Properties of some nonlinear partial dynamic equations on time scales

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Abstract

The aim of the present paper is to study some basic qualitative properties of solutions of certain nonlinear partial dynamic equations on time scales. The tools employed are application of Banach fixed point theorem and a variant of certain fundamental integral inequality with explicit estimates on time scales.

Keywords: Dynamic equations, time scales, qualitative properties, inequalities with explicit estimates.

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1 Introduction

During the past few years many authors have obtained the time scale analogue of well known partial dynamic equations see [1, 6, 10, 11]. Recently in [2, 3, 9] authors have obtained inequalities on two independent variables on time scales. In the present paper we establish some basic qualitative properties of solutions of certain partial dynamic equation on time scales. We use certain fundamental integral inequalities with explicit estimates to establish our results. We assume understanding of time scales and its notation. Excellent information about introduction to time scales can be found in [4, 5].

In what follows $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ the set of integers and $\mathbb{T}$ denotes arbitrary time scales. Let C_{rd} be the set of all rd continuous functions. We assume $\mathbb{T}_1$ and $\mathbb{T}_2$ are two time scales and $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$. The delta partial derivative of a function $z(x,y)$ for $(x,y) \in \Omega$ with respect to x, y and xy are denoted by $z^{\Delta_1}(x,y)$, $z^{\Delta_2}(x,y)$ and $z^{\Delta_1\Delta_2}(x,y) = z^{\Delta_2\Delta_1}(x,y)$ with the given boundary conditions.

$$u^{\Delta_2\Delta_1}(x,y) = f\left(x, y, u(x,y), u^{\Delta_1}(x,y)\right) \tag{1.1}$$

with the given initial boundary conditions

$$u(x, y_0) = \alpha(x), u(x_0, y) = \beta(y), u(x_0, y_0) = 0, \tag{1.2}$$

for $(x,y) \in \Omega$, where $f \in C_{rd}(\Omega \times R^n \times R^n, R_+)$, $\alpha, \beta \in (R_+, R^n)$.

In this paper we study the existence, uniqueness and other properties of the solutions of (1.1)-(1.2) under some suitable conditions on the functions involved in (1.1)-(1.2). The main tool employed is based on application of Banach fixed point theorem [8] coupled with Bielecki-type norm [7] and suitable integral inequality with explicit estimate.

2 Preliminaries and Basic Inequality

For a function $u(x,y)$ and its delta derivative $u^{\Delta_1}(x,y)$ in $C_{rd}(\Omega, R^n)$ we denote by $|u(x,y)|_W = |u(x,y)| + \left|u^{\Delta_1}(x,y)\right|$. For $(t,s) \in \Omega$ the notation $a(t,s) = O(b(t,s))$ then there exists a constant $q > 0$ such that

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$\left| \frac{a(t,s)}{b(t,s)} \right| \leq q$ right-hand neighborhood. Let G be the space whose functions $(\phi(x,y), \phi^{\Delta}(x,y)) \in W$ which are rd-continuous for $(x,y) \in \Omega$ and satisfy the condition

$$|\phi(x,y)|_W = O(e_\lambda(x,y)), \tag{2.1}$$

for $(x,y) \in \Omega$, where $\lambda > 0$ is a constant. In space G we define the norm

$$|\phi|_G = \sup_{(x,y)\in\Omega} [|\phi(x,y)|_W \, e_{\ominus\lambda}(x,y)]. \tag{2.2}$$

It is easy to see that G with norm defined in (2.2) is a Banach space. The condition (2.1) implies that there is a constant $N \geq 0$ such that

$$|\phi(x,y)|_W \leq Ne_\lambda(x,y). \tag{2.3}$$

Using the fact that in (2.2) we observe that

$$|\phi(x,y)|_W \leq N. \tag{2.4}$$

By a solution of $(1.1)-(1.2)$ we means a function $u(x,y) \in C_{rd}(\Omega, R^n)$ which satisfies the equation $(1.1)-(1.2)$. It is easy to see that solution $u(x,y)$ of $(1.1)-(1.2)$ satisfies the following equation

$$u^{\Delta_1}(x,y) = \alpha^{\Delta}(x) + \int_{y_0}^{y} f\left(x,t,u(x,t),u^{\Delta_1}(x,t)\right) \Delta t, \tag{2.5}$$

and

$$u(x,y) = \alpha(x) + \beta(y) + \int_{x_0}^{x}\int_{y_0}^{y} f\left(s,t,u(s,t),u^{\Delta_1}(s,t)\right) \Delta t \Delta s, \tag{2.6}$$

for $(x,y) \in \Omega$.

