Malaya Journal of Matematik 3(1)(2013) 34–43

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Some oscillation theorems for second order nonlinear neutral type difference equations

E. Thandapani[a,*] and V. Balasubramanian[b]

[a] Ramanujan Institute For Advanced Study in Mathematics, University of Madras, Chennai-600005, Tamil Nadu, India.

[b] Department of Mathematics, Periyar University, Salem-636 011, Tamil Nadu, India.

Abstract

In this paper some new sufficient conditions for the oscillatory behavior of second order nonlinear neutral type difference equation of the form

$$\Delta\Big(a_n\Delta(x_n + p_n x_{n-k})\Big) + q_n f(x_{\sigma(n+1)}) = 0$$

where $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are real sequences, $\{\sigma(n)\}$ is a sequence of integers, k is a positive integer and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$ are established. Examples are provided to illustrate the main results.

Keywords: Second order, nonlinear, neutral type difference equation, oscillation.

2010 MSC: 39A11.

1 Introduction

In this paper we study the oscillatory behavior of second order neutral type difference equation of the form

$$\Delta\Big(a_n\Delta(x_n + p_n x_{n-k})\Big) + q_n f(x_{\sigma(n+1)}) = 0, \quad n \in \mathbb{N}(n_0) \tag{1.1}$$

where k is a positive integer, $\{a_n\}$, $\{p_n\}$, $\{q_n\}$ are real sequences defined on $\mathbb{N}(n_0)$ and $\sigma(n+1)$ is a sequence of integers. We assume the following conditions without further mention:

(H_1) $\{a_n\}$ is a positive real sequence with $\displaystyle\sum_{n=n_0}^{\infty}\frac{1}{a_n} < \infty$;

(H_2) $\{p_n\}$ is a real sequence with $p_n \geq 1$ for all $n \in \mathbb{N}(n_0)$;

(H_3) $\{q_n\}$ is a positive real sequence for all $n \in \mathbb{N}(n_0)$;

(H_4) $\{\sigma(n)\}$ is an increasing sequence of integers such that $\sigma(n) \leq n$ and $\sigma(n) \to \infty$ as $n \to \infty$;

(H_5) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a constant $L > 0$ such that $\frac{f(u)}{u^\alpha} \geq L$ for all $u \neq 0$, where α is a ratio of odd positive integers.

Let $\theta = max\Big\{k, min_{n\in\mathbb{N}(n_0)}\sigma(n)\Big\}$. By a solution of equation (1.1), we mean a nontrivial real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfying the equation (1.1) for all $n \in \mathbb{N}(n_0)$. A solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

In recent years, there has been much research concerning the oscillation of delay and neutral type difference equations. In most of the papers, the authors considered the case $\sum_{n=n_0}^{\infty}\frac{1}{a_n} = \infty$ and either $-1 < p \leq p_n \leq 0$ or $0 \leq p_n \leq p < 1$, see for example [3-6, 9, 10, 13-16]. In [7, 8, 11, 12] the authors considered equation (1.1) under the assumptions $\sum_{n=n_0}^{\infty}\frac{1}{a_n} < \infty$ and $0 \leq p_n \leq p < 1$ and established sufficient conditions for the

*Corresponding author.

E-mail addresses: ethandapani@yahoo.co.in (E. Thandapani) and vbmaths@gmail.com (V. Balasubramanian)

oscillation of all solutions of equation (1.1).

Motivated by this observation in this paper we present some sufficient conditions for the oscillation of all solutions of equation (1.1) under the conditions $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$ and $p_n \geq 1$ for all $n \in \mathbb{N}(n_0)$. In Section 2, we present some preliminary lemmas, and in Section 3 we obtain some sufficient conditions for the oscillation of all solutions of equation (1.1). In Section 4, we provide some examples to illustrate the main results.

2 Some preliminary lemmas

Throughout this paper we use the following notation without further mention:

$$\begin{aligned} z_n &= x_n + p_n x_{n-k} , \\ A(n) &= a_{\sigma(n)} \sum_{s=n_0}^{n} \frac{1}{a_{\sigma(s)}} , \qquad R(n) = \sum_{s=n_0}^{n-1} \frac{1}{a_s} , \\ B(n) &= \frac{1}{p_{n+k}} \Big(1 - \frac{R(n+2k)}{R(n+k) p_{n+2k}} \Big) > 0 , \\ C(n) &= \frac{1}{p_{n+k}} \Big(1 - \frac{1}{p_{n+2k}} \Big) , \quad \text{and} \quad E(n) = \sum_{s=\tau(n)}^{\infty} \frac{1}{a_s} , \end{aligned}$$

where $\{\tau(n)\}$ is defined later. Note that from the assumptions it is enough to state and prove the lemmas and theorems for the case $\{x_n\}$ is eventually positive since the opposite case is proved similarly. To prove our main results we need the following lemmas.

