Malaya Journal of Matematik 4(1)(2013) 20–29

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Time dependent solution of Non-Markovian queue with two phases of service and general vacation time distribution

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Abstract

We consider an $M^{[x]}/G/1$ queue with two phases of service, with different general (arbitrary) service time distributions. The first phase of service is essential for all customers, as soon as the first service of a customer is completed, then with probability θ, he may opt for the second service or else with probability $(1-\theta)$, he leaves the system. At each service completion, the server will take compulsory vacation. The vacation period of the server has two heterogeneous phases. Phase one is compulsory and phase two follows the phase one vacation in such a way that the server may take phase two vacation with probability p or return back to the system with probability $(1-p)$. The service and vacation periods are assumed to be general. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly. Also the average number of customers in the queue and the waiting time are also derived.

Keywords: Batch arrival, optional service, second optional vacation, stability condition, average queue size, average waiting time.

2010 MSC: 60K25, 60K30.

1 Introduction

The modelling analysis for the queueing systems with vacations has been done by a considerable amount of work in the past and successfully used in various applied problems such as production/inventory system, communication systems, computer networks etc. A comprehensive and excellent study on the vacation models can be found in Levy and Yechiali [11], Doshi [6], Takagi [15], Lee et al. [10], Krishna Reddy et al. [9], Hur and Paik [7] and others. Batch arrival $M^{[x]}/G/1$ queueing systems with multiple vacations were first studied by Baba [1]. Badamchi Zadeh [2] studied a batch arrival queuening system with two phases of heterogenous service with optional second service and restricted admissibility with single vacation policy.

Recently, there have been several contributions considering queueing system of M/G/1 type in which the server may provide a second phase service. Such queueing situations occur in day-to-day life, for example in many applications such as hospital services, production systems, bank services, computer and communication networks there is two phase of services such that the first phase is essential for all customers, but as soon as the essential services completed, it may leave the system or may immediately go for the second phase of service. One may refer to Medhi [14], Krishnakumar et al. [8], Choudhury [4], Madan and Choudhury [13], Choudhury and Paul [5]. Badamchi and Shankar [3] have also studied a single server queue with two phase queueing system with Bernoulli feedback and Bernoulli schedule server vacation. Madan and Choudhury [12] proposed an queueing system with restricted admissibilty of arriving batches and Bernoulli schedule server vacation.

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In this paper we consider batch arrival queue with two phases of service and optional server vacation. The first phase of service is essential for all customers, as soon as the first service of a customer is completed, then with probability θ, he may opt for the second service or else with probability $(1-\theta)$, he leaves the system. After completion of each service, the server will take compulsory vacation. The vacation periods of the server has two heterogeneous phases. However, after returning from phase one compulsory vacation the server may take one more optional vacation with probability p or return back to the system with probability $(1-p)$.

This paper is organized as follows. The mathematical description of our model is given in section 2. Definitions and equations governing the system are given in section 3. The time dependent solution have been obtained in section 4 and corresponding steady state results have been derived explicitly in section 5. Average queue size and average waiting time are computed in section 6.

2 Mathematical description of the model

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ $(i = 1, 2, \ldots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$ and $\sum\limits_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) There is a single server who provides the first phase of essential service for all customers, as soon as the essential service of a customer is completed, then with probability θ, he may opt for the second service or else with probability $(1-\theta)$, he leave the system.

c) The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x+dx]$, given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}, \quad i = 1, 2,$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int\limits_0^s \mu_i(x)dx}, \quad i = 1, 2.$$

d) After completion of each service, the server will take compulsory vacation of random length. The vacation time has two phases with phase one is compulsory. However, after phase one vacation, the server takes phase two optional vacation with probability p or may return back to the system with probability $(1-p)$.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V_i(t)$ and density function $v_i(t)$. Let $\gamma_i(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x, so that

$$\gamma_i(x) = \frac{v_i(x)}{1-V_i(x)}, \quad i = 1, 2,$$

and therefore,

$$v_i(t) = \gamma_i(t) e^{-\int\limits_0^t \gamma_i(x)dx} \quad i = 1, 2.$$

f) Various stochastic processes involved in the system are assumed to be independent of each other.

3 Definitions and equations governing the system

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing essential service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x.

Consequently $P_n^{(1)}(t) = \int\limits_0^\infty P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the essential service irrespective of the value of x.
$P_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing second optional service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $P_n^{(2)}(t) = \int\limits_0^\infty P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the second optional service irrespective of the value of x.