We require the following integral inequality.

Lemma 2.1. *Let $u,a,b \in C_{rd}(\Omega, R^n)$*

$$u(x,y) \leq c + \int_{y_0}^{y} a(x,t)u(x,t)\Delta t + \int_{x_0}^{x}\int_{y}^{y} b(s,t)u(s,t)\,\Delta t \Delta s, \tag{2.7}$$

for $(x,y) \in \Omega$, then

$$u(x,y) \leq cH(x,y)\, e_{\int_{y_0}^{y} b(s,t)H(s,t)\Delta t}(x,x_0), \tag{2.8}$$

for $(x,y) \in \Omega$, where

$$H(x,y) = e_a(y,y_0), \tag{2.9}$$

for $(x,y) \in \Omega$.

Proof. Define a function $n(x,y)$ by

$$n(x,y) = c + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)u(s,t)\,\Delta t \Delta s, \tag{2.10}$$

then (2.7) can be restated as

$$u(x,y) \leq n(x,y) + \int_{y_0}^{y} a(x,t)u(x,t)\Delta t. \tag{2.11}$$

Clearly $n(x,y)$ is nonnegative for $(x,y) \in \Omega$ and nondecreasing for x. Treating (2.11) as a one-dimensional integral inequality and a suitable application of inequality given in Theorem 3.1 [12] yields

$$u(x,y) \leq n(x,y)H(x,y), \tag{2.12}$$

where $H(x, y)$ is given by (2.9). From (2.10) and (2.12) we have

$$n(x, y) \leq c + \int_{x_0}^{x} \int_{y_0}^{y} b(s, t) H(s, t) n(s, t) \Delta t \Delta s. \tag{2.13}$$

Now a suitable application of Theorem 2.1 [9] to (2.13) yields

$$n(x, y) \leq c e_{\int_{y_0}^{y} b(s,t) H(s,t) \Delta t}(x, x_0). \tag{2.14}$$

Using (2.14) into (2.12) we get the required inequality in (2.8). $\qquad\square$

3 Main results

Our main results are given in the following theorems.

Theorem 3.1. *Suppose that*
(i) the function f in (1.1) satisfies the condition

$$|f(x, y, u, v) - f(x, y, \overline{u}, \overline{v})| \leq p(x, y) [|u - \overline{u}| + |v - \overline{v}|], \tag{3.1}$$

where $p \in C_{rd}(\Omega, R^n)$,
(ii) for λ as in (2.1)
(a) there exists a nonnegative constant γ such that $\gamma < 1$ and

$$\int_{y_0}^{y} p(x, t) e_\lambda(x, t) \Delta t + \int_{x_0}^{x} \int_{y_0}^{y} p(s, t) e_\lambda(s, t) \Delta t \Delta s \leq \gamma e_\lambda(x, y), \tag{3.2}$$

(b) there exists a nonnegative constant η such that

$$|\alpha(x)| + |\beta(y)| + |\alpha^\Delta(x)| + \int_{y_0}^{y} |f(x, t, 0, 0)| \Delta t$$

$$+ \int_{x_0}^{x} \int_{y_0}^{y} |f(s, t, 0, 0)| \Delta t \Delta s \leq \eta e_\lambda(x, y), \tag{3.3}$$

for $(x, y) \in \Omega$ where α, β are the functions given in (1.2). Then $(1.1) - (1.2)$ has a unique solution in G.