Lemma 2.1. *Let $\{x_n\}$ be an eventually positive solution of equation* (1.1). *Then one of the following two cases holds for all sufficiently large n:*

(I) $z_n > 0, \quad a_n \Delta z_n > 0, \quad \Delta(a_n \Delta z_n) \leq 0;$

(II) $z_n > 0, \quad a_n \Delta z_n < 0, \quad \Delta(a_n \Delta z_n) \leq 0.$

Proof. The proof of the lemma can be found in [11]. □

Lemma 2.2. *Let $\{x_n\}$ be an eventually positive solution of equation* (1.1) *and suppose case (I) of Lemma 2.1 holds. Then there exists $N \in \mathbb{N}(n_0)$ such that*

$$x_n \geq B(n) z_n, \quad \textit{for all} \quad n \geq N. \tag{2.1}$$

Proof. From the definition of z_n, we have

$$\frac{z_{n+k}}{p_{n+k}} = \frac{x_{n+k}}{p_{n+k}} + x_n$$

or

$$x_n = \frac{1}{p_{n+k}} (z_{n+k} - x_{n+k}). \tag{2.2}$$

On the other hand

$$z_n = z_{n_0} + \sum_{s=n_0}^{n-1} \frac{a_s \Delta z_s}{a_s} \;\geq\; a_n R(n) \Delta z_n$$

or

$$R(n) \Delta z_n - z_n \Delta R(n) \leq 0.$$

or

$$\frac{R(n) \Delta z_n - z_n \Delta R(n)}{R(n) R(n+1)} \leq 0.$$

or

$$\Delta \Big(\frac{z_n}{R(n)} \Big) \leq 0.$$

Thus z_n is increasing and $\frac{z_n}{R(n)}$ is nonincreasing. Further

$$x_{n+k} \leq \frac{1}{p_{n+2k}} R(n+2k) \frac{z_{n+2k}}{R(n+2k)} \leq \frac{R(n+2k)}{p_{n+2k}} \Big(\frac{z_{n+k}}{R(n+k)} \Big). \tag{2.3}$$

From (2.2) and (2.3) we obtain (2.1). This completes the proof. □

Lemma 2.3. *Let $\{x_n\}$ be an eventually positive solution of equation* (1.1) *and suppose case (II) of Lemma 2.1 holds. Then there exists $N \in \mathbb{N}(n_0)$ such that*

$$x_n \geq C(n) z_{n+k}, \quad for \quad all \quad n \geq N. \tag{2.4}$$

Proof. From the proof of Lemma 2.2, we have (2.2). From $\Delta z_n < 0$ we have

$$x_{n+k} \leq \frac{z_{n+2k}}{p_{n+2k}} \leq \frac{z_{n+k}}{p_{n+2k}}. \tag{2.5}$$

Using (2.5) in (2.2), we obtain (2.4). This completes the proof. □

Lemma 2.4. *Let $\{x_n\}$ be an eventually positive solution of equation* (1.1) *and suppose case (I) of Lemma 2.1 holds. Then there exists $N \in \mathbb{N}(n_0)$ such that*

$$z_{\sigma(n+1)} \geq A(n) \Delta z_{\sigma(n)}, \quad forall \quad n \geq N. \tag{2.6}$$

Proof. Since $\Delta(a_n \Delta z_n) \leq 0$ and $\Delta\sigma(n) > 0$, we see that

$$z_{\sigma(n+1)} = z_{\sigma(N)} + \sum_{s=N}^{n} \Delta z_{\sigma(s)} \geq a_{\sigma(n)} \Delta z_{\sigma(n)} \sum_{s=N}^{n} \frac{1}{a_{\sigma(s)}}.$$

The proof is now complete. □

3 Oscillation results

In this section we obtain some new sufficient conditions for the oscillation of all solutions of equation (1.1).

