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New existence and uniqueness results for an α order boundary value problem

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Abstract

This paper is concerned with the existence of solutions for a non local fractional boundary value problem with integral conditions. New existence and uniqueness results are established using Banach fixed point theorem. Other existence results are obtained using Schauder and Krasnoselskii theorems. As an application, we give an example to illustrate our results.

Keywords: Caputo derivative, fixed point theorem, boundary value problem.

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1 Introduction

Differential equations of fractional order occur more frequently in different research areas such as engineering, physics, chemistry, economics, etc. Indeed, we can find numerous applications in visco-elasticity, electrochemistry control, porous media, electromagnetic and signal processing, etc. [3, 4, 5]. For an extensive collection of results about this type of equations, we refer the reader to [1, 2, 9, 11] and the references therein. In this paper, we are concerned with the following fractional differential problem

$$\begin{gathered} D^{\alpha}u(t) = f(t, u(t), u'(t)),\ t \in J, 2 < \alpha < 3, \\ u(0) = 0, au'(0) - bu''(0) = \int_0^1 u(t)A(t)dt := \delta\,[u]\,, \\ cu'(1) + du''(1) = \int_0^1 u(t)B(t)dt := \beta\,[u] \end{gathered} \tag{1.1}$$

where, A, B are two continuous functions on $J := [0,1]\,,\ A_1 = \sup_{t\in J}|A(t)|, B_1 = \sup_{t\in J}|B(t)|$, $f \in C\,(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and a, b, c, d are nonnegative constants with $\rho := -2\,(ac + ad + bc)$.

2 Notations and Preliminaries

In the following, we give the necessary notation and basic definitions which will be used in this paper:

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as*

$$J^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \alpha > 0, \\ \\ f(t), \alpha = 0, \end{cases} \tag{2.1}$$

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where
$\Gamma(\alpha) = \int\limits_0^{+\infty} e^{-t}t^{\alpha-1}dt.$

Definition 2.2. *The fractional derivative of $f \in C^n([0,\infty[)$ in the sense of Caputo is defined as*

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, n-1<\alpha<n, n \in \mathbb{N} \\ \\ \frac{d^n}{dt^n} f(t), \alpha = n. \end{cases} \quad (2.2)$$

Details on Caputo's derivative can be found in [8, 10].
We give also the following lemmas [5, 7].

Lemma 2.1. *The general solution of the fractional differential equation*

$$D^\alpha x(t) = 0, \alpha > 0 \quad (2.3)$$

is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + ...t^{n-1}, \quad (2.4)$$

where $c_i \in \mathbb{R}, i = 0,1,2,...n-1, n = [\alpha]+1.$

Lemma 2.2. *Let $\alpha > 0$, then*

$$J^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + t^{n-1} \quad (2.5)$$

for some $c_i \in \mathbb{R}, i = 0,1,2,...n-1, n = [\alpha]+1.$

Let us now introduce the space

$$\widetilde{C}(J,\mathbb{R}) = \{u \in C(J,\mathbb{R}), u' \in C(J,\mathbb{R})\} \quad (2.5)$$

On $\widetilde{C}(J,\mathbb{R})$, we define the norm

$$\|u\|_1 := \max(\|u\|, \|u'\|); ||u|| = \sup_{t\in J}|u(t)|, ||u'|| = \sup_{t\in J}|u'(t)|. \quad (2.6)$$

It is clear that $\left(\widetilde{C}(J,\mathbb{R}), \|.\|_1\right)$ is a Banach space.
The following lemma is crucial to prove our results.