Proof. Let $u(x, y) \in G$ and define operator S by

$$(Su)(x, y) = |\alpha(x)| + |\beta(y)| + \int_{x_0}^{x} \int_{y_0}^{y} f(s, t, u(s, t), u^{\Delta_1}(s, t)) \Delta t \Delta s. \tag{3.4}$$

Delta differentiating both sides of (3.4) with respect to x we get

$$(Su)^{\Delta_1}(x, y) = |\alpha^\Delta(x)| + \int_{y_0}^{y} f(x, t, u(x, t), u^{\Delta_1}(x, t)) \Delta t. \tag{3.5}$$

First we show that Su maps G into itself. Evidently (Su) is rd-continuous . We verify that (2.1) is fulfilled.

From (3.4) and (3.5) using the hypotheses and (2.3) we have

$$|(Su)(x,y)| + \left|(Su)^{\Delta_1}(x,y)\right|$$

$$\leq |\alpha(x)| + |\beta(y)| + |\alpha^{\Delta}(x)| + \int_{x_0}^{x}\int_{y_0}^{y} \left|f(s,t,u(s,t),u^{\Delta_1}(s,t)) - f(s,t,0,0)\right|\Delta t\Delta s$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y} |f(s,t,0,0)|\Delta t\Delta s + \int_{y_0}^{y} \left|f(x,t,u(s,t),u^{\Delta_1}(x,t)) - f(x,t,0,0)\right|\Delta t$$

$$+ \int_{y_0}^{y} |f(x,t,0,0)|\Delta t$$

$$\leq \eta e_\lambda(x,y) + \int_{y_0}^{y} p(x,t) e_\lambda(x,t) |u(x,t)|_G \, e_{\ominus\lambda}(x,t)\,\Delta t$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y} p(s,t)e_\lambda(s,t) |u(s,t)|_G \, e_{\ominus\lambda}(s,t)\,\Delta t\Delta s$$

$$\leq \eta e_\lambda(x,y) + |u|_s \left[\int_{y_0}^{y} p(x,t) e_\lambda(x,t)\,\Delta t + \int_{x_0}^{x}\int_{y_0}^{y} p(s,t)e_\lambda(s,t)\,\Delta t\Delta s\right]$$

$$\leq [\eta + N\gamma] e_\lambda(x,y). \tag{3.6}$$

From (3.6) it follows that $Su \in G$. This proves S maps G into itself.

Now we verify S is a contraction map. Let $u(x,y), v(x,y) \in G$. From (3.4) and (3.5) and using hypotheses we have

$$|(Su)(x,y) - (Sv)(x,y)| + \left|(Su)^{\Delta_1}(x,y) - (Sv)^{\Delta_1}(x,y)\right|$$

$$\leq \int_{x_0}^{x}\int_{y_0}^{y} \left|f\left(s,t,u(s,t),u^{\Delta_1}(s,t)\right) - f\left(s,t,v(s,t),v^{\Delta_1}(s,t)\right)\right|\Delta t\Delta s$$

$$+ \int_{y_0}^{y} \left|f\left(x,t,u(x,t),u^{\Delta_1}(x,t)\right) - f\left(x,t,v(x,t),v^{\Delta_1}(x,t)\right)\right|\Delta t$$

$$\leq \int_{y_0}^{y} p(x,t) e_\lambda(x,t) |u(x,t) - v(x,t)|_G \, e_{\ominus\lambda}(x,t)\,\Delta t$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y} p(s,t)e_\lambda(s,t) |u(s,t) - v(s,t)|_G \, e_{\ominus\lambda}(s,t)\,\Delta t\Delta s$$

$$\leq |u - v|_G \, \gamma e_\lambda(x,y). \tag{3.7}$$

Consequently from (3.7) we have

$$|Su - Sv|_G \leq \gamma |u - v|_G.$$

Since $\gamma < 1$, it follows from Banach fixed point theorem that S has a unique fixed point in G. The fixed point of S is however solution of $(1.1) - (1.2)$. The proof is complete. $\square$

Now we give theorem concerning the uniqueness of solutions of $(1.1) - (1.2)$ without existence.