Theorem 3.1. *Assume that $\alpha \geq 1$, and there exists a sequence of integers $\{\tau(n)\}$ such that $\tau(n) \geq n$, $\Delta\tau(n) > 0$ and $\sigma(n) \leq \tau(n) - k$. If there exists a positive increasing real sequence $\{\rho_n\}$ such that for all constants $M > 0$ and $D > 0$ one has*

$$\sum_{n=N}^{\infty} \left[L \rho_n q_n B^{\alpha}(\sigma(n+1)) - \frac{1}{4\alpha M^{\alpha-1}} \frac{(\Delta\rho_n)^2 a_{\sigma(n)}}{\rho_n} \right] = \infty \tag{3.1}$$

and

$$\sum_{n=N}^{\infty} \left[L q_n E^{\alpha}(n+1) C^{\alpha}(\sigma(n+1)) - \frac{\alpha}{D^{\alpha-1} E(n) a_{\tau(n)}} \right] = \infty \tag{3.2}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume to the contrary that there exists a nonoscillatory solution $\{x_n\}$ of equation (1.1). Without loss of generality we may assume that $x_{n-\theta} > 0$ for all $n \geq N \in \mathbb{N}(n_0)$, where N is chosen so that one of the cases of Lemma 2.1 hold for all $n \geq N$. We shall show that in each case we are led to a contradiction.
Case(I). From Lemma 2.2 and equation (1.1), we have

$$\Delta(a_n \Delta z_n) + L q_n B^{\alpha}(\sigma(n+1)) z^{\alpha}_{\sigma(n+1)} \leq 0, \quad n \geq N. \tag{3.3}$$

Define

$$w_n = \rho_n \frac{a_n \Delta z_n}{z^{\alpha}_{\sigma(n)}}, \quad n \geq N,$$

we have

$$\begin{aligned} \Delta w_n \quad = \quad & \frac{\rho_n \Delta(a_n \Delta z_n)}{z^{\alpha}_{\sigma(n+1)}} + \Delta\rho_n a_{\frac{n+1 \Delta z_{n+1}}{z^{\alpha}_{\sigma(n+1)}}} - \rho_n \frac{a_n \Delta z_n}{z^{\alpha}_{\sigma(n+1)} z^{\alpha}_{\sigma(n)}} \Delta z^{\alpha}_{\sigma(n)} - L \rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\ & - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z^{\alpha}_{\sigma(n)}}{z^{\alpha}_{\sigma(n)}} \end{aligned} \tag{3.4}$$

for $n \geq N$. By Mean value theorem

$$\Delta z^{\alpha}_{\sigma(n)} = \alpha t^{\alpha-1} \Delta z_{\sigma(n)},$$

where $z_{\sigma(n)} < t < z_{\sigma(n+1)}$. Since $\alpha \geq 1$, we have

$$\Delta z^{\alpha}_{\sigma(n)} \geq \alpha z^{\alpha-1}_{\sigma(n)} \Delta z_{\sigma(n)}. \tag{3.5}$$

Using (3.5) in (3.3) we obtain for $n \geq N$

$$\Delta w_n \leq -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha\rho_n}{\rho_{n+1}} w_{n+1} \frac{z^{\alpha-1}_{\sigma(n)}\Delta z_{\sigma(n)}}{z^{\alpha}_{\sigma(n)}}. \tag{3.6}$$

Since z_n increasing and $a_n\Delta z_n$ is nonincreasing we have from (3.6)

$$\Delta w_n \leq -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha\rho_n}{\rho^2_{n+1}} \frac{M^{\alpha-1}}{a_{\sigma(n)}} w^2_{n+1} \tag{3.7}$$

where $M = z_{\sigma(N)}$. Summing the last inequality from N to $n-1$ and using completing the square we have

$$0 < w_n \leq w_N - \sum_{s=N}^{n-1} \Big[L\rho_s q_s B^{\alpha}(\sigma(s+1)) - \frac{1}{4\alpha M^{\alpha-1}} \frac{(\Delta\rho_s)^2 a_{\sigma(s)}}{\rho_s} \Big].$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.1).
Case(II). Define

$$v_n = \frac{a_n\Delta z_n}{z^{\alpha}_{\sigma(n)}}\ , \quad n \geq N. \tag{3.8}$$

Then $v_n < 0$ for $n \geq N$. Since $\{a_n\Delta z_n\}$ is nonincreasing, we have

$$\Delta z_s \leq \frac{a_n\Delta z_n}{a_s}\ , \quad s \geq n.$$

Summing the last inequality from $\tau(n)$ to ∞, we obtain

$$z_{\infty} \leq z_{\tau(n)} + a_n\Delta z_n \sum_{s=\tau(n)}^{\infty} \frac{1}{a_s}.$$

Since $z_n > 0$ for all sufficiently large n we have

$$0 \leq z_{\infty} \leq z_{\tau(n)} + a_n\Delta z_n E(n)\ , \quad n \geq N,$$

or

$$\frac{a_n\Delta z_n E(n)}{z_{\tau(n)}} \geq -1\ , \quad n \geq N.$$

Thus

$$-\frac{a_n\Delta z_n(-a_n\Delta z_n)^{\alpha-1}}{z^{\alpha}_{\tau(n)}} E^{\alpha}(n) \leq 1.$$