$V_n^{(1)}(x,t)$ = Probability that at time t, the server is under phase one compulsory vacation with elapsed vacation time x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(1)}(t)= \int\limits_0^\infty V_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase one compulsory vacation irrespective of the value of x.

$V_n^{(2)}(x,t)$ = Probability that at time t, the server is under phase two optional vacation with elapsed vacation time x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(2)}(t)=\int\limits_0^\infty V_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase two optional vacation irrespective of the value of x.
Q(t) = Probability that at time t, there are no customers in the queue and the server is idle but available in the system.

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + [\lambda + \mu_1(x)]P_0^{(1)}(x,t) = 0 \tag{3.1}$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + [\lambda + \mu_1(x)]P_n^{(1)}(x,t) = \lambda \sum_{k=1}^{n} c_k P_{n-k}^{(1)}(x,t), \ \ n \geq 1 \tag{3.2}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial t}P_0^{(2)}(x,t) + [\lambda + \mu_2(x)]P_0^{(2)}(x,t) = 0 \tag{3.3}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + [\lambda + \mu_2(x)]P_n^{(2)}(x,t) = \lambda \sum_{k=1}^{n} c_k P_{n-k}^{(2)}(x,t), \quad n \geq 1 \tag{3.4}$$

$$\frac{\partial}{\partial x}V_0^{(1)}(x,t) + \frac{\partial}{\partial t}V_0^{(1)}(x,t) + [\lambda + \gamma_1(x)]V_0^{(1)}(x,t) = 0 \tag{3.5}$$

$$\frac{\partial}{\partial x}V_n^{(1)}(x,t) + \frac{\partial}{\partial t}V_n^{(1)}(x,t) + [\lambda + \gamma_1(x)]V_n^{(1)}(x,t) = \lambda \sum_{k=1}^{n} c_k V_{n-k}^{(1)}(x,t), \quad n \geq 1 \tag{3.6}$$

$$\frac{\partial}{\partial x}V_0^{(2)}(x,t) + \frac{\partial}{\partial t}V_0^{(2)}(x,t) + [\lambda + \gamma_2(x)]V_0^{(2)}(x,t) = 0 \tag{3.7}$$

$$\frac{\partial}{\partial x}V_n^{(2)}(x,t) + \frac{\partial}{\partial t}V_n^{(2)}(x,t) + [\lambda + \gamma_2(x)]V_n^{(2)}(x,t) = \lambda\beta \sum_{k=1}^{n} c_k V_{n-k}^{(2)}(x,t), \ \ n \geq 1 \tag{3.8}$$

$$\frac{d}{dt}Q(t) + \lambda Q(t) = (1-p)\int_0^\infty \gamma_1(x)V_0^{(1)}(x,t)dx + \int_0^\infty \gamma_2(x)V_0^{(2)}(x,t)dx \tag{3.9}$$

The above equations are to be solved subject to the following boundary conditions:

$$P_n^{(1)}(0,t) = \alpha\lambda C_{n+1}Q(t) + (1-p)\int_0^\infty \gamma_1(x)V_{n+1}^{(1)}(x,t)dx + \int_0^\infty \gamma_2(x)V_{n+1}^{(2)}(x,t)dx, \quad n \geq 0 \tag{3.10}$$

$$P_n^{(2)}(0,t) = \theta \int_0^\infty \mu_1(x)P_n^{(1)}(x,t)dx, \quad n \geq 0 \tag{3.11}$$

$$V_n^{(1)}(0,t) = (1-\theta)\int_0^\infty \mu_1(x)P_n^{(1)}(x,t)dx + \int_0^\infty \mu_2(x)P_n^{(2)}(x,t)dx, \quad n \geq 0 \tag{3.12}$$

$$V_n^{(2)}(0,t) = p\int_0^\infty \gamma_1(x)V_n^{(1)}(x,t)dx, \quad n \geq 0 \tag{3.13}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$\begin{aligned} &V_0^{(i)}(0) = V_n^{(i)}(0) = 0, \quad Q(0) = 1 \text{ and} \\ &P_n^{(i)}(0) = 0 \text{ for } n = 0,1,2,..., \quad i = 1,2. \end{aligned} \tag{3.14}$$

4 Generating functions of the queue length: The time-dependent solution

In this section we obtain the transient solution for the above set of dfferential-difference equations.

