv 2(mod 5)$, then $\gamma_t^d(G) = |S| = \lceil (2(n-x-y+1)/5) \rceil$ and otherwise $\gamma_t^d(G) = |S| = \lceil (2(n-x-y+1)/5) \rceil + 1$ are obtained. ∎

Definition 2.7. *[17] The graph obtained by joining a pendant edge at each vertex of a path P_n is called a comb graph and is denoted by $P_n \square K_1$. The graph $P_5 \square K_1$ is illustrated in Figure 3.*
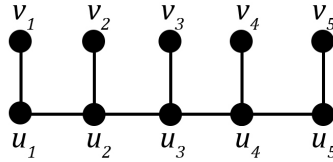

Figure 3: The comb graph $P_5 \square K_1$.

Theorem 2.8. *If $G \cong P_n \square K_1$ of order $2n$, then $\gamma_t^d(G) = 4 + \lfloor (n-4)/2 \rfloor$.*

Proof. Note that $V(G) = \{u_i, v_i \mid 1 \leq i \leq n\}$ and $E(G) = \{u_i v_i \mid 1 \leq i \leq n\} \cup \{u_i u_{i+1} \mid 1 \leq i \leq n-1\}$. It is obvious that $|V(G)| = 2n$, $|E(G)| = 2n-1$, $deg(v_i) = 1$ where $i \in \{1, ..., n\}$, $deg(u_1) = deg(u_n) = 2$ and $deg(u_i) = 3$ where $i \in \{2, ..., n-1\}$. We set the upper limits to the disjunctive total domination number of G, first. Suppose D is a DTD-set in G. According to degree of vertices of G, some u_i-vertices ($i \in \{2, ..., n-1\}$) must be taken to the set D. To disjunctively totally dominate the vertices v_1 and v_n, it must be $\{u_1, u_n\} \subseteq D$. Since $N_G(u_1) = \{v_1, u_2\}$ and $N_G(u_n) = \{v_n, u_{n-1}\}$, $\{u_2, u_{n-1}\} \subset D$ should be to disjunctively totally dominate the vertices u_1 and u_n. Thus, the set D is as follows:

$$
D = \bigcup_{i=0}^{\lfloor \frac{n-4}{2} \rfloor + 1} \{u_{2i+4}\} \cup \{u_1, u_2, u_{n-1}, u_n\}.
$$

Clearly, the set D is a DTD-set for every $n \geq 5$. Furthermore, we get $|D| = 4 + \lfloor (n-4)/2 \rfloor$, also is an upper bound. Thus, $\gamma_t^d(G) \leq 4 + \lfloor (n-4)/2 \rfloor$ is obtained.
To prove the inverse of equality, let the set T be a $\gamma_t^d(G)$-set of G. Assume that the two vertices are adjacent in T. Furthermore, the set S is as follows:

$$
S = \bigcup_{i=0}^{\lfloor \frac{n}{3} \rfloor - 1} \{u_{3i+1}, u_{3i+2}\}
$$

, where $S \subseteq T$. If $n \equiv 1(mod3)$, we have $T = S \cup \{v_{n-1}, v_n\}$; if $n \equiv 0(mod3)$, we have $T = S \cup \{v_n\}$ and if $n \equiv 2(mod3)$, we have $T = S$. Thus, we obtain $|T| = 2\lfloor n/3 \rfloor + 2$ for $n \equiv 1(mod3)$, $|T| = 2\lfloor n/3 \rfloor + 1$ for $n \equiv 0(mod3)$ and $|T| = 2\lfloor n/3 \rfloor$ for $n \equiv 2(mod3)$. These results contradict the previous upper bound for $n \geq 7$.

Furthermore, we get $\{u_1, u_2, u_{n-1}, u_n\} \subset T$. However, apart from these vertices, no two vertices in T should be adjacent to each other. If the distance between the two vertices is at least three, all vertices in G cannot be disjunctively totally dominated. So, the distance between two vertices must be exactly 2. Thus, it is easy to see that $\gamma_t^d(G) \geq 4 + \lfloor (n-4)/2 \rfloor$, also we have $\gamma_t^d(G) = 4 + \lfloor (n-4)/2 \rfloor$. ∎

Corollary 2.9. If $G \cong P_n \square K_1$ of order $2n$, then $\gamma_t^d(G) = 4 + \gamma(P_{n-4})$.