Lemma 2.3. *Let $2 < \alpha < 3$. The unique solution of the problem (1.1) is given by:*

$$u(t) = J^\alpha f(t,u(t),u'(t)) - c_0 - c_1 t - c_2 t^2, t \in J, \quad (2.7)$$

where

$$\begin{aligned} c_0 &= 0, \quad J^\alpha f(1,u(1),u'(1)) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau,u(\tau),u'(\tau))d\tau, \\ c_1 &= \frac{2(c+d)\delta[u] - 2b\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta[u]\right]}{\rho}, \\ c_2 &= \frac{-a\left[cJ^{\alpha-1}f(1,u(1),u'(1)) + dJ^{\alpha-2}f(1,u(1),u'(1)) - \beta[u]\right] - c\delta[u]}{\rho}. \end{aligned} \quad (2.8)$$

Proof. Let $u \in \widetilde{C}(J,\mathbb{R})$, then we have

$$D^\alpha u(t) = f(t,u(t),u'(t)), t \in J. \quad (2.9)$$

Applying J^{α} for both sides of (2.9), and using the identity

$$J^{\alpha}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2, t \in J, \tag{2.10}$$

then using the initial conditions of (1.1), we obtain:

$$\begin{aligned} u(t) &= J^{\alpha} f(t, u(t), u'(t)) \\ &+ \frac{-2\,(c+d)\,\delta\,[u] + 2b\left[cJ^{\alpha-1} f(1, u(1), u'(1)) + dJ^{\alpha-2} f(1, u(1), u'(1)) - \beta\,[u]\right]}{\rho} t \\ &+ \frac{a\left[cJ^{\alpha-1} f(1, u(1), u'(1)) + dJ^{\alpha-2} f(1, u(1), u'(1)) - \beta\,[u]\right] + c\delta\,[u]}{\rho} t^2. \end{aligned} \tag{2.11}$$

□

Now, let us define the operator $T : \widetilde{C}\,(J,\mathbb{R}) \to \widetilde{C}\,(J,\mathbb{R})$ as follows:

$$\begin{aligned} Tu(t) &= J^{\alpha} f(t, u(t), u'(t)) \\ &+ \frac{-2\,(c+d)\,\delta\,[u] + 2b\left[cJ^{\alpha-1} f(1, u(1), u'(1)) + dJ^{\alpha-2} f(1, u(1), u'(1)) - \beta\,[u]\right]}{\rho} t \\ &+ \frac{a\left[cJ^{\alpha-1} f(1, u(1), u'(1)) + dJ^{\alpha-2} f(1, u(1), u'(1)) - \beta\,[u]\right] + c\delta\,[u]}{\rho} t^2. \end{aligned} \tag{2.12}$$

It is clear that

$$\begin{aligned} (Tu)'\,(t) &= J^{\alpha-1} f(t, u(t), u'(t)) \\ &+ \frac{-2\,(c+d)\,\delta\,[u] + 2b\left[cJ^{\alpha-1} f(1, u(1), u'(1)) + dJ^{\alpha-2} f(1, u(1), u'(1)) - \beta\,[u]\right]}{\rho} \\ &+ \frac{2a\left[cJ^{\alpha-1} f(1, u(1), u'(1)) + dJ^{\alpha-2} f(1, u(1), u'(1)) - \beta\,[u]\right] + 2c\delta\,[u]}{\rho} t. \end{aligned} \tag{2.13}$$

3 Main Results

The following conditions are essential to prove our results:

(H_1) : Suppose that $|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq k \max(|u_1 - u_2|, |v_1 - v_2|)$, for all $t \in J$, and $u_1, v_1, u_2, v_2 \in \mathbb{R}$.

(H_2) : The function f is continuous on $J \times \mathbb{R} \times \mathbb{R}$.

(H_3) : There exists a positive constant N, such that $|f(t, u, v)| \leq N$, for all $t \in J, u, v \in \mathbb{R}$.

Our first result is based on the Banach fixed point theorem. We have:

Theorem 3.1. *Suppose that the condition (H_1) is satisfied. If*

$$\frac{|\rho|\,k + [(4c+2d)A_1 + 2(a+b)B_1]\,\Gamma(\alpha) + 2(a+b)\alpha k\,(c + d(\alpha-1))}{|\rho|\,\Gamma(\alpha)} < 1, \tag{3.1}$$

then the boundary value problem (1.1) has a unique solution on J.