So, by $\Delta(-a_n\Delta z_n) > 0$ and (3.8), we have

$$-\frac{1}{D^{\alpha-1}} \leq v_n E^{\alpha}(n) \leq 0\ , \quad n \geq N, \tag{3.9}$$

where $D = -a_N\Delta z_N$. From (3.8), we have

$$\Delta v_n = \frac{\Delta(a_n\Delta z_n)}{z^{\alpha}_{\tau(n+1)}} - \frac{a_n\Delta z_n}{z^{\alpha}_{\tau(n)} z^{\alpha}_{\tau(n+1)}} \Delta z^{\alpha}_{\tau(n)}.$$

By Mean Value Theorem,

$$\Delta z^{\alpha}_{\tau(n)} = \alpha t^{\alpha-1}\Delta z_{\tau(n)}$$

where $z_{\tau(n+1)} < t < z_{\tau(n)}$. Since $\alpha \geq 1$ and $\Delta z_{\tau(n)} < 0$, we have

$$\Delta z^{\alpha}_{\tau(n)} \leq \alpha z^{\alpha-1}_{\tau(n+1)} \Delta z_{\tau(n)}.$$

Therefore

$$\Delta v_n \leq -\frac{Lq_n x^{\alpha}_{\sigma(n+1)}}{z^{\alpha}_{\tau(n+1)}} - \frac{\alpha a_n \Delta z_n}{z^{\alpha}_{\tau(n)} z_{\tau(n+1)}} \Delta z_{\tau(n)}. \tag{3.10}$$

From (2.4) and by $\sigma(n) \leq \tau(n) - k$, we have

$$\frac{x^{\alpha}_{\sigma(n+1)}}{z^{\alpha}_{\tau(n+1)}} \geq C^{\alpha}(\sigma(n+1)). \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\Delta v_n + Lq_n C^{\alpha}(\sigma(n+1)) \leq 0, \quad n \geq N. \tag{3.12}$$

Multiplying (3.12) by $E^{\alpha}(n+1)$ and then summing it from N to $n-1$, we have

$$\sum_{s=N}^{n-1} E^{\alpha}(s+1)\Delta v_s + \sum_{s=N}^{n-1} LE^{\alpha}(s+1) q_s C^{\alpha}(\sigma(s+1)) \leq 0.$$

Summation by parts formula yields

$$\sum_{s=N}^{n-1} E^{\alpha}(s+1)\Delta v_s = E^{\alpha}(n)v_n - E^{\alpha}(N)v_N - \sum_{s=N}^{n-1} v_s \Delta E^{\alpha}(s).$$

Using Mean Value Theorem, we obtain

$$\Delta E^{\alpha}(s) \geq -\frac{\alpha E^{\alpha-1}(s)}{a_{\tau(s)}}.$$

Since $v_n < 0$, we have

$$\sum_{s=N}^{n-1} E^{\alpha}(s+1)\Delta v_s \geq E^{\alpha}(n)v_n - E^{\alpha}(N)v_N + \sum_{s=N}^{n-1} \frac{\alpha v_s E^{\alpha-1}(s)}{a_{\tau(s)}} ,$$

or

$$E^{\alpha}(n)v_n - E^{\alpha}(N)v_N + \sum_{s=N}^{n-1} \frac{\alpha v_s E^{\alpha-1}(s)}{a_{\tau(s)}} + \sum_{s=N}^{n-1} Lq_s E^{\alpha}(s+1) C^{\alpha}(\sigma(s+1)) \leq 0. \tag{3.13}$$

Therefore, from (3.9) and (3.13), we obtain

$$-\frac{1}{D^{\alpha-1}} \quad \leq \quad E^{\alpha}(n)v_n \leq E^{\alpha}(N)v_N - \sum_{s=N}^{n-1} \Big[Lq_s E^{\alpha}(s+1) C^{\alpha}(\sigma(s+1)) - \frac{\alpha}{D^{\alpha-1} E(s) a_{\tau(s)}} \Big].$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.2). This completes the proof. □