Definition 2.10. *[9] Let $p_1, p_2, \ldots, p_n$ be non-negative integers and the graph G be such a graph, where $|V(G)| = n$. The thorn graph of the graph G with parameters $p_1, p_2, \ldots, p_n$ is obtained by attaching p_i new vertices of degree one to the vertex u_i of the graph G, where $i = \overline{1, n}$. The thorn graph of the graph G will be denoted by G^* or if the respective parameters need to be specified, by $G^*(p_1, p_2, \ldots, p_n)$. The graph $P_7^*(2, 1, 1, 3, 2, 1, 4)$ is illustrated in Figure 4.*
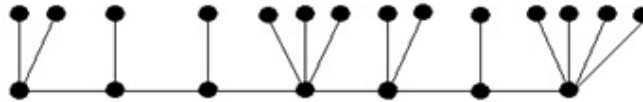

Figure 4: The thorn graph $P_7^*(2, 1, 1, 3, 2, 1, 4)$.

Theorem 2.11. *If $G \cong P_n^*$ is a thorn graph of P_n with $p_i \geq 2$, then $\gamma_t^d(G) = \gamma_t^d(P_n \square K_1)$.*

Proof. The proof is quite close to that of Theorem 2.8, so we omit it. ∎

Definition 2.12. *[18] $C_{(t,0)}P_n$ is a generalized Caterpillar obtained from the path graph P_n by attaching t vertices of degree one to each vertex of degree two of P_n. The tree $C_{(t,1)}P_n$ is a generalized Caterpillar obtained from the path graph P_n by attaching m vertices of degree two to each vertex of degree two of P_n. The graph $C_{(3,0)}P_7$ and $C_{(3,1)}P_7$ are illustrated in Figure 5.*
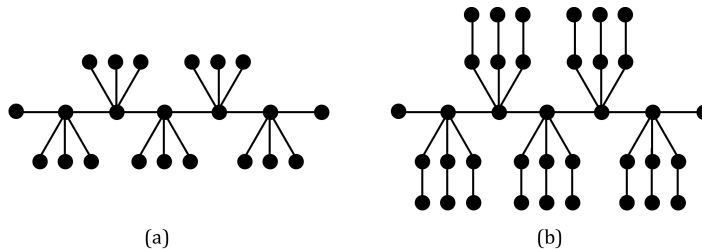

(a) (b)

Figure 5: **(a)** The graph $C_{(3,0)}P_7$ and **(b)** the graph $C_{(3,1)}P_7$.

Theorem 2.13. *Let $G \cong C_{(t,0)}P_n$ be generalized caterpillar graph with $(n + t(n-2))$-vertices. Then, for $t \geq 2$ and $p_i = t$, $\gamma_t^d(G) = \gamma_t^d(P_{n-2}^*)$.*

Proof. The proof is quite close to that of Theorem 2.8, so we omit it. ∎

Theorem 2.14. *Let $G \cong C_{(t,1)}P_n$ be generalized caterpillar graph with $(n+t(n-2))$-vertices. Then, for $t \geq 3$, $\gamma_t^d(G) = t(n-2)$.*

Proof. The graph G has $(n+t(n-2))$-vertices. Let $V(G) = V_1 \cup V_2 \cup V_3$, where $V_1 = \{u_i \in V(P_n) \mid 1 \leq i \leq n\}$, $V_2 = \{v_i \in (V(G)-V(P_n)) \mid deg(v_i) = 2 \text{ and } 1 \leq i \leq t(n-2)\}$ and $V_3 = \{w_i \in (V(G)-V(P_n)) \mid deg(w_i) = 1 \text{ and } 1 \leq i \leq t(n-2)\}$. Clearly, we have $deg(u_1) = deg(u_n) = 1$ and $deg(u_i) = 2$ for the vertices of V_1, where $i \in \{2, ..., n-1\}$. Suppose S is a $\gamma_t^d(G)$-set of G. Since $deg(w_i) = 1$ for each vertex $w_i \in V_3$, the all vertices of V_2 must be taken to the set S. Therefore, each vertex of V_1 is disjunctively totally dominated by the set S. Thus, all vertices in G are disjunctively totally dominated by S. It is easily seen that the set S is unique and there is no other set of $\gamma_t^d(G)$-set. Note that $|V_2| = t(n-2)$. Then, $|S| = t(n-2)$. Hence, we get $\gamma_t^d(G) = t(n-2)$. Thus, the proof holds. ∎