Proof. To prove this theorem, we need to prove that the operator T has a fixed point on $\widetilde{C}\,(J,\mathbb{R})$. So, we shall prove that T is a contraction mapping on $\widetilde{C}\,(J,\mathbb{R})$.

Let $u, v \in \widetilde{C}(J,\mathbb{R})$. Then for all $t \in J$, we can write

$$\begin{aligned}
|Tu(t)-Tv(t)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&-\frac{2(c+d)t}{\rho}\int_0^1 A(\tau)\left(u(\tau)-v(\tau)\right)d\tau \\
&+\frac{2bct}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&+\frac{2bdt}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&-\frac{2bt}{\rho}\int_0^1 B(\tau)\left(u(\tau)-v(\tau)\right)d\tau \qquad (3.2)\\
&+\frac{act^2}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&+\frac{adt^2}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left[f(\tau,u(\tau),u'(\tau))-f(\tau,v(\tau),v'(\tau))\right]d\tau \\
&-\frac{at^2}{\rho}\int_0^1 B(\tau)\left(u(\tau)-v(\tau)\right)d\tau+\frac{ct^2}{\rho}\int_0^1 A(\tau)\left(u(\tau)-v(\tau)\right)d\tau\Big|.
\end{aligned}$$

Thanks to (H_1), we obtain

$$\begin{aligned}
\|Tu-Tv\| &\le \frac{k}{\Gamma(\alpha+1)}\|u-v\|_1+\left[\frac{(3c+2d)A_1+(a+2b)\,B_1}{|\rho|}\right]\|u-v\| \\
&+\frac{(a+2b)\,k}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\|u-v\|_1. \qquad (3.3)
\end{aligned}$$

Since$\|u-v\|\le\|u-v\|_1$, then we get

$$\|Tu-Tv\|\le\frac{|\rho|\,k+[(3c+2d)A_1+(a+2b)B_1]\,\Gamma(\alpha+1)+(a+2b)k\alpha\,[c+d(\alpha-1)]}{|\rho|\,\Gamma(\alpha+1)}\|u-v\|_1. \qquad (3.4)$$

On the other hand, we have

$$\left\|(Tu)'-(Tv)'\right\|\le\frac{|\rho|\,k+[(4c+2d)A_1+2(a+b)B_1]\,\Gamma(\alpha)+2(a+b)k\alpha\,[c+d(\alpha-1)]}{|\rho|\,\Gamma(\alpha)}\|u-v\|_1. \qquad (3.5)$$

By the condition (3.1), we conclude that T is a contraction mapping. Hence, by Banach fixed point theorem, there exists a unique fixed point $u\in\widetilde{C}(J,\mathbb{R})$ which is a solution of the problem (1.1). □

Our second result is the following:

Theorem 3.2. *Suppose that the conditions* (H_2) *and* (H_3) *are satisfied. If*

$$|\rho|>(4c+2d)A_1+2\,(a+b)\,B_1, \qquad (3.6)$$

then the problem (1.1) has at least a solution in $\widetilde{C}\,(J,\mathbb{R})$.

Proof. We use Schaefer's fixed point theorem to prove that T has a fixed point on $\widetilde{C}\,(J,\mathbb{R})$.

Let us first choose ν such that

$$\nu\ge\max\left(\frac{|\rho|\,N+N(a+2b)\alpha\,[c+d(\alpha-1)]}{\Gamma(\alpha+1)\,(|\rho|-[(3c+2d)A_1+(a+2b)B_1])},\frac{|\rho|\,N+2N(a+b)\,[c+d(\alpha-1)]}{\Gamma(\alpha)\,(|\rho|-[(4c+2d)A_1+2(a+b)B_1])}\right) \qquad (3.7)$$

and set $\widetilde{C}_{\nu}=\{u\in C(J,\mathbb{R}),||u||_1\leq\nu\}$. It is clear that $\widetilde{C}_{\nu}$ is a closed and convex subset.