Theorem 3.2. *Assume that* $\alpha \geq 1$ *and there exists a sequence* $\{\tau(n)\}$ *of integers such that* $\tau(n) \geq n$, $\Delta\tau(n) > 0$ *and* $\tau(n) \leq \sigma(n) - k$. *If there exists a positive increasing real sequence* $\{\rho_n\}$ *such that for every constants* $M > 0$,*and* $D > 0$, (3.1) *holds, and*

$$\sum_{n=N}^{\infty} \Big[q_n E^{\alpha+1}(n+1) C^{\alpha}(\sigma(n+1)) - \frac{\alpha+1}{D^{\alpha-1} a_{\tau(n)}} \Big] = \infty, \tag{3.14}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1, we see that Lemma 2.1 holds for $n \geq N \in \mathbb{N}(n_0)$.
Case(I). Proceeding as in the proof of Theorem 3.1(Case(I)) we obtain a contradiction to (3.1).
Case(II). Proceeding as in the proof of Theorem 3.1(Case(II)) we obtain (3.9) and (3.12). Multiplying (3.12) by $E^{\alpha+1}(n+1)$ and then summing it from N to $n-1$ we have

$$\sum_{s=N}^{n-1} E^{\alpha+1}(s+1)\Delta v_s + \sum_{s=N}^{n-1} Lq_s E^{\alpha+1}(s+1) C^{\alpha}(\sigma(s+1)) \leq 0.$$

Using the summation by parts formula in the first term of the last inequality and then rearranging, we obtain

$$E^{\alpha+1}(n)v_n - E^{\alpha+1}(N)v_N + \sum_{s=N}^{n-1}\frac{(\alpha+1)v_sE^{\alpha}(s)}{a_{\tau(s)}} + \sum_{s=N}^{n-1}Lq_sE^{\alpha+1}(s+1)C^{\alpha}(\sigma(s+1)) \leq 0. \quad (3.15)$$

In view of (3.9), we have $-v_nE^{\alpha+1}(n) \leq \frac{1}{D^{\alpha-1}}E(n) < \infty$ as $n \to \infty$, and

$$\sum_{s=N}^{n-1}Lq_sE^{\alpha+1}(s+1)C^{\alpha}(\sigma(s+1)) \leq E^{\alpha+1}(N)v_N - E^{\alpha+1}(n)v_n + \frac{(\alpha+1)}{D^{\alpha-1}}\sum_{s=N}^{n-1}\frac{1}{a_{\tau(s)}}.$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.14). This completes the proof. □

Theorem 3.3. *Assume that $\alpha \geq 1$, and there exists a sequence $\{\tau(n)\}$ of integers such that $\tau(n) \geq n$, $\Delta\tau(n) > 0$ and $\sigma(n) \leq \tau(n) - k$. If there exists a positive increasing real sequence $\{\rho_n\}$ such that for every constant $M > 0$, (3.1) holds, and*

$$\sum_{n=N}^{\infty}\frac{1}{a_n}\sum_{s=N}^{n-1}q_sE^{\alpha}(s+1)C^{\alpha}(\sigma(s+1)) = \infty, \quad (3.16)$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1, we see that Lemma 2.1 holds and Case(I) is eliminated by the condition (3.1).

Case(II) Proceeding as in the proof of Theorem 3.1(Case(II)), we have

$$z_{\tau(n)} \geq -a_n\Delta z_nE(n) \geq -a_N\Delta z_NE(n) = dE(n)$$

where $d = -a_N\Delta z_N$. From equation (1.1), we have

$$\Delta(-a_n\Delta z_n) \geq Lq_nx^{\alpha}_{\sigma(n+1)},$$

and

$$\frac{x_{\sigma(n+1)}}{z_{\tau(n+1)}} \geq C(\sigma(n+1)).$$

Hence

$$\Delta(-a_n\Delta z_n) \geq d^{\alpha}Lq_nC^{\alpha}(\sigma(n+1))E^{\alpha}(n+1).$$

Summing the last inequality from N to $n-1$, we obtain

$$\begin{aligned}-a_n\Delta z_n &\geq -a_N\Delta z_N + d^{\alpha}L\sum_{s=N}^{n-1}q_sC^{\alpha}(\sigma(s+1))E^{\alpha}(s+1)\\ &\geq Ld^{\alpha}\sum_{s=N}^{n-1}q_sC^{\alpha}(\sigma(s+1))E^{\alpha}(s+1).\end{aligned}$$

Again summing the last inequality from N to $n-1$, we have

$$z_N \geq z_N - z_n \geq Ld^{\alpha}\sum_{s=N}^{n-1}\frac{1}{a_s}\sum_{t=N}^{s-1}q_tC^{\alpha}(\sigma(t+1))E^{\alpha}(t+1).$$

Letting $n \to \infty$ in the above inequality, we obtain

$$Ld^{\alpha}\sum_{n=N}^{\infty}\frac{1}{a_n}\sum_{t=N}^{n-1}q_tC^{\alpha}(\sigma(t+1))E^{\alpha}(t+1) \leq z_N$$

a contradiction to (3.16). This completes the proof. □

Next, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1) when $0 < \alpha \leq 1$.