Theorem 4.1. *The system of differential difference equations to describe an $M^{[x]}/G/1$ queue with first essential service, second optional service, first phase of vacation and optional vacation are given by equations (3.1) to (3.13) with initial conditions (3.14) and the generating functions of transient solution are given by equation (4.50) to (4.53).*

Proof. We define the probability generating functions ,

$$P^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x,t); \quad P^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(t), \text{for } i = 1,2. \tag{4.1}$$

$$V^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n V_n^{(i)}(x,t); V^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n V_n^{(i)}(t), C(z) = \sum_{n=1}^{\infty} c_n z^n \tag{4.2}$$

which are convergent inside the circle given by z $\leq$ 1 and define the Laplace transform of a function f(t) as

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt, \quad \Re(s) > 0 \tag{4.3}$$

Taking the Laplace transform of equations (3.1) to (3.13) and using (3.14), we obtain

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s) + (s+\lambda+\mu_1(x))\bar{P}_0^{(1)}(x,s) = 0 \tag{4.4}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + (s+\lambda+\mu_1(x))\bar{P}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(1)}(x,s), n \geq 1 \tag{4.5}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s) + (s+\lambda+\mu_2(x))\bar{P}_0^{(2)}(x,s) = 0 \tag{4.6}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + (s+\lambda+\mu_2(x))\bar{P}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(2)}(x,s), n \geq 1 \tag{4.7}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(1)}(x,s) + (s+\lambda+\gamma_1(x))\bar{V}_0^{(1)}(x,s) = 0 \tag{4.8}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,s) + (s+\lambda\beta+\gamma_1(x))\bar{V}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(1)}(x,s), n \geq 1 \tag{4.9}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(2)}(x,s) + (s+\lambda+\gamma_2(x))\bar{V}_0^{(2)}(x,s) = 0 \tag{4.10}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,s) + (s+\lambda+\gamma_2(x))\bar{V}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(2)}(x,s), n \geq 1 \tag{4.11}$$

$$(s+\lambda)\bar{Q}(s)=1+(1-p)\int_0^\infty \gamma_1(x)\bar{V}_0^{(1)}(x,s)dx+\int_0^\infty \gamma_2(x)\bar{V}_0^{(2)}(x,s)dx \tag{4.12}$$

$$\bar{P}_n^{(1)}(0,s)=\alpha\lambda c_{n+1}\bar{Q}(s)+(1-p)\int_0^\infty \gamma_1(x)\bar{V}_{n+1}^{(1)}(x,s)dx+\int_0^\infty \gamma_2(x)\bar{V}_{n+1}^{(2)}(x,s)dx,\quad n\geq 0 \tag{4.13}$$

$$\bar{P}_n^{(2)}(0,s)=\theta\int_0^\infty \mu_1(x)\bar{P}_n^{(1)}(x,s)dx,\quad n\geq 0 \tag{4.14}$$

$$\bar{V}_n^{(1)}(0,s)=(1-\theta)\int_0^\infty \mu_1(x)\bar{P}_n^{(1)}(x,s)dx+\int_0^\infty \mu_2(x)\bar{P}_n^{(2)}(x,s)dx,\ n\geq 0 \tag{4.15}$$

$$\bar{V}_n^{(2)}(0,s)=p\int_0^\infty \gamma_1(x)\bar{V}_n^{(1)}(x,s)dx,\quad n\geq 0 \tag{4.16}$$

Now multiplying equations (4.5), (4.7), (4.9) and (4.11) by z^n and summing over n from 1 to ∞, adding to equation (4.4), (4.6), (4.8) and (4.10) using the generating functions defined in (4.1) and (4.2) we get

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_1(x)]\bar{P}^{(1)}(x,z,s)=0 \tag{4.17}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,z,s)+[s+\lambda(1-C(z))+\mu_2(x)]\bar{P}^{(2)}(x,z,s)=0 \tag{4.18}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma_1(x)]\bar{V}^{(1)}(x,z,s)=0 \tag{4.19}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma_2(x)]\bar{V}^{(2)}(x,z,s)=0 \tag{4.20}$$

For the boundary conditions, we multiply both sides of equation (4.13) by z^n sum over n from 0 to ∞, and use the equation (4.12), we get

$$\begin{aligned}z\bar{P}^{(1)}(0,z,s)&=[1-s\bar{Q}(s)]+\lambda[C(z)-1]\bar{Q}(s)\\&+(1-p)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx+\int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx\end{aligned} \tag{4.21}$$