Definition 2.15. *[6] The binomial tree B_n with root R is the tree defines as follows:*

  i. *If $n = 0$, then $B_n = B_0 = R$, i.e., the binomial tree of order zero consists of a single root R.*

  ii. *If $n > 0$, then $B_n = R, B_0, B_1, \ldots, B_{n-1}$, i.e., the binomial tree of order $n > 0$ comprises the root R and n binomial subtrees $B_0, B_1, \ldots, B_{n-1}$.*

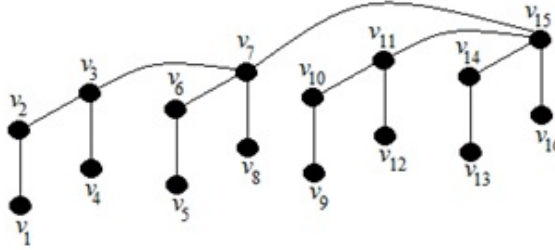*In Figures 6 and 7, the binomial trees B_4 and B_5 are illustrated.*
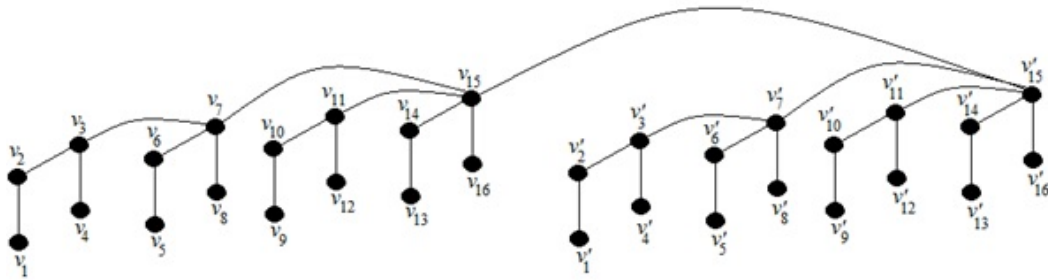

Figure 6: The binomial tree B_4.

Figure 7: The binomial tree B_5.

Theorem 2.16. *If $G \cong B_n$ of order 2^n with $n \geq 5$, then $\gamma_t^d(G) = 7(2^{n-4})$.*

Proof. The binomial tree B_n has 2^n-vertices. It is clear that $\gamma_t^d(B_0) = 1$, $\gamma_t^d(B_1) = \gamma_t^d(B_2) = 2$, and $\gamma_t^d(B_3) = 4$ for $n \leq 3$. Let $n = 4$, and let S_1 be a $\gamma_t^d(G)$-set of B_4. It is easily seen that $S_1 = \{v_2, v_3, v_6, v_{10}, v_{11}, v_{14}, v_{15}\}$ (see Figure 6). Then, we get $\gamma_t^d(B_4) = 7$.

   Suppose S is a $\gamma_t^d(G)$-set of B_5. Since B_5 has two copies of B_4, the vertices $\{v_2, v_3, v_6, v_{10}, v_{11}, v_{14}, v_{15}\}$ and $\{v_2', v_3', v_6', v_{10}', v_{11}', v_{14}', v_{15}'\}$ in the first and second copies of B_4, respectively, must be taken to the set S

(see Figure 7). Thus, we have $S = \{v_2, v_3, v_6, v_{10}, v_{11}, v_{14}, v_{15}, v_2', v_3', v_6', v_{10}', v_{11}', v_{14}', v_{15}'\}$. Then, we obtain $\gamma_t^d(B_5) = 2\gamma_t^d(B_4) = 14$. With the same thought, $\gamma_t^d(B_6) = 2\gamma_t^d(B_5) = 28$ and $\gamma_t^d(B_7) = 2\gamma_t^d(B_6) = 56$ are obtained. If this continue for $n \geq 5$, we get the following recurrence formula:

$$\gamma_t^d(B_n) = 2\gamma_t^d(B_{n-1}) \text{ for } n \geq 5. \tag{1}$$

From this formula, we have:

$$\gamma_t^d(B_n) = 2\gamma_t^d(B_{n-1}) = 2(2\gamma_t^d(B_{n-2})) = 2^2\gamma_t^d(B_{n-1}) = ... = 2^{n-4}\gamma_t^d(B_4).$$

Furthermore, we obtain

$$\gamma_t^d(B_n) = 2^i\gamma_t^d(B_{n-i}), i \in \{1, 2, n-1\}. \tag{2}$$

This equality can be seen by the induction method.