Theorem 3.4. *Assume that $0 < \alpha \le 1$, and there exist a real sequence $\{\tau(n)\}$ of integers such that $\tau(n) \ge n$, $\Delta\tau(n) > 0$ and $\sigma(n) \le \tau(n) - k$. If there exists a positive nondecreasing sequence $\{\rho_n\}$ such that for all constants $M_1 > 0$ and $M_2 > 0$, one has*

$$\sum_{n=N}^{\infty}\Big[L\rho_n q_n B^{\alpha}(\sigma(n+1)) - M_1^{1-\alpha}\frac{\Delta\rho_n a_{\sigma(n)}^{\alpha}}{A^{\alpha}(n)}\Big] = \infty \tag{3.17}$$

and

$$\sum_{n=N}^{\infty}\Big[LM_2^{\alpha-1}q_n E(n+1)C^{\alpha}(\sigma(n+1)) - \frac{1}{4a_{\tau(n)}E(n+1)}\Big] = \infty, \tag{3.18}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1, we see that Lemma 2.1 holds for all $n \ge N \in \mathbb{N}(n_0)$.
Case(I). Define

$$w_n = \rho_n\frac{a_n\Delta z_n}{z_{\sigma(n)}^{\alpha}}, \quad n \ge N.$$

Then $w_n > 0$ and from equation (1.1) and from (2.1), we have

$$\Delta w_n \le -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \Delta\rho_n\frac{a_{\sigma(n)}\Delta z_{\sigma(n)}}{z_{\sigma(n+1)}^{\alpha}}, \quad n \ge N.$$

Using (2.6) in the last inequality, we obtain

$$\Delta w_n \le -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n a_{\sigma(n)}^{\alpha}}{A^{\alpha}(n)}(a_{\sigma(n)}\Delta z_{\sigma(n)})^{1-\alpha}, \quad n \ge N.$$

From the monotonicity of $\{a_n\Delta z_n\}$ and $0 < \alpha \le 1$, we have from the last inequality

$$\Delta w_n \le -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n a_{\sigma(n)}^{\alpha}}{A^{\alpha}(n)}M_1^{1-\alpha}, \quad n \ge N, \tag{3.19}$$

where $M_1 = a_{\sigma(N)}\Delta z_{\sigma(N)}$. Summing the inequality (3.19) from N to $n-1$, we obtain

$$0 < w_n \le w_N - \sum_{s=N}^{n-1}\Big(L\rho_s q_s B^{\alpha}(\sigma(s+1)) - \frac{M_1^{1-\alpha}a_{\sigma(s)}^{\alpha}\Delta\rho_s}{A^{\alpha}(s)}\Big). \tag{3.20}$$

Letting $n \to \infty$ in (3.20), we obtain a contradiction to (3.17).
Case(II). Define

$$v_n = \frac{a_n\Delta z_n}{z_{\tau(n)}}, \quad n \ge N. \tag{3.21}$$

Then $v_n < 0$ for $n \ge N$. Further, we have

$$a_s\Delta z_s \le a_n\Delta z_n, \quad s \ge n.$$

Dividing the last inequality by a_s and then summing it from $\tau(n)$ to ℓ, we obtain

$$z_{\ell+1} - z_{\tau(n)} \le a_n\Delta z_n\sum_{s=\tau(n)}^{\ell}\frac{1}{a_s}.$$

Letting $\ell \to \infty$, we obtain

$$0 \le z_{\tau(n)} + a_n\Delta z_n E(n)$$

or

$$-1 \le v_n E(n), \quad n \ge N. \tag{3.22}$$

From (3.21) and equation (1.1), we have

$$\Delta v_n \le -\frac{Lq_n x_{\sigma(n+1)}^{\alpha}}{z_{\tau(n+1)}} - \frac{a_n\Delta z_n\Delta z_{\tau(n)}}{z_{\tau(n)}z_{\tau(n+1)}}.$$

Since $\tau(n) \geq n$ and $a_n\Delta z_n$ is negative and decreasing, we have

$$a_{\tau(n)}\Delta z_{\tau(n)} \leq a_n \Delta z_n.$$

Therefore

$$\Delta v_n \leq -Lq_n \frac{x^{\alpha}_{\sigma(n+1)}}{z_{\tau(n+1)}} - \frac{(a_n\Delta z_n)^2}{a_{\tau(n)} z_{\tau(n)} z_{\tau(n+1)}}, \quad n \geq N.$$

Since z_n is positive and decreasing, we have $z_{\tau(n+1)} \leq z_{\tau(n)}$ for $n \geq N$. Combining the last two inequalities, we obtain

$$\Delta v_n \leq -Lq_n \frac{x^{\alpha}_{\sigma(n+1)}}{z_{\tau(n+1)}} - \frac{v_n^2}{a_{\tau(n)}}, \quad n \geq N. \tag{3.23}$$