Performing similar operation on equations (4.14) to (4.16) we get,

$$\bar{P}^{(2)}(0,z,s)=\theta\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx \tag{4.22}$$

$$\bar{V}^{(1)}(0,z,s)=(1-\theta)\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx+\int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx \tag{4.23}$$

$$\bar{V}^{(2)}(0,z,s)=p\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \tag{4.24}$$

Integrating equation (4.17) between 0 to x, we get

$$\bar{P}^{(1)}(x,z,s)=\bar{P}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \mu_1(t)dt} \tag{4.25}$$

where $P^{(1)}(0,z,s)$ is given by equation (4.21).
Again integrating equation (4.25) by parts with respect to x yields,

$$\bar{P}^{(1)}(z,s)=\bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{4.26}$$

where

$$\bar{B}_1(s+\lambda-\lambda C(z))=\int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB_1(x) \tag{4.27}$$

is the Laplace-Stieltjes transform of the first phase of essential service time $B_1(x)$. Now multiplying both sides of equation (4.25) by $\mu_1(x)$ and integrating over x we obtain

$$\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda(1-c(z))] \tag{4.28}$$

Similarly, on integrating equations (4.18) to (4.20) from 0 to x, we get

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \mu_2(t)dt} \tag{4.29}$$

$$\bar{V}^{(1)}(x,z,s) = \bar{V}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \gamma_1(t)dt} \tag{4.30}$$

$$\bar{V}^{(2)}(x,z,s) = \bar{V}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \gamma_2(t)dt} \tag{4.31}$$

where $\bar{P}^{(2)}(0,z,s)$, $\bar{V}^{(1)}(0,z,s)$ and $\bar{V}^{(2)}(0,z,s)$ are given by equations
(4.22) to (4.24). Again integrating equations (4.29) to (4.31) by parts with respect to x yields,

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda c(z)}\right] \tag{4.32}$$

$$\bar{V}^{(1)}(z,s) = \bar{V}^{(1)}(0,z,s)\left[\frac{1-\bar{V}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{4.33}$$

$$\bar{V}^{(2)}(z,s) = \bar{V}^{(2)}(0,z,s)\left[\frac{1-\bar{V}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{4.34}$$

where

$$\bar{B}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB_2(x) \tag{4.35}$$

$$\bar{V}_1(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV_1(x) \tag{4.36}$$

$$\bar{V}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV_2(x) \tag{4.37}$$

are the Laplace-Stieltjes transform of the second optional service time $B_2(x)$, first phase of vacation time $V_1(x)$ and second optional vacation $V_2(x)$ respectively. Now multiplying both sides of equation (4.29) to (4.31) by $\mu_2(x)$, $\gamma_1(x)$ and $\gamma_2(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)] \tag{4.38}$$

$$\int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx = \bar{V}^{(1)}(0,z,s)\bar{V}_1[s+\lambda-\lambda C(z)] \tag{4.39}$$

$$\int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx = \bar{V}^{(2)}(0,z,s)\bar{V}_2[s+\lambda-\lambda C(z)] \tag{4.40}$$

Using equations (4.28) and (4.38), we can write equation (4.23) as

$$\bar{V}^{(1)}(0,z,s) = (1-\theta)\bar{B}_1(R)\bar{P}^{(1)}(0,z,s) + \bar{B}_2(R)\bar{P}^{(2)}(0,z,s) \tag{4.41}$$

Using equation (4.28) in (4.22), we get

$$\bar{P}^{(2)}(0,z,s) = \theta\bar{P}^{(1)}(0,z,s)\bar{B}_1(R) \tag{4.42}$$

By using equation (4.42), equation (4.41) reduces to

$$\bar{V}^{(1)}(0,z,s) = \bar{B}_1(R)[1-\theta+\theta\bar{B}_2(R)]\bar{P}^{(1)}(0,z,s) \tag{4.43}$$

Using equations (4.39) and (4.43) in (4.24), we get

$$\bar{V}^{(2)}(0,z,s) = p\bar{B}_1(R)\bar{V}_1(R)[1-\theta+\theta\bar{B}_2(R)]\bar{P}^{(1)}(0,z,s) \tag{4.44}$$