Let $i = 1$.

Thus, we have $\gamma_t^d(B_n) = 2\gamma_t^d(B_{n-1})$, also is clear by Eq. **(1)**. We prove this statement with induction on i. When $i = 1$, we have $\gamma_t^d(B_n) = 2^i\gamma_t^d(B_{n-i})$ and via the Eq. **(1)**, this is valid. We suppose that the result is true for $i = k$ and prove it for $i = k + 1$. By induction hypothesis and Eq. **(1)**, we get

$$\gamma_t^d(B_n) = 2^k\gamma_t^d(B_{n-k}) = 2^k(2\gamma_t^d(B_{n-k-1})) = 2^{k+1}\gamma_t^d(B_{n-k-1}).$$

This means the claim is valid where $i = k + 1$. Hence, we get

$\gamma_t^d(B_n) = 2^i\gamma_t^d(B_{n-i})$, where $1 \leq i \leq n - 4$,

$\gamma_t^d(B_n) = 7$.

Since the first case obtained when $i = n - 4$ is $n = 4$, the following is obtained from Eq. **(2)**.

$\gamma_t^d(B_n) = 2^{n-4}(B_{n-(n-4)}) = 2^{n-4}(\gamma_t^d(B_4)) = 7(2^{n-4})$. ∎

Definition 2.17. *[6] The complete k-ary tree of height h, T_h^k, is a rooted tree with each leaf having the same depth and each vertex except the leaves in degree k. T_h^k in which every non-leaf vertex has exactly k-children and the distance from the root to each leaf is exactly h. The complete k-ary tree for $k \geq 2$ has $\dfrac{k^{h+1} - 1}{k - 1}$ vertices and $\dfrac{k^{h+1} - 1}{k - 1} - 1 = \dfrac{k^{h+1} - k}{k - 1}$ edges. The complete binary tree is the complete k-ary tree with $k = 2$. Figure 8 shows an example of a complete k-ary tree T_4^2 ($k = 2$ and $h = 4$). In Figure 8, the value of L expresses the depth of each vertex.*
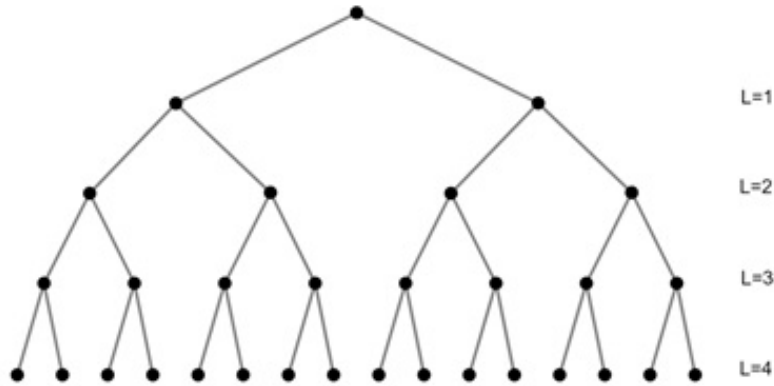

Figure 8: The complete 2-ary tree T_4^2 ($k = 2$ and $h = 4$).

Theorem 2.18. *If $G \cong T_h^k$ of order $\dfrac{k^{h+1} - 1}{k - 1}$, where $h \geq 4$ and $k \geq 3$, then*

$$
\gamma_t^d(G) = \begin{cases}
\left(\displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+3} \right) + 1 & , h \equiv 0 \, (mod 4); \\[4ex]
\left(\displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+4} \right) + 2 & , h \equiv 1 \, (mod 4); \\[4ex]
\left(\displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+1} \right) & , h \equiv 2 \, (mod 4); \\[4ex]
\left(\displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+2} \right) & , h \equiv 3 \, (mod 4).
\end{cases}
$$

Proof. By the definition of complete k-ary tree T_h^k, we know that there are $k^0, k^1, ..., k^h$ vertices in the levels $0^{th}, 1^{th}, ..., h^{th}$, respectively. Suppose S is a DTD-set in T_h^k for $h \geq 4$ and $k \geq 3$. We know that the vertices that in the level h^{th} are called leaf vertices. Since the degree of each leaf vertex is 1, the set S must include support vertices in the level $(h-1)^{th}$. Furthermore, the distance between each vertex in S and at least two vertices in S is two. Thus, the vertices which are in the levels h^{th}, $(h-2)^{th}$ and $(h-3)^{th}$ are disjunctively totally dominated by the set S. Then, the vertices that in the level $(h-5)^{th}$ must be added to the set S. It is easy to see that distance of the any two vertices which are in distinct levels is four. With the same thought, the set will be occurred. But, we have four cases according to h.