Now using (3.11) in (3.23), we have

$$\Delta v_n \leq -Lq_n \frac{C^{\alpha}(\sigma(n+1))}{M_2^{1-\alpha}} - \frac{v_n^2}{a_{\tau(n)}}$$

for some constant $M_2 = z_{\tau(N+1)} > 0$. That is,

$$\Delta v_n + LM_2^{\alpha-1} q_n C^{\alpha}(\sigma(n+1)) + \frac{v_n^2}{a_{\tau(n)}} \leq 0, \quad n \geq N. \tag{3.24}$$

Multiplying (3.23) by $E(n+1)$, and then summing it from N to $n-1$, we have

$$\sum_{s=N}^{n-1} E(s+1)\Delta v_s \quad + \quad \sum_{s=N}^{n-1} LM_2^{\alpha-1} q_s E(s+1) C^{\alpha}(\sigma(s+1)) + \sum_{s=N}^{n-1} \frac{E(s+1)v_s^2}{a_{\tau(s)}} \leq 0. \tag{3.25}$$

Using the summation by parts formula in the first term of (3.25) and then rearranging, we obtain

$$E(n)v_n - E(N)v_N \quad + \quad \sum_{s=N}^{n-1} LM_2^{\alpha-1} q_s E(s+1) C^{\alpha}(\sigma(s+1)) + \sum_{s=N}^{n-1} \Big(\frac{v_s}{a_{\tau(s)}} + \frac{v_s^2 E(s+1)}{a_{\tau(s)}}\Big) \leq 0.$$

Using completing the square in the last term of the above inequality, we obtain

$$\begin{aligned} E(n)v_n - E(N)v_N &+ \sum_{s=N}^{n-1} LM_2^{\alpha-1} q_s E(s+1) C^{\alpha}(\sigma(s+1)) \\ &+ \sum_{s=N}^{n-1} \frac{E(s+1)}{a_{\tau(s)}} \Big(v_s + \frac{1}{2E(s+1)}\Big)^2 - \sum_{s=N}^{n-1} \frac{1}{4a_{\tau(s)}E(s+1)} \leq 0 \end{aligned}$$

or

$$E(n)v_n \quad \leq \quad E(N)v_N - \sum_{s=N}^{n-1} \Big(LM_2^{\alpha-1} q_s E(s+1) C^{\alpha}(\sigma(s+1)) - \frac{1}{4a_{\tau(s)}E(s+1)}\Big).$$

Letting $n \to \infty$ in the last inequality and using (3.22), we obtain a contradiction to (3.18). The proof is now complete. □

4 Examples

In this section, we present some examples to illustrate the main results.

Example 4.1. Consider the neutral difference equation

$$\Delta\Big(2^{n+1}\Delta(x_n + 2x_{n-2})\Big) + 9 \times 2^{n+2} x_{n-1} = 0, \quad n \in \mathbb{N}(0). \tag{4.1}$$

Here $a_n = 2^{n+1}$, $p_n = 2$, $k = 2$, $\sigma(n+1) = n-1$, $\alpha = 1$, $q_n = 36(2^n)$ and $\tau(n) = n+2$. Then $R(n) = \frac{2^n-1}{2^n}$, $E(n) = \frac{1}{2^{n+2}}$, $C(n) = \frac{1}{4}$ and $B(n) = \frac{1}{16}\Big(\frac{4\,(2^{n+2})-7}{2^{n+2}-1}\Big)$. By taking $\rho_n = 1$, we see that all conditions of Theorem 3.1 are satisfied and hence every solution of equation (4.1) is oscillatory. In fact

$\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (4.1) since it satisfies the given equation.

Example 4.2. Consider the neutral difference equation

$$\Delta\Big(2^{n+1}\Delta(x_n + 2x_{n-3})\Big) + 1905 \times 2^{5(n-3)} x_{n-2}^3 = 0, \quad n \in \mathbb{N}(0). \tag{4.2}$$

Here $a_n = 2^{n+1}$, $p_n = 2$, $k = 3$, $\sigma(n+1) = n-2$, $\alpha = 3$, $q_n = \left(\frac{1905}{32768}\right)2^{5n}$, $L = 1$ and $\tau(n) = n+2$. Then $R(n) = \frac{2^n-1}{2^n}$, $E(n) = \frac{1}{2^{n+2}}$, $C(n) = \frac{1}{4}$ and $B(n) = \frac{1}{32}\left(16 - \frac{2^{n+6}-1}{2^{n+1}-1}\right)$. By taking $\rho_n = 1$, we see that all conditions of Theorem 3.2 are satisfied and hence every solution of equation (4.2) is oscillatory. In fact $\{x_n\} = \{\frac{(-1)^n}{4^n}\}$ is one such oscillatory solution of equation (4.2) since it satisfies the given equation.