Similarly using equations (4.39), (4.40), (4.43) and (4.44) in (4.21), we get

$$\bar{P}^{(1)}(0,z,s) = \frac{[1-s\bar{Q}(s)] + \lambda[(C(z)-1]\bar{Q}(s)}{DR} \tag{4.45}$$

where

$$DR = z - \bar{B}_1(R)\bar{V}_1(R)[1-\theta+\theta\bar{B}_2(R)](1-p+p\bar{V}_2(R)), \tag{4.46}$$

$R = s+\lambda-\lambda C(z)$. Substituting the value of $\bar{P}^{(1)}(0,z,s)$ from equation (4.45) into equations (4.42), (4.43) and (4.44), we get

$$\bar{P}^{(2)}(0,z,s) = \theta\frac{\bar{B}_1(R)[(1-s\bar{Q}(s)) + \lambda\alpha(C(z)-1)\bar{Q}(s)]}{DR} \tag{4.47}$$

$$\bar{V}^{(1)}(0,z,s) = \bar{B}_1(R)(1-\theta+\theta\bar{B}_2(R))\frac{[(1-s\bar{Q}(s)) + \lambda\alpha(C(z)-1)\bar{Q}(s)]}{DR} \tag{4.48}$$

$$\bar{V}^{(2)}(0,z,s) = p\bar{B}_1(R)(1-\theta+\theta\bar{B}_2(R))\bar{V}_1(R)\frac{[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{DR} \tag{4.49}$$

Using equations (4.45), (4.47), (4.48) and (4.49) in (4.26), (4.32), (4.33) and (4.34), we get

$$\bar{P}^{(1)}(z,s) = \frac{[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{DR}\frac{[1-\bar{B}_1(R)]}{R} \tag{4.50}$$

$$\bar{P}^{(2)}(z,s) = \frac{\theta\bar{B}_1(R)[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{DR}\frac{[1-\bar{B}_2(R)]}{R} \tag{4.51}$$

$$\bar{V}^{(1)}(z,s) = \frac{[1-\theta+\theta\bar{B}_2(R)]\bar{B}_1(R)}{DR} [(1-s\bar{Q}(s))(\lambda C(z)-\lambda)\bar{Q}(s)]\frac{[1-\bar{V}_1(R)]}{R} \tag{4.52}$$

$$\bar{V}^{(2)}(z,s) = p\bar{B}_1(R)\bar{V}_1(R)\frac{[1-\theta+\theta\bar{B}_2(R)]}{DR} [(1-s\bar{Q}(s)) + (\lambda C(z)-\lambda)\bar{Q}(s)]\frac{[1-\bar{V}_2(R)]}{R} \tag{4.53}$$

where DR is given by equation (60). Thus $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ are completely determined from equations (4.50) to (4.53) which completes the proof of the theorem. □

5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \tag{5.1}$$

In order to determine $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (4.50) to (4.53). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + V^{(1)}(1) + V^{(2)}(1) + Q = 1 \tag{5.2}$$

Theorem 5.1. *The steady state probabilities for an $M^{[x]}/G/1$ first essential service, second optional service, first phase of vacation and optional vacation are given by*

$$P^{(1)}(1) = \frac{\lambda E(I)E(B_1)Q}{dr} \tag{5.3}$$

$$P^{(2)}(1) = \frac{\theta\lambda E(I)E(B_2)Q}{dr} \tag{5.4}$$

$$V^{(1)}(1) = \frac{\lambda E(I)E(V_1)Q}{dr} \tag{5.5}$$

$$V^{(2)}(1) = \frac{p\lambda E(I)E(V_2)Q}{dr} \tag{5.6}$$

where

$$dr = 1 - \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)], \tag{5.7}$$

$P^{(1)}(1)$, $P^{(2)}(1)$, $V^{(1)}(1)$, $V^{(2)}(1)$ *and* Q *are the steady state probabilities that the server is providing first essential service, second optional service, server under first phase of vacation, optional vacation, server under idle respectively without regard to the number of customers in the system.*

Proof. Multiplying both sides of equations (4.50) to (4.53) by s, taking limit as $s \to 0$, applying property (5.1) and simplifying, we obtain

$$P^{(1)}(z) = \frac{[\bar{B}_1(T) - 1]Q}{D(z)} \tag{5.8}$$

$$P^{(2)}(z) = \frac{\theta\bar{B}_1(T)[\bar{B}_2(T) - 1]Q}{D(z)} \tag{5.9}$$

$$V^{(1)}(z) = \frac{\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)][\bar{V}_1(T) - 1]Q}{D(z)} \tag{5.10}$$

$$V^{(2)}(z) = \frac{p\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)]\bar{V}_1(f_3(z))[\bar{V}_2(T) - 1]Q}{D(z)} \tag{5.11}$$

where

$$D(z) = z - \bar{V}_1(T)\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)][1 - p + p\bar{V}_2(T)], \tag{5.12}$$