Case 1. $h \equiv 0 \, (mod 4)$.
Let $L = \displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} \{4i + 3\}$, where the elements of L is the levels of tree T_h^k. Let the set S includes the vertices which are in the levels in L. Thus, we get $|S| = \displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+3}$. But the root vertex of T_h^k is not disjunctively totally dominated by S. If any vertex in the first level is added to S, then root vertex is disjunctively totally dominated. So, $\gamma_t^d(T_h^k) = |S| = \left(\displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+3} \right) + 1$ is obtained.

Case 2. $h \equiv 1 \, (mod 4)$.
Let $L = \displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} \{4i + 4\}$, where the elements of L is the levels of tree T_h^k. Let the set S includes the vertices which are in the levels in L. Thus, we get $|S| = \displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+4}$. But the root vertex and the vertices which are in first level of T_h^k are not disjunctively totally dominated by S. If any vertex in the first level and the root vertex are added to S, then all vertices of T_h^k are disjunctively totally dominated by S. So, $\gamma_t^d(T_h^k) = |S| = \left(\displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+4} \right) + 2$ is obtained.

Case 3. $h \equiv 2(mod4)$.

Let $L = \displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} \{4i + 1\}$, where the elements of L is the levels of tree T_h^k. Let the set S includes the vertices

which are in the levels in L. Thus, we get $|S| = \displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+1}$. Clearly, all vertices of T_h^k are disjunctively

totally dominated by S. So, $\gamma_t^d(T_h^k) = |S| = \displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+1}$ is obtained.

Case 4. $h \equiv 3(mod4)$.

Let $L = \displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} \{4i + 2\}$, where the elements of L is the levels of tree T_h^k. This case is similar to the *Case 3*.

So, $\gamma_t^d(T_h^k) = |S| = \displaystyle\sum_{i=0}^{\lceil h/4 \rceil - 1} k^{4i+2}$ is obtained.

Thus, the proof of theorem is completed by the *Cases* 1, 2, 3 and 4.

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Theorem 2.19. *If* $G \cong T_h^k$ *of order* $\dfrac{k^{h+1} - 1}{k - 1}$, *where* $h \leq 3$ *and* $k \geq 3$, *then* $\gamma_t^d(G) = k^{h-1}$.

Proof. Suppose S is a DTD-set in T_h^k. Due to leaf vertices of T_h^k, the all support vertices must be taken to the set S. It is clear that all vertices of T_h^k are disjunctively totally dominated by S. So, we get $\gamma_t^d(T_h^k) = k^{h-1}$.

∎

Theorem 2.20. *If* $G \cong T_h^2$ *is a complete 2-ary tree of order* $2^{h+1} - 1$, *where* $h \geq 4$, *then*

$$\gamma_t^d(G) = \begin{cases} \dfrac{2(2^h - 1)}{3} & , \text{if } t \equiv 0(mod2); \\ \\ 2^{h-1} + 2^{h-3} & , \text{otherwise.} \end{cases}$$

Proof. By the definition of complete k-ary tree T_h^k, we know that T_h^2 consists of 2 copies of T_{h-1}^2, also T_{h-1}^2 consists of 2 copies of T_{h-2}^2, etc. Clearly, we get $T_1^2 \cong S_{1,2}$, where $S_{1,2}$ is a star graph. It is easily seen that $\gamma_t^d(T_1^2) = 2$. Let D be a $\gamma_t^d(G)$-set of T_h^2 for $h \leq 3$. To be disjunctively totally dominated each vertex of T_h^2, all vertices which are in $(h - 1)^{th}$ level must be added to the set D. Thus, all vertices except the vertices in D are disjunctively totally dominated. Therefore, the vertex which in level zero must be added to D. So, all vertices of T_h^2 are disjunctively totally dominated by the set D. Clearly, $|D| = 2^{h-1} + 1$, also the set D is the minimum DTD-set. As a result, $\gamma_t^d(T_h^2) = 2^{h-1} + 1$ for $h \leq 3$.

