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Fitting ellipsoids to objects by the first order polarization tensor

Taufiq K. A. Khairuddin,[a,b,*] and William R. B. Lionheart[a]

[a] *School of Mathematics, The University of Manchester, UK.*

[b] *Department of Mathematical Sciences, Universiti Teknologi Malaysia, Johor, Malaysia.*

Abstract

This article present the manual to determine ellipsoids that has the same first order polarization tensor to any conducting objects included in electrical field. Given the first order polarization tensor for an object at specified conductivity, the analytical formula of the first order polarization tensor for ellipsoid in the integral form is firstly expressed as system of nonlinear equation by the trapezium rule. We will then discuss how the derived equations are simultaneously solved by appropriated numerical method to uniquely compute all semi principal axes of the ellipsoid. Few examples to use the proposed technique in this study are also provided in three different situations. In each case, the first order polarization tensor for the obtained ellipsoid can be calculated back from the analytical formula to examine the effectiveness of the method.

Keywords: Integral operator, multi-indices, matrices, numerical integration, eigenvalues.

2010 MSC: 65D30, 34A34, 05B20.

1 Introduction

There has been many efforts for examples in [3, 6, 10, 11, 12] done to describe and present shape of objects theoretically as they are very essential in science and engineering applications. Most shapes discussed are very unique and special and has their own mathematical properties. However, certain properties of two different shapes sometimes can be mathematically related to each other. For example, the length of semi principal axes of an ellipsoid each can be equal to half of the length of every side of a cuboid. This suggest that both cuboid and ellipsoid are physically similar in some sense.

Extensive studies by [3, 12, 14] indicate that shape of conducting objects included in an electrical field can be recognized and described through their first order polarization tensor. In electrical imaging especially, instead of fully reconstructing the shape, fitting the shape with its first order polarization tensor could also be very useful to describe the shape as it offers lower computational cost. Indeed, it is also almost impossible to mathematically and correctly obtain the first order polarization tensor for most shapes unless by using well established methodology such as proposed by Pólya and Szegő [12] or Ammari and Kang [3].

Furthermore, for some applications [14, 16], it might be essential to know an ellipsoid which have identical first order polarization tensor with the other shape. This is possible to achieve as the analytical formula of the first order polarization tensor for the ellipsoid in these applications exists and clearly explained in [3]. Therefore, the main purpose of this paper is to discuss procedures to determine such ellipsoid with examples for future relevant applications.

Basically, this ellipsoid is obtained after all of its semi principal axes which are included in the analytical formula of its first order polarization tensor are determined. Thus, we will derive three nonlinear equations with three unknowns from the analytical formula after setting the formula equal to the first order polarization tensor for a known object and then simultaneously solve them. Since these equations can not be directly and

*Corresponding author.

E-mail addresses: taufiq@utm.my (Taufiq Khairuddin)

analytically obtained, some ideas and numerical properties to derive the equations will become the main focus of this study.

For convenience, this paper is organized into six sections. Section 2 mentions mathematical background about the first order polarization tensor and some of its applications. Section 3 then will provide framework and method on how to mathematically obtain the elliposid which have similar first order polarization tensor with other shape. After that, few numerical examples for this purpose will be included in Section 4. Finally, discussion and conlusions about this study are stated in the last two sections of this paper.

2 First order PT

The concept of polarization tensor (PT) that arise from a transmission problem discussed by many literatures will be firstly stated here. Following [3], consider a Lipschitz bounded domain B in $\mathbb{R}^3$ such that the origin O is in B and let the conductivity of B be equal to k where $0 < k \neq 1 < +\infty$. Suppose that H be a harmonic function in $\mathbb{R}^3$ and u be the solution to the following problem

$$\begin{cases} \operatorname{div}(1+(k-1)\chi(B)\operatorname{grad}(u)) = 0 \text{ in } \mathbb{R}^3 \\ u(x) - H(x) = O(1/|x|^2) \text{ as } |x| \to \infty \end{cases} \tag{2.1}$$

where χ denotes the characteristic function of B. This mathematical formulation (2.1) actually appears in many industrial applications such as medical imaging, landmine detector and material sciences [1, 3, 8, 12]. The PT is then defined through the following far-field expansion of u by [3] as