$T = \lambda - \lambda C(z)$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (5.8) to (5.11) we obtain

$$W_q(z) = P^{(1)}(z) + P^{(2)}(z) + V^{(1)}(z) + V^{(2)}(z)$$

$$\begin{aligned} W_q(z) = {} & \frac{[\bar{B}_1(T) - 1]Q}{D(z)} \\ & + \frac{\theta\bar{B}_1(T)[\bar{B}_2(T) - 1]Q}{D(z)} \\ & + \frac{\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)][\bar{V}_1(T) - 1]Q}{D(z)} \\ & + \frac{p\bar{B}_1(T)[1 - \theta + \theta\bar{B}_2(T)]\bar{V}_1(T)[\bar{V}_2(T) - 1]Q}{D(z)} \end{aligned} \tag{5.13}$$

where C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers,
$-\bar{B}_i'(0) = E(B_i), -\bar{V}_i'(0) = E(V_i)$, for i = 1, 2.

In order to find Q, we use the normalization condition $W_q(1)$ + Q = 1. We see that for z=1, $W_q(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying we get,

$$W_q(1) = \frac{\alpha\lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1 + pE(V_2))]}{1 - \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)]}Q \tag{5.14}$$

Therefore adding Q to equation (5.14), equating to 1 and simplifying, we get

$$Q = 1 - \rho \tag{5.15}$$

and hence the utilization factor ρ of the system is given by

$$\rho = \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)] \tag{5.16}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (5.15) gives the probability that the server is idle. Substituting Q from (5.15) into (5.13), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size. □

6 The mean queue size and the mean system size

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz} W_q(z) \text{ at } z = 1$$

Since this formula gives 0/0 form, then we write $W_q(z)$ given in (5.13) as $W_q(z) = \frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are numerator and denominator of the right hand side of (5.13) respectively. Then we use

$$L_q = \lim_{z \to 1} \frac{d}{dz} W_q(z) = \lim_{z \to 1} \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right] \tag{6.1}$$

where primes and double primes in (6.1) denote first and second derivative at $z = 1$, respectively. Carrying out the derivative at $z = 1$ we have

$$N'(1) = \lambda\alpha\beta E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)]Q \tag{6.2}$$

$$\begin{aligned} N''(1) = [&\lambda^2(E(I))^2[E(B_1^2) + \theta E(B_2^2) + E(V_1^2) + pE(V_2^2))] \\ &+ \lambda E(I(I-1))[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2] \\ &+ 2\lambda^2(E(I))^2[E(B_1)E(V_1) + p\theta E(B_2)E(V_2)] \\ &+ 2\lambda^2(E(I))^2(E(B_1) + E(V_1))(\theta E(B_2) + pE(V_2))]Q \end{aligned} \tag{6.3}$$

$$D'(1) = 1 - \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)] \tag{6.4}$$

$$\begin{aligned} D''(1) = &-\lambda^2(E(I))^2[E(B_1) + \theta E(B_2) + E(V_1 + pE(V_2] \\ &-2\lambda^2(E(I))^2[E(B_1)E(V_1) + p\theta E(B_2)E(V_2)] \\ &+2\lambda^2(E(I))^2[E(B_1) + E(V_1)][\theta E(B_2) + pE(V_2)] \\ &+\lambda E(I(I-1))[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)], \end{aligned} \tag{6.5}$$

where $E(V^2)$ are the second moment of the vacation time. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (6.2) to (6.5) into equation (6.1) we obtain L_q in the closed form.

Further, we find the mean system size L using Little's formula. Thus we have

$$L = L_q + \rho \tag{6.6}$$

where L_q has been found by equation (6.1) and ρ is obtained from equation (5.16).

7 The average waiting time

Let W_q and W denote the mean waiting time in the queue and in the system respectively. Then using Little's formula, we obtain,

$$W_q = \frac{L_q}{\lambda} \tag{7.1}$$

$$W = \frac{L}{\lambda}, \tag{7.2}$$

where L_q and L have been found in equations (6.1) and (6.6).

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Received: April 18, 2013; *Accepted*: June 18, 2013