Step1: T is continuous:

Let $(u_n)_n$ be a sequence such that $u_n\to u$, $n\to+\infty$ in $\widetilde{C}(J,\mathbb{R})$. For each $t\in J$, we have

$$\begin{aligned}|Tu_n(t)-Tu(t)| &\leq \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\right| \\ &+\left[\frac{2(c+d)A_1+2bB_2}{|\rho|}\right]|u_n(t)-u(t)|t \\ &+\frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\right|t \\ &+\left[\frac{aB_1+cA_1}{|\rho|}\right]|u_n(t)-u(t)|t^2 \\ &+\frac{a}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\right|t^2 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\left|(Tu_n)'(t)-(Tu)'(t)\right| &\leq \frac{t^{\alpha}}{\Gamma(\alpha)}\left|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\right| \\ &+\left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]|u_n(t)-u(t)| \\ &+\frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\right| \\ &+\left[\frac{2aB_1+2cA_1}{|\rho|}\right]|u_n(t)-u(t)|t \\ &+\frac{2a}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\right|t. \end{aligned} \tag{3.9}$$

Since f is a continuous function, the right-hand sides of (3.8) (3.9) tend to zero as n tends to $+\infty$. Then

$$||T(u_n)-T(u)||_1\to 0,\ n\to+\infty. \tag{3.10}$$

Step2: We shall prove that $T(\widetilde{C}_{\nu})\subset\widetilde{C}_{\nu}$:

Let us take $u\in\widetilde{C}_{\nu}$. Then for each $t\in J$, we have

$$\begin{aligned}|Tu(t)| &\leq \frac{1}{\Gamma(\alpha+1)}\sup_{t\in J}|f(t,u(t),u'(t))| \\ &+\left[\frac{(3c+2d)A_1+(a+2b)\,B_1}{|\rho|}\right]\|u\| \\ &+\frac{(a+2b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\sup_{t\in J}|f(t,u(t),u'(t))| \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\left|(Tu)'(t)\right| &\leq \frac{1}{\Gamma(\alpha)}\sup_{t\in J}|f(t,u(t),u'(t))| \\ &+\left[\frac{(4c+2d)A_1+2\,(a+b)\,B_1}{|\rho|}\right]\|u\| \\ &+\frac{2\,(a+b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\sup_{t\in J}|f(t,u(t),u'(t))|. \end{aligned} \tag{3.12}$$

By (H_3), we obtain

$$\begin{aligned}|\,|Tu|| &\leq \frac{N}{\Gamma(\alpha+1)}+\left[\frac{(3c+2d)A_1+(a+2b)\,B_1}{|\rho|}\right]\nu+\frac{(a+2b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]N \\ &\leq \nu \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\left\|(Tu)'\right\| &\leq \frac{N}{\Gamma(\alpha)}+\left[\frac{(4c+2d)A_1+2\left(a+b\right)B_1}{|\rho|}\right]\nu+\frac{2N\left(a+b\right)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\\ &\leq \nu. \end{aligned} \tag{3.14}$$

Consequently,

$$\|Tu\|_1 \leq \nu. \tag{3.15}$$

Step3: T **maps bounded sets into equi-continuous sets of** $\widetilde{C}(J,\mathbb{R})$:

Let $t_1,\, t_2 \in J,\, t_1 < t_2, u \in \widetilde{C}_\nu$. Then, we can write

$$\begin{aligned}|Tu(t_2)-Tu(t_1)| &\leq \frac{1}{\Gamma(\alpha+1)}\left(t_2^\alpha-t_1^\alpha\right)\sup_{t\in J}|f(t,u(t),u'(t))|+\left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u\|\left(t_2-t_1\right)\\ &+\frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2-t_1\right)\sup_{t\in J}|f(t,u(t),u'(t))|\\ &+\left[\frac{aB_1+cA_1}{|\rho|}\right]\|u\|\left(t_2^2-t_2^2\right)\\ &+\frac{a}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2^2-t_2^2\right)\sup_{t\in J}|f(t,u(t),u'(t))|. \end{aligned} \tag{3.16}$$