Theorem 3.2. *Assume that the function f in (1.1) satisfies the condition (3.1). Then $(1.1) - (1.2)$ has at most one solution on Ω.*

Proof. Let $u_1(x, y)$ and $u_2(x, y)$ be two solutions of $(1.1) - (1.2)$. Then by hypotheses we have

$$
\left| u_1(x, y) - u_2(x, y) \right| + \left| u_1^{\Delta_1}(x, y) - u_2^{\Delta_1}(x, y) \right|
$$

$$
\leq \int_{x_0}^{x} \int_{y_0}^{y} \left| f\left(s, t, u_1(s, t), u_1^{\Delta_1}(s, t)\right) - f\left(s, t, u_2(s, t), u_2^{\Delta_1}(s, t)\right) \right| \Delta t \Delta s
$$

$$
+ \int_{y_0}^{y} \left| f\left(x, t, u_1(x, t), u_1^{\Delta_1}(x, t)\right) - f\left(x, t, u_2(x, t), u_2^{\Delta_1}(x, t)\right) \right| \Delta t
$$

$$
\leq \int_{y_0}^{y} p(x, t) \left[\left| u_1(x, t) - u_2(x, t) \right| + \left| u_1^{\Delta_1}(x, t) - u_2^{\Delta_1}(x, t) \right| \right] \Delta t
$$

$$
+ \int_{x_0}^{x} \int_{y_0}^{y} p(s, t) \left[\left| u_1(s, t) - u_2(s, t) \right| + \left| u_1^{\Delta_1}(s, t) - u_2^{\Delta_1}(s, t) \right| \right] \Delta t \Delta s. \tag{3.8}
$$

Now an application of Lemma 2.1 (with $a(x, y) = b(x, y) = p(x, y)$ and $c = 0$) to (3.8) yields.

$$
\left| u_1(x, y) - u_2(x, y) \right| + \left| u_1^{\Delta_1}(x, y) - u_2^{\Delta_1}(x, y) \right| \leq 0, \tag{3.9}
$$

which implies $u_1(x, y) = u_2(x, y)$ for $(x, y) \in \Omega$. Thus there is at most one solution of $(1.1) - (1.2)$ on Ω. $\quad\square$

4 Boundedness and continuous dependence

In this section we study the boundedness of solution of $(1.1) - (1.2)$ and the continuous dependence of solutions of equation (1.1) on the given initial boundary values, the function f involved therein and also the continuous dependence of solutions of equations of the (1.1) on parameters.

The following theorem concerning the estimate on the solution $(1.1) - (1.2)$ holds.

Theorem 4.1. *Assume that*

$$
|\alpha(x)| + |\beta(y)| + \left| \alpha^{\Delta}(x) \right| \leq k, \tag{4.1}
$$

$$
|f(x, y, u, v)| \leq r(x, y) \left[|u| + |v| \right], \tag{4.2}
$$

where $k \geq 0$ is a constant $r \in C_{rd}(\Omega, R^n)$. If $u(x, y), (x, y) \in \Omega$ is any solution of $(1.1) - (1.2)$ then

$$
\left| u(x, y) \right| + \left| u^{\Delta_1}(x, y) \right| \leq kq(x, y) e_{\int_{y_0}^{y} r(s, t) q(s, t) \Delta t}(x, x_0), \tag{4.3}
$$

for $(x, y) \in \Omega$, where

$$
q(x, y) = e_r(y, y_0), \tag{4.4}
$$

for $(x, y) \in \Omega$.