Let $h \geq 4$, and let S_1 be a $\gamma_t^d(G)$-set of T_4^2. Clearly, the set S_1 has the vertices of v_1, v_2, v_i, where $i \in \{7, 8, ..., 14\}$ in the Figure 8. So, all vertices in T_4^2 and S_1 are disjunctively totally dominated by S_1. It is easy to see that $|S_1| = 2^3 + 2 = 10$. Hence, $\gamma_t^d(T_4^2) = 10$. Let S be $\gamma_t^d(G)$-set of T_5^2. Since the tree T_5^2 has k-copies of T_4^2, all vertices which are correspond to vertices of S_1 in all copies of T_4^2 must be added to the set S. That is, $S = \bigcup_{j=1}^2 S_1$. Thus, we get $|S| = 2|S_1|$. So, $\gamma_t^d(T_5^2) = 2(\gamma_t^d(T_4^2)) = 2^4 + 2^2$ is obtained.

With the same thought, we consider the tree T_6^2. If the vertices of $\gamma_t^d(G)$-set of the tree T_5^2 which is copy of T_6^2, all vertices of T_6^2 are not disjunctively totally dominated. Because, the vertices which are in the DTD-set are not dominated. To disjunctively total dominate of these vertices, the vertices which are in 1-level must be taken in to the $\gamma_t^d(G)$-set. Thus, we obtain $\gamma_t^d(T_6^2) = 2(\gamma_t^d(T_5^2)) + 2$. Furthermore, we get following recursive formulas for $\gamma_t^d(T_h^2)$, where $h \geq 5$:

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$$\gamma_t^d(T_h^2) = \begin{cases} 2\gamma_t^d(T_{h-1}^2) + 2 & ,if\ h \equiv 0(mod2); \\[2mm] \gamma_t^d(T_{h-1}^2) & ,otherwise. \end{cases}$$

Let $h = 2k + 1$, where $k \in Z^+$. Then we have

$$\gamma_t^d(T_h^2) = 2\gamma_t^d(T_{h-1}^2) = 2(2\gamma_t^d(T_{h-2}^2)) = 2^2\gamma_t^d(T_{h-2}^2) = ... = 2^{n-4}(T_4^2).$$

Let $h = 2k$, where $k \in Z^+$. Then we have

$$\gamma_t^d(T_h^2) = 2\gamma_t^d(T_{h-1}^2) + 2$$

$$= 2(2\gamma_t^d(T_{h-2}^2) + 2) = 2^2(\gamma_t^d(T_{h-2}^2)) + 2$$

$$= 2^2(2\gamma_t^d(T_{h-3}^2) + 2 = 2^3(\gamma_t^d(T_{h-3}^2)) + 2^3 + 2$$

$$.$$
$$.$$
$$.$$

$$= 2^{h-4}\gamma_t^d(T_4^2) + 2^{h-5} + 2^{h-7} + ... + 2^{h-(h-3)} + 2^{h-(h-1)}.$$

Clearly, we get following result for $1 \le i \le h - 4$.

$$\gamma_t^d(T_h^2) = 2^i\gamma_t^d(T_{h-i}^2), \text{if } h = 2k + 1;$$

$$\gamma_t^d(T_h^2) = 2^i\gamma_t^d(T_{h-i}^2) + \sum_{i=0}^{\frac{h-4}{2}-1} 2^{2i+1}, \text{if } h = 2k.$$

If we use geometric series for $h = 2k$, then we have $\gamma_t^d(T_h^2) = 2^i\gamma_t^d(T_{h-i}^2) + 2\left(\frac{2^i-1}{2^2-1}\right)$.

These equalities can be proved by induction method, also remaining of the proof is similar to the proof of Theorem 2.16. So, the remaining of proof is omitted. Thus, the proof holds.

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3. Conclusion

Various measures to determine the network robustness were suggested in the literature, and a number of graph-theoretical parameters were used to assess network reliability. We have discussed the disjunctive total domination number for some tree networks in this work. Suppose one can break a more complex network into smaller networks, then under some conditions. In that case, the optimization problem's solutions on the smaller networks can be combined to solve the optimization problem on the larger network. Thus, calculation of the disjunctive total domination number for simple graph types is important.

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