Example 4.3. Consider the neutral difference equation

$$\Delta\Big((n+1)(n+2)\Delta(x_n + 3x_{n-1})\Big) + 8(n+2)^2 x_{n-2}^{1/3} = 0, \quad n \in \mathbb{N}(1). \tag{4.3}$$

Here $a_n = (n+1)(n+2)$, $p_n = 3$, $k = 1$, $\sigma(n+1) = n-2$, $\alpha = \frac{1}{3}$, $q_n = 8(n+2)^2$ and $\tau(n) = n$. Then $R(n) = \frac{n-1}{2(n+1)}$, $E(n) = \frac{1}{(n+1)}$, $C(n) = \frac{2}{9}$ and $B(n) = \frac{2}{9}\left(\frac{n^2+3n-1}{n(n+3)}\right)$. By taking $\rho_n = 1$, we see that all conditions of Theorem 3.4 are satisfied and hence every solution of equation (4.3) is oscillatory. In fact $\{x_n\} = \{(-1)^{3n}\}$ is one such oscillatory solution of equation (4.3) since it satisfies the given equation.
We conclude this paper with the following remark.

Remark 4.1. *The results obtained in this paper are new and complement to that of in [8, 11, 12].*

5 Acknowledgment

The authors thank the referee for his/her helpful comments which improved the content of the paper.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities, Theory, Methods and Applications*, Second Edition, Marcel Dekker, New York, 2000.

[2] R.P. Agarwal, M. Bohner S.R. Grace, and D.O'Regan, *Discrete Oscillation Theory*, Hindawi Publ. Corp., New York, 2005.

[3] R.P.Agarwal, M.M.S. Manuel and E. Thandapani, Oscillatory and nonoscillatory behavior of second order neutral delay difference equations, *Math. Comput. Model.,* 24(1)(1996), 5-11.

[4] R.P. Agarwal, M.M.S. Manuel and E. Thandapani, Oscillatory and nonoscillatory behavior of second order neutral delay difference equations II, *Appl. Math. Lett.,* 10(2)(1997), 103-109.

[5] D.Bainov and D.P.Mishev, Classification and existence of positive solutions of second order nonlinear neutral difference equations, *Funk. Ekavac.,* 40(1997), 371-393.

[6] S. Elizabeth, J.R. Graef, P. Sundaram and E. Thandapani, Classifying nonoscillatory solutions and oscillation of neutral difference equations, *J. Diff. Eqns. Appl.,* 11(7)(2005), 605-618.

[7] J. Jiang, Oscillation of second order nonlinear neutral delay difference equations, *Appl. Math. Comput.,* 146(2003), 791-801.

[8] E. Thandapani and K. Mahalingam, Oscillation and nonoscillation of second order neutral delay difference equations, *Czech. Math.J.,* 53(128)(2003), 935-947.

[9] E. Thandapani and M.M.S. Manuel, Asymptotic and oscillatory behavior of second order neutral delay difference equations, *Engin. Simul.,* 15(1998), 423-430.

[10] E. Thandapani and P. Mohankumar, Oscillation and nonoscillation of nonlinear neutral delay difference equations, *Tamkang J. Math.,* 38(4) (2007), 323-333.

[11] E. Thandapani and S. Selvarangam, Oscillation of second order Emden-Fowler type neutral difference equations, *Dyn. Cont. Dis. Impul. Sys.,* 19(2012), 453-469.

[12] D.M. Wang and Z.T. Xu, Oscillation of second order quasilinear neutral delay difference equations, *Acta Math. Appl. Sinica.,* 27(1)(2011), 93-104.

[13] B.G.Zhang, Oscillation and asymptotic behavior of second order difference equations, *J.Math. Anal. Appl.,* 173(1993), 58-68.

[14] B.G. Zhang and S.H. Saker, Kamenev-type oscillation criteria for nonlinear neutral delay difference equations, *Indian J.Pure Appl. Math.,* 34(11)(2003), 1571-1584.

[15] G. Zhang and Y. Geo, *Oscillation Theory for Difference Equations*, Publishing House of Higher Education, Beijing, 2001.

[16] Z. Zhang, J. Chen and C. Zhang, Oscillation of solutions for second order nonlinear difference equations with nonlinear neutral term, *Comput. Math. Appl.,* 21(2001), 1487-1494.

Received: April 04, 2013; *Accepted*: May 12, 2013

UNIVERSITY PRESS