$$(u-H)(x) = \sum_{|i|,|j|=1}^{+\infty} \frac{(-1)^{|i|}}{i!j!} \partial_x^i \Gamma(x) M_{ij}(k,B) \partial^j H(0) \text{ as } |x| \to +\infty \tag{2.2}$$

for $i = (i_1, i_2, i_3)$, $j = (j_1, j_2, j_3)$ multi indices, Γ is a fundamental solution of the Laplacian and $M_{ij}(k,B)$ is the generalized polarization tensor (GPT).

Generally, the GPT is referred as the dipole in electromagnetic applications by physicists probably because it shows the conductivity distribution of the object. Furthermore, the definition of GPT in (2.2) is extended by Ammari and Kang [3] through an integral operator over the boundary of B by

$$M_{ij} = \int_{\partial B} y^j \phi_i(y) d\sigma(y) \tag{2.3}$$

where $\phi_i(y)$ is given by

$$\phi_i(y) = (\lambda I - \mathcal{K}_B^*)^{-1}(v_x \cdot \nabla x^i)(y) \tag{2.4}$$

for $x, y \in \partial B$ with v_x is the outer unit normal vector to the boundary ∂B at x and λ is defined by $\lambda = (k+1)/2(k-1)$. $\mathcal{K}_B^*$ is a singular integral operator defined with Cauchy principal value $P.V.$ by

$$\mathcal{K}_B^* \phi(x) = \frac{1}{4\pi} P.V. \int_{\partial B} \frac{\langle x-y, v_x \rangle}{|x-y|^3} \phi(y) d\sigma(y). \tag{2.5}$$

Consequently, the first order PT can be evaluated by using (2.3) for i, j=$(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ only so that $|i| = i_1 + i_2 + i_3 = 1 = |j|$. By combining all possible values of i and j, the first order PT of an object B is a real 3×3 matrix in the form

$$M = \begin{bmatrix} M_{(1,0,0)(1,0,0)} & M_{(1,0,0)(0,1,0)} & M_{(1,0,0)(0,0,1)} \\ M_{(0,1,0)(1,0,0)} & M_{(0,1,0)(0,1,0)} & M_{(0,1,0)(0,0,1)} \\ M_{(0,0,1)(1,0,0)} & M_{(0,0,1)(0,1,0)} & M_{(0,0,1)(0,0,1)} \end{bmatrix}. \tag{2.6}$$

Furthermore, if B is an ellipsoid centered at origin in the Cartesian coordinate system represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where a, b and c each is the length of semi principal axes of B, the first order PT of B when the conductivity is k is given by [3] as

$$M(k,B) = (k-1)|B| \begin{bmatrix} \frac{1}{(1-P)+kP} & 0 & 0 \\ 0 & \frac{1}{(1-Q)+kQ} & 0 \\ 0 & 0 & \frac{1}{(1-R)+kR} \end{bmatrix} \tag{2.7}$$

where $|B|$ is the volume of B, P, Q and R are constants defined by

$$
\begin{aligned}
P &= \frac{bc}{a^2}\int_1^{+\infty} \frac{1}{t^2\sqrt{t^2-1+(\frac{b}{a})^2}\sqrt{t^2-1+(\frac{c}{a})^2}}dt,\\
Q &= \frac{bc}{a^2}\int_1^{+\infty} \frac{1}{(t^2-1+(\frac{b}{a})^2)^{\frac{3}{2}}\sqrt{t^2-1+(\frac{c}{a})^2}}dt,\\
R &= \frac{bc}{a^2}\int_1^{+\infty} \frac{1}{\sqrt{t^2-1+(\frac{b}{a})^2}(t^2-1+(\frac{c}{a})^2)^{\frac{3}{2}}}dt.
\end{aligned}
\tag{2.8}
$$

Generally, if given any object then the first order PT of the object can be obtained by formula (2.3) - (2.6) while the first order PT for an ellipsoid can be alternatively calculated by using simpler formula (2.7) - (2.8). Both approaches are possible only by using numerical method as the integrand in (2.3), (2.5) and (2.8) can not be analytically determined. In this study however, our purpose is to find an ellipsoid by using (2.7) and (2.8) if given to us the first order PT of any object in the form of (2.6).