Proof. Using the fact that $u(x, y)$ is a solution of $(1.1) - (1.2)$ and conditions $(4.1) - (4.2)$ we have

$$
\left| u(x, y) \right| + \left| u^{\Delta_1}(x, y) \right|
$$

$$
\leq |\alpha(x)| + |\beta(y)| + \left| \alpha^{\Delta}(x) \right|
$$

$$
+ \int_{x_0}^{x} \int_{y_0}^{y} f\left(s, t, u(s, t), u^{\Delta_1}(s, t)\right) \Delta t \Delta s + \int_{y_0}^{y} f\left(x, t, u(x, t), u^{\Delta_1}(x, t)\right) \Delta t
$$

$$
\leq k + \int_{y_0}^{y} r(x, t) \left[\left| u(x, t) \right| + \left| u^{\Delta_1}(x, t) \right| \right] \Delta t
$$

$$
+ \int_{x_0}^{x} \int_{y_0}^{y} r(s, t) \left[\left| u(s, t) \right| + \left| u^{\Delta_1}(s, t) \right| \right] \Delta t \Delta s. \tag{4.5}
$$

Now a suitable application of Lemma 2.1 to (4.5) yields (4.3). $\quad\square$

Remark 4.1. *The estimates obtained in* (4.3) *yields not only the bound on the solution* $u(x,y)$ *of* (1.1) − (1.2) *but also the bound on* $u^{\Delta_1}(x,y)$. *If the estimate on the right side in* (4.3) *is bounded then the solution of* $u(x,y)$ *of* (1.1) − (1.2) *and also* $u^{\Delta_1}(x,y)$ *are bounded on* Ω.

Our next result deals with the continuous dependence of solutions of equation (1.1) and given initial boundary conditions.

Theorem 4.2. *Let* $u_1(x,y)$ *and* $u_2(x,y)$ *be the solution of equation* (1.1) *and given initial boundary conditions*

$$u_1(x,y_0) = \alpha_1(x), \quad u_1(x_0,y) = \beta_1(y), \quad u_1(x_0,y_0) = 0, \tag{4.6}$$

$$u_2(x,y_0) = \alpha_2(x), \quad u_2(x_0,y) = \beta_2(y), \quad u_2(x_0,y_0) = 0, \tag{4.7}$$

respectively where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (R_+, R^n)$. *Suppose that the function* f *in equation* (1.1) *satisfies the condition* (3.1) *and*

$$|\alpha_1(x) + \beta_1(y) - \alpha_2(x) - \beta_2(y)| + \left|\alpha_1^{\Delta}(x) - \alpha_2^{\Delta}(x)\right| \le d, \tag{4.8}$$

for $(x,y) \in \Omega$, *where* $d \ge 0$ *is a constant. Then*

$$|u_1(x,y) - u_2(x,y)| + \left|u_1^{\Delta_1}(x,y) - u_2^{\Delta_2}(x,y)\right| \le d\bar{q}(x,y)\, e_{\underset{y_0}{\overset{y}{\int} p(s,t)\bar{q}(s,t)\Delta t\Delta s}}(x,x_0), \tag{4.9}$$

for $(x,y) \in \Omega$, *where* $\bar{q}(x,y)$ *is defined by the right hand side of* (4.4) *replacing* $r(x,y)$ *by* $p(x,y)$ *for* $(x,y) \in \Omega$.

Proof. From the hypotheses it is easy to observe that

$$
\begin{aligned}
&|u_1(x,y) - u_2(x,y)| + \left|u_1^{\Delta_1}(x,y) - u_2^{\Delta_2}(x,y)\right| \\
&\le |\alpha_1(x) + \beta_1(y) - \alpha_2(x) - \beta_2(y)| + \left|\alpha_1^{\Delta}(x) - \alpha_2^{\Delta}(x)\right| \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} \left|f\left(s,t,u_1(s,t),u_1^{\Delta_1}(s,t)\right) - f\left(s,t,u_2(s,t),u_2^{\Delta_1}(s,t)\right)\right| \Delta t \Delta s \\
&+ \int_{y_0}^{y} \left|f\left(x,t,u_1(x,t),u_1^{\Delta_1}(x,t)\right) - f\left(x,t,u_2(x,t),u_2^{\Delta_1}(x,t)\right)\right| \Delta t \\
&\le d + \int_{y_0}^{y} p(x,t)\left[|u_1(x,t) - u_2(x,t)| - \left|u_1^{\Delta_1}(x,t) - u_2^{\Delta_1}(x,t)\right|\right] \Delta t \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} p(s,t)\left[|u_1(s,t) - u_2(s,t)| + \left|u_1^{\Delta_1}(s,t) - u_2^{\Delta_1}(s,t)\right|\right] \Delta t \Delta s. \tag{4.10}
\end{aligned}
$$