Using (H_3), we obtain the following result

$$\begin{aligned}|Tu(t_2)-Tu(t_1)| &\leq \frac{N}{\Gamma(\alpha+1)}\left(t_2^\alpha-t_1^\alpha\right)+\nu\left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\left(t_2-t_1\right)\\ &+\frac{2bN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2-t_1\right)+\nu\left[\frac{aB_1+cA_1}{|\rho|}\right]\left(t_2^2-t_2^2\right)\\ &+\frac{aN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2^2-t_2^2\right), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\left|(Tu)'\left(t_2\right)-(Tu)'\left(t_1\right)\right| &\leq \frac{N}{\Gamma(\alpha)}\left(t_2^\alpha-t_1^\alpha\right)+2\nu\left[\frac{aB_1+cA_1}{|\rho|}\right]\left(t_2-t_2\right)\\ &+\frac{2aN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\left(t_2-t_2\right). \end{aligned} \tag{3.18}$$

As $t_2 \to t_1$, the right-hand sides of (3.17) and (3.18) tend to zero. Then, as a consequence of Steps $1,2,3$ together with the Arzela-Ascoli theorem, we conclude that T is completely continuous.

Step4: The set B is bounded:

Now, we prove that the set $B=\{u\in\widetilde{C}(J,\mathbb{R}), u=\lambda T(u), 0<\lambda<1\}$ is bounded.

Let $u\in B$, then $u=\lambda T(u)$, for some $0<\lambda<1$. Hence, for each $t\in J$, we have

$$\begin{aligned}\frac{|u(t)|}{\lambda} &\leq \frac{Nt^\alpha}{\Gamma(\alpha+1)}+\|u\|\left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]t+\frac{2bN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]t\\ &+\|u\|\left[\frac{aB_1+cA_1}{|\rho|}\right]t^2+\frac{aN}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]t^2. \end{aligned} \tag{3.19}$$

Since $t\in J$, hence we can write

$$|u(t)|\leq \frac{\lambda N}{|\rho|-\lambda\left[(3c+2d)A_1+(a+2b)\,B_1\right]}\left[\frac{|\rho|}{\Gamma(\alpha+1)}+(a+2b)\left(\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right)\right] \tag{3.20}$$

and

$$|u'(t)|\leq \frac{\lambda N}{|\rho|-\lambda\left[(4c+2d)A_1+2\left(a+b\right)B_1\right]}\left[\frac{|\rho|}{\Gamma(\alpha)}+2\left(a+b\right)\left(\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right)\right]. \tag{3.21}$$

Thanks to 3.6, we get

$$\|u\|_1<\infty. \tag{3.22}$$

This shows that the set is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that T has a fixed point which is a solution of the problem (1.1). □

We state a result due to Krasnoselskii [6] which is needed to prove the existence of at least one solution of the problem (1.1).

Theorem 3.3. *(Krasnoselskii fixed point theorem) Let S be a closed convex and nonempty subset of a Banach space X. Let P, Q be the operators such that*
(i) $Px + Qy \in S$; whenever $x, y \in S$
(ii) P is compact and continuous;
(iii) Q is a contraction mapping. Then there exists x^* such that $x^* = Px^* + Qx^*$.

We have:

Theorem 3.4. *Suppose that there exist ω and θ two positives real numbers such that $0 < \omega < 1, \theta > 0$. If the following conditions are satisfied*

$$\frac{N}{\Gamma(\alpha+1)} + \frac{(a+2b)\,N}{|\rho|}\left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] \quad \leq \quad (1-\omega)\theta, \tag{3.23}$$

$$\left[\frac{(3c+2d)A_1 + (a+2b)\,B_1}{|\rho|}\right] \quad \leq \quad \omega, \tag{3.24}$$

and

$$\frac{|\rho|\,k + [2(c+d)A_1 + 2bB_1]\,\Gamma(\alpha) + 2bk\,[c + d(\alpha-1)]}{|\rho|\,\Gamma(\alpha)} < 1, \tag{3.25}$$

then (1.1) has a solution u such that $\|u\|_1 \leq \theta$.