3 Mathematical properties of the first order PT for ellipsoid

This section review some mathematical properties of the formula (2.7) which are very useful in determining the ellipsoid when the first order PT is given. For this purpose, every integrand in (2.8) is firstly denoted as function $f_1(t,a,b,c)$, $f_2(t,a,b,c)$ and $f_3(t,a,b,c)$ such that

$$
\begin{aligned}
P &= \frac{bc}{a^2}\int_1^{+\infty} f_1(t,a,b,c)dt,\\
Q &= \frac{bc}{a^2}\int_1^{+\infty} f_2(t,a,b,c)dt,\\
R &= \frac{bc}{a^2}\int_1^{+\infty} f_3(t,a,b,c)dt.
\end{aligned}
$$

Lemma 3.1. *For any $t > 0$,*

1. *if $0 < a \le b \le c$ then $f_1(t,a,b,c) \le 1/t^4$.*
2. *if $0 < c \le b \le a$ then there exist positive constant K so that $f_1(t,a,b,c) \le K/t^4$.*

Proof. In order to prove (1), starting from $a \le b$, it is easy to show that for any $t > 0$

$$t \le \sqrt{t^2-1+(b/a)^2}$$

while $a \le c$ implies

$$t \le \sqrt{t^2-1+(c/a)^2}$$

for any $t > 0$. Multiplying both inequalities yield to

$$t^2 \le \sqrt{t^2-1+(b/a)^2}\sqrt{t^2-1+(c/a)^2}$$

and hence $t^4 \le (f_1(t,a,b,c))^{-1}$. This completes the proof of (1).

Similarly, $b \le a$ and $c \le a$ imply that for any $t > 0$

$$\sqrt{t^2-1+(b/a)^2} \le t \text{ and } \sqrt{t^2-1+(c/a)^2} \le t \text{ respectively.}$$

This leads to

$$\sqrt{t^2-1+(b/a)^2}\sqrt{t^2-1+(c/a)^2} \le t^2 \text{ and } 1/t^4 \le f_1(t,a,b,c).$$

As $0 < 1/t^4 \le f_1(t,a,b,c)$, multiply the right hand-sided of $1/t^4 \le f_1(t,a,b,c)$ with a positive constant K to complete the proof of the lemma. □

The above lemma explains that $f_1(t,a,b,c)$ is bounded for both $0<a\le b\le c$ and $0<c\le b\le a$ and is also true if $f_1(t,a,b,c)$ is replaced by $f_2(t,a,b,c)$ and $f_3(t,a,b,c)$. This lemma is important to derive three nonlinear equations with three variables from (2.7) and (2.8) by appropriate numerical method. Next, we prove the following lemma which might be essential to solve the obtained system of nonlinear equations later on.

Lemma 3.2. *Let the constants $C_1, C_2, C_3 > 1$. For $a,b,c>0$ and $i=1,2,3$, the function $f_i(t,a,b,c)$ is continuous in the interval $1<t<C_i$.*

Proof. The continuity of $f_1(t,a,b,c)$ is firstly shown. Notice that for $C_1>1$,

$$\int_1^{+\infty} f_1(t,a,b,c)dt = \int_1^{C_1} f_1(t,a,b,c)dt + \int_{C_1}^{+\infty} f_1(t,a,b,c)dt.$$

Since $\int_1^{+\infty} f_1(t,a,b,c)dt$ exist for any $a,b,c>0$ by formula (2.7), then $\int_1^{C_1} f_1(t,a,b,c)dt$ also exist by [9] for $a,b,c>0$. This concludes that $f_1(t,a,b,c)$ is continuous in the interval $1<t<C_1$. Similar steps can be repeated to show that each $f_2(t,a,b,c)$ and $f_3(t,a,b,c)$ is continuous for $1<t<C_2$ and $1<t<C_3$ respectively. □

4 Numerical setup and methodology

The discussion in this section based on the previous properties is divided into two parts. Firstly, this section explains the numerical integration method used to derive three nonlinear equations with three unknowns from (2.7). After that, the procedure to solve these equations is briefly discussed.