Now a suitable application of Lemma 2.1 to (4.10) yields the bound in (4.9), which shows dependency on solution of equation (1.1) on given initial boundary values. $\qquad\square$

Consider the initial boundary value problem (1.1)-(1.2) and the corresponding initial boundary value problem

$$v^{\Delta_2\Delta_1}(x,y) = F\left(x,y,v(x,y),v^{\Delta_1}(x,y)\right), \tag{4.11}$$

$$v(x,y_0) = \overline{\alpha}(x), \quad v(x_0,y) = \overline{\beta}(y), \quad v(x_0,y_0) = 0, \tag{4.12}$$

for $(x,y) \in \Omega$ where $v \in C_{rd}(\Omega, R_+)$, $\overline{\alpha}, \overline{\beta} \in (R_+, R^n)$ and $F \in C_{rd}(\Omega \times R^n \times R^n, R^n)$.

The next theorem deals with continuous dependence solutions of initial boundary value problem (1.1)-(1.2) on the functions involved therein.

Theorem 4.3. *Assume that the function f in equation* (1.1) *satisfies the condition* (3.1). *Let $v(x,y)$ for* $(x,y) \in \Omega$ *be a solution of initial boundary value problem* (4.11) − (4.12) *and*

$$\left| \alpha\left(x\right) + \beta\left(y\right) - \overline{\alpha}\left(x\right) - \overline{\beta}\left(y\right) \right| + \left| \alpha^\Delta\left(x\right) - \overline{\alpha}^\Delta\left(x\right) \right|$$

$$+ \int_{y_0}^{y} \left| f\left(x,t,v\left(x,t\right),v^{\Delta_1}\left(x,t\right)\right) - F\left(x,t,v\left(x,t\right),v^{\Delta_1}\left(x,t\right)\right) \right| \Delta t$$

$$+ \int_{x_0}^{x} \int_{y_0}^{y} \left| f\left(s,t,v\left(s,t\right),v^{\Delta_1}\left(s,t\right)\right) - F\left(s,t,v\left(s,t\right),v^{\Delta_1}\left(s,t\right)\right) \right| \Delta t \Delta s \leq \epsilon, \tag{4.13}$$

for $(x,y) \in \Omega$ *where* $\overline{\alpha}, \overline{\beta}, F$ *are functions involved in initial boundary value problem* (1.1) − (1.2) *and initial boundary value problem* (4.11) − (4.12) *and* $\epsilon \geq 0$ *is a constant. Then the solution $u(x,y)$ on initial boundary value problem* (1.1) − (1.2) *depends continuously on the functions involved therein.*

Proof. Since $u(x,y)$ and $v(x,y)$ are solutions of initial boundary value problem (1.1)-(1.2) and initial boundary value problem (4.11) − (4.12) and the conditions (3.1), (4.13) we get

$$\left| u\left(x,y\right) - v(x,y) \right| + \left| u^{\Delta_1}\left(x,y\right) - v^{\Delta_1}(x,y) \right|$$

$$\leq \left| \alpha\left(x\right) + \beta\left(y\right) - \overline{\alpha}\left(x\right) - \overline{\beta}\left(y\right) \right| + \left| \alpha\left(x\right) - \overline{\alpha}^\Delta\left(x\right) \right|$$

$$+ \int_{x_0}^{x} \int_{y_0}^{y} \left| f\left(s,t,u\left(s,t\right),u^{\Delta_1}\left(s,t\right)\right) - f\left(s,t,v\left(s,t\right),v^{\Delta_1}\left(s,t\right)\right) \right| \Delta t \Delta s$$

$$+ \int_{x_0}^{x} \int_{y_0}^{y} \left| f\left(s,t,v\left(s,t\right),v^{\Delta_1}\left(s,t\right)\right) - F\left(s,t,v\left(s,t\right),v^{\Delta_1}\left(s,t\right)\right) \right| \Delta t \Delta s$$