Proof. Let $B_\theta = \{u \in C(J, \mathbb{R}), ||u||_1 \leq \theta\}$. We define the operator R as follows:

$$\begin{aligned} Ru(t) \quad : \quad &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau), u'(\tau))d\tau + (\frac{-2(c+d)}{\rho}\int_0^1 A(t)\,(u(t)-v(t))\,dt \\ &+ \frac{2bc}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f(\tau, u(\tau), u'(\tau))d\tau \\ &+ \frac{2bd}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f(\tau, u(\tau), u'(\tau))d\tau - \frac{2b}{\rho}\int_0^1 B(t)\,(u(t)-v(t))\,dt)t, \end{aligned} \tag{3.26}$$

It is clear that

$$\begin{aligned} (Ru)'(t) \quad : \quad &= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2} f(\tau, u(\tau), u'(\tau))d\tau - \frac{2(c+d)}{\rho}\int_0^1 A(t)u(t)dt \\ &+ \frac{2bc}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f(\tau, u(\tau), u'(\tau))d\tau + \frac{2bd}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f(\tau, u(\tau), u'(\tau))d\tau \\ &- \frac{2b}{\rho}\int_0^1 B(\tau)u(\tau)d\tau, \end{aligned} \tag{3.27}$$

We also define the operator S by:

$$\begin{aligned} Su(t) \quad : \quad &= \frac{act^2}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f(\tau, u(\tau), u'(\tau))d\tau + \frac{adt^2}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f(\tau, u(\tau), u'(\tau))d\tau \\ &- \frac{at^2}{\rho}\int_0^1 B(\tau)u(\tau)d\tau + \frac{ct^2}{\rho}\int_0^1 A(\tau)u(\tau)d\tau. \end{aligned} \tag{3.28}$$

Then,

$$
\begin{aligned}
(Su)'(t) \quad : \quad &= \frac{2act}{\rho\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f(\tau,u(\tau),u'(\tau))d\tau + \frac{2adt}{\rho\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f(\tau,u(\tau),u'(\tau))d\tau \\
&-\frac{2at}{\rho}\int_0^1 B(\tau)u(\tau)d\tau + \frac{2ct}{\rho}\int_0^1 A(\tau)u(\tau)d\tau. \qquad (3.29)
\end{aligned}
$$

(1*) Let $u, v \in B_\theta$. We have

$$
\begin{aligned}
|Ru(t)+Sv(t)| \quad &\leq \quad \frac{1}{\Gamma(\alpha+1)}|f(t,u(t),u'(t))| + \left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u\| \qquad (3.30)\\
&+\left[\frac{cA_1+aB_1}{|\rho|}\right]\|v\| + \frac{(a+2b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]|f(t,u(t),u'(t))|,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|(Ru)'(t)+(Sv)'(t)\right| \quad &\leq \quad \frac{1}{\Gamma(\alpha)}|f(t,u(t),u'(t))| + \left[\frac{2(c+d)A_1+bB_1}{|\rho|}\right]\|u\| \qquad (3.31)\\
&+\left[\frac{2cA_1+2aB_1}{|\rho|}\right]\|v\| + \frac{2(a+b)}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]|f(t,u(t),u'(t))|.
\end{aligned}
$$

Thanks to (3.23) and (3.24), we can write

$$
\begin{aligned}
\|Ru+Sv\| \quad &\leq \quad \frac{N}{\Gamma(\alpha+1)} + \theta\left[\frac{(3c+2d)A_1+(2b+a)B_1}{|\rho|}\right] + \frac{(a+2b)N}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right] \qquad (3.32)\\
&\leq \quad \omega\theta+(1-\omega)\theta=\theta.
\end{aligned}
$$

Consequently,

$$
Ru+Sv \in B_\theta. \qquad (3.33)
$$

(2*) Now we prove the contraction of R.