4.1 Formulating the nonlinear equations

The discussion begins with the explanation on how integral equations in (2.7) are estimated. Since all integrals involve infinite interval, a common approach to estimate them is by truncating the limits where the infinite range is replaced by a sufficiently large value L so that the integrals becomes finite and then can be approximated by a standard numerical integration method of finite interval [5]. Therefore, it is neccessary to properly choose L to avoid inaccurate result if L is underestimated or expending needless effort if L is overestimated.

Before proceeding further, consider the problem to numerically determine

$$I = \int_1^{\infty} (K/t^4)dt \text{ for } K>0. \tag{4.9}$$

By following [5],

$$I = \int_C^{+\infty} (K/t^4)dt \tag{4.10}$$

where the constant $C>1$ is firstly estimated. For $t\ge C$ then $t^4 \ge Ct^3$. Hence,

$$\int_C^{+\infty} (K/t^4)dt \le \int_C^{+\infty} (K/Ct^3)dt = K/2C^3. \tag{4.11}$$

Thus, if $K=20$ and $C=100$ for example then $K/2C^3 \approx 10^{-5}$ so (4.9) can be approximated to four figures of computation by $I=\int_1^{100}(20/t^4)dt$ with $K=20$.

Suppose that now we want to approximate $\int_1^{+\infty} f_1(t,a,b,c)dt$ by sufficiently $\int_1^{C_1} f_1(t,a,b,c)dt$. Since $\int f_1(t,a,b,c)dt$ can not be analytically integrated then it is impossible to investigate C_1. However, by using Lemma 3.1, we have

$$\int_1^{C_1} f_1(t,a,b,c)dt \le \int_1^{C_1} (1/t^4)dt \tag{4.12}$$

for $0<a\le b\le c$ and

$$\int_1^{C_1} f_1(t,a,b,c)dt \le \int_1^{C_1} (K/t^4)dt \tag{4.13}$$

where K positive constant for $0<c\le b\le a$. Furthermore, $\int_1^{C_1}(1/t^4)dt \le \int_1^{C_1}(K/t^4)dt$. Therefore, we may refer C in (4.10) to guess C_1 when doing computation for both cases.

Similar approach is also used to approximate $\int_1^{+\infty} f_2(t,a,b,c)dt$ and $\int_1^{+\infty} f_3(t,a,b,c)dt$ by $\int_1^{C_2} f_2(t,a,b,c)dt$ and $\int_1^{C_3} f_3(t,a,b,c)dt$ respectively. This means our problem now becomes to derive the nonlinear equations from

$$\begin{aligned}\int_1^{C_1} f_1(t,a,b,c)dt &= \hat{f}_1(a,b,c),\\ \int_1^{C_2} f_2(t,a,b,c)dt &= \hat{f}_2(a,b,c),\\ \int_1^{C_3} f_3(t,a,b,c)dt &= \hat{f}_3(a,b,c).\end{aligned} \tag{4.14}$$

In this study, the trapezoidal rule is implemented to achieve this such that

$$\begin{aligned}\hat{f}_1(a,b,c) &= h_1\left[\frac{f_1(1,a,b,c)+f_1(C_1,a,b,c)}{2}+\sum_{k=1}^{n_1-1} f_1(1+kh_1,a,b,c)+R_1\right],\\ \hat{f}_2(a,b,c) &= h_2\left[\frac{f_2(1,a,b,c)+f_2(C_2,a,b,c)}{2}+\sum_{k=1}^{n_2-1} f_2(1+kh_2,a,b,c)+R_2\right],\\ \hat{f}_3(a,b,c) &= h_3\left[\frac{f_3(1,a,b,c)+f_3(C_3,a,b,c)}{2}+\sum_{k=1}^{n_3-1} f_3(1+kh_3,a,b,c)+R_3\right].\end{aligned} \tag{4.15}$$

where for $i = 1,2,3$, R_i is the small error in the approximation and $h