$$+ \int_{y_0}^{y} \left| f\left(x,t,u\left(x,t\right),u^{\Delta_1}\left(x,t\right)\right) - f\left(x,t,v\left(x,t\right),v^{\Delta_1}\left(x,t\right)\right) \right| \Delta t$$

$$+ \int_{y_0}^{y} \left| f\left(x,t,v\left(x,t\right),v^{\Delta_1}\left(x,t\right)\right) - F\left(x,t,v\left(x,t\right),v^{\Delta_1}\left(x,t\right)\right) \right| \Delta t$$

$$\leq \varepsilon + \int_{y_0}^{y} p\left(x,t\right) \left[\left| u\left(x,t\right) - v\left(x,t\right) \right| + \left| u^{\Delta_1}\left(x,t\right) - v^{\Delta_1}\left(x,t\right) \right| \right] \Delta t$$

$$+ \int_{x_0}^{x} \int_{y_0}^{y} p\left(x,t\right) \left[\left| u\left(s,t\right) - v\left(x,t\right) \right| + \left| u^{\Delta_1}\left(s,t\right) - v^{\Delta_1}\left(s,t\right) \right| \right] \Delta t \Delta s. \tag{4.14}$$

Now a suitable application of Lemma 2.1 to (4.14) yields

$$\left| u\left(x,y\right) - v(x,y) \right| + \left| u^{\Delta_1}\left(x,y\right) - v^{\Delta_1}(x,y) \right| \leq \varepsilon \overline{q}\left(x,y\right) e_{\int_{y_0}^{y} p(s,t)\overline{q}(s,t)}\left(x,x_0\right). \tag{4.15}$$

$\square$

Now we consider following equation

$$z^{\Delta_2 \Delta_1}\left(x,y\right) = f\left(x,t,z\left(x,t\right),z^{\Delta_1}\left(x,t\right),\mu\right), \tag{4.16}$$

$$z^{\Delta_2 \Delta_1}\left(x,y\right) = f\left(x,t,z\left(x,t\right),z^{\Delta_1}\left(x,t\right),\mu_0\right), \tag{4.17}$$

with the given initial boundary conditions

$$z\left(x,y_0\right) = \tau\left(x\right), \quad z\left(x_0,y\right) = \psi\left(y\right), \quad z\left(x_0,y_0\right) = 0, \tag{4.18}$$

where $z \in C_{rd}\left(\Omega, R_+\right)$, $\tau, \psi \in \left(R_+, R^n\right)$, $f \in C_{rd}\left(\Omega \times R^n \times R^n \times R, R^n\right)$ and μ, μ_0 are real parameters.

Finally, we present following theorem which deals with continuous dependency of solutions of initial boundary value problem (4.16) − (4.18) and initial boundary value problem (4.17) − (4.18) on parameters.

Theorem 4.4. *Assume that the function f in (4.16) and (4.17) satisfy the conditions*

$$|f(x, y, u, v, \mu) - f(x, y, \overline{u}, \overline{v}, \mu)| \leq h(x, y)[|u - \overline{u}| + |v - \overline{v}|], \tag{4.19}$$

$$|f(x, y, u, v, \mu) - f(x, y, u, v, \mu_0)| \leq m(x, y)|\mu - \mu_0|, \tag{4.20}$$

for $(x, y) \in \Omega$ where $n, m \in C_{rd}(\Omega, R_+)$ and

$$\int_{y_0}^{y} m(x, y)\Delta t + \int_{x_0}^{x}\int_{y_0}^{y} m(s, t)\,\Delta t \Delta s \leq \delta, \tag{4.21}$$

where $\delta \geq 0$ is a constant. Let $z_1(x, y)$ and $z_2(x, y)$ be the solutions of initial boundary value problem $(4.16) - (4.18)$ and initial boundary value problem $(4.17) - (4.18)$ respectively. Then

$$
\begin{aligned}
&|z_1(x, y) - z_2(x, y)| + \left|z_1^{\Delta_1}(x, y) - z_2^{\Delta_1}(x, y)\right| \\
&\leq |\mu - \mu_0|\,\delta Q(x, y)\, e_{\int_{y_0}^{y} h(s,t)Q(s,t)\Delta t}(x, x_0),
\end{aligned}
\tag{4.22}
$$

for $x, y \in \Omega$, where

$$Q(x, y) = \int_{y_0}^{y} h(x, t)\Delta t, \tag{4.23}$$

for $x, y \in \Omega$.