$$
\begin{aligned}
|Ru(t)-Rv(t)| \quad &\leq \quad \frac{1}{\Gamma(\alpha+1)}|f(t,u(t),u'(t))-f(t,v(t),v'(t))| \\
&+\left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u-v\| \qquad (3.34)\\
&+\frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]|f(t,u(t),u'(t))-f(t,v(t),v'(t))|,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|(Ru)'(t)-(Rv)'(t)\right| \quad &\leq \quad \frac{1}{\Gamma(\alpha)}|f(t,u(t),u'(t))-f(t,v(t),v'(t))| + \left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u-v\| \\
&+\frac{2b}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]|f(t,u(t),u'(t))-f(t,v(t),v'(t))|, \qquad (3.35)
\end{aligned}
$$

By the hypothesis (H_1), we have

$$
\begin{aligned}
|Ru-Rv| \quad &\leq \quad \frac{k}{\Gamma(\alpha+1)}\|u-v\|_1 + \left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u-v\| \\
&+\frac{2bk}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\|u-v\|_1, \qquad (3.36)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|(Ru)'(t)-(Rv)'(t)\right| \quad &\leq \quad \frac{k}{\Gamma(\alpha)}\|u-v\|_1 + \left[\frac{2(c+d)A_1+2bB_1}{|\rho|}\right]\|u-v\| \\
&+\frac{2bk}{|\rho|}\left[\frac{c}{\Gamma(\alpha)}+\frac{d}{\Gamma(\alpha-1)}\right]\|u-v\|_1, \qquad (3.37)
\end{aligned}
$$

Hence, by (3.25), R is a contraction mapping.

(3*) The Continuity of f implies that the operator S is continuous.

(4*) The compactness of S :

Let us take $u \in B_\theta, t_1, t_2 \in J, t_1 < t_2$. We have

$$|Su(t_1) - Su(t_2)| \leq \nu \left[\frac{aB_1 + cA_1}{|\rho|}\right] \left(t_2^2 - t_1^2\right) + \frac{aN}{|\rho|} \left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] \left(t_2^2 - t_1^2\right), \tag{3.38}$$

and

$$\left|(Su)'(t_1) - (Su)'(t_2)\right| \leq 2\nu \left[\frac{aB_1 + cA_1}{|\rho|}\right] (t_2 - t_1) + \frac{2aN}{|\rho|} \left[\frac{c}{\Gamma(\alpha)} + \frac{d}{\Gamma(\alpha-1)}\right] (t_2 - t_1). \tag{3.39}$$

The right hand side of (3.38) and (3.39) are independent of u. Hence S is equicontinuous. And as $t_1 \to t_2$, the left hand sides of (3.38) and (3.39) tend to 0; so $S(B_\theta)$ is relatively compact and then by Ascolli-Arzella theorem, the operator S is compact. Finally, by Krasnoselskii theorem, we conclude that there exists a solution to (1.1). Theorem 3.4 is thus proved. □

4 Example

Consider the three-point BVP

$$\begin{cases} D^{\frac{5}{2}}u(t) = \frac{u(t)+u'(t)}{64}e^{-t^2} + \frac{1}{1+t^2}, t \in [0,1], \\ u(0) = 0, u'(0) - u''(0) = \int_0^1 u(t)\frac{e^{-t}}{64}dt, \\ 2u'(1) + 2u''(1) = \int_0^1 u(t)\frac{e^{-t^2}}{64}dt, \end{cases} \tag{4.1}$$

In this example, we have $a = b = 1, c = d = 2,\ A(t) = \frac{e^{-t}}{64}, B(t) = \frac{e^{-t^2}}{64}, N = \frac{1}{64}, A_1 = B_1 = k.$

The condition (3.1) is given by

$$\frac{|\rho|\, k + [(4c + 2d)A_1 + 2(a + b)B_1]\, \Gamma(\alpha) + 2(a + b)\alpha k\, (c + d(\alpha - 1))}{|\rho|\, \Gamma(\alpha)} = \frac{31 + 6\sqrt{\pi}}{288\sqrt{\pi}} < 1.$$

Then, the problem (4.1) has a solution on $[0, 1]$.

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