Proof. Since $z_1(x, y)$ and $z_2(x, y)$ be the solutions of initial boundary value problem $(4.16) - (4.18)$ and and initial boundary value problem $(4.17) - (4.18)$ and conditions $(4.19) - (4.21)$ we have

$$
\begin{aligned}
&|z_1(x, y) - z_2(x, y)| + \left|z_1^{\Delta_1}(x, y) - z_2^{\Delta_1}(x, y)\right| \\
&\leq \int_{x_0}^{x}\int_{y_0}^{y} \left|f\left(s, t, z_1(s, t), z_1^{\Delta_1}(s, t), \mu\right) - f\left(s, t, z_2(s, t), z_2^{\Delta_1}(s, t), \mu\right)\right|\Delta t \Delta s \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} \left|f\left(s, t, z_2(s, t), z_2^{\Delta_1}(s, t), \mu\right) - f\left(s, t, z_2(s, t), z_2^{\Delta_1}(s, t), \mu_0\right)\right|\Delta t \Delta s \\
&+ \int_{y_0}^{y} \left|f\left(x, t, z_1(x, t), z_1^{\Delta_1}(x, t), \mu\right) - f\left(s, t, z_2(x, t), z_2^{\Delta_1}(x, t), \mu\right)\right|\Delta t \\
&+ \int_{y_0}^{y} \left|f\left(x, t, z_1(x, t), z_1^{\Delta_1}(x, t), \mu\right) - f\left(s, t, z_2(x, t), z_2^{\Delta_1}(x, t), \mu_0\right)\right|\Delta t \\
&\leq \int_{x_0}^{x}\int_{y_0}^{y} h(s, t)\left[|z_1(s, t) - z_2(s, t)| + \left|z_1^{\Delta_1}(s, t) - z_2^{\Delta_1}(s, t)\right|\right]\Delta t \Delta s \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} m(s, t)[|\mu - \mu_0|]\Delta t \Delta s \\
&+ \int_{y_0}^{y} h(x, t)\left[|z_1(x, t) - z_2(x, t)| + \left|z_1^{\Delta_1}(x, t) - z_2^{\Delta_1}(x, t)\right|\right] \\
&+ \int_{y_0}^{y} m(x, t)[|\mu - \mu_0|]\Delta t \\
&\leq |\mu - \mu_0|\,\delta + \int_{y_0}^{y} h(x, t)\left[|z_1(x, t) - z_2(x, t)| + \left|z_1^{\Delta_1}(x, t) - z_2^{\Delta_1}(x, t)\right|\right]\Delta t
\end{aligned}
$$

$$+ \int\limits_{x_0}^{x} \int\limits_{y_0}^{y} h\left(s,t\right) \left[\left| z_1\left(s,t\right) - z_2\left(s,t\right) \right| + \left| z_1^{\Delta_1}\left(s,t\right) - z_2^{\Delta_1}\left(s,t\right) \right| \right] \Delta t \Delta s. \qquad (4.24)$$

Now a suitable application of Lemma to (4.24) yields (4.22) which shows the dependency of solutions of initial boundary value problem $(4.16) - (4.17)$ and initial boundary value problem $(4.17) - (4.18)$ on parameters. $\quad\square$

Remark 4.2. *The results obtained above can be very easily extended to the following partial dynamic equation on time scales*

$$u^{\Delta_2 \Delta_1}\left(x,y\right) = f\left(x,y,u(x,y),u^{\Delta_2}(x,y)\right) \qquad (1.1)$$

with the initial boundary conditions in (1.2) *by modifying suitably the inequality given in Lemma* 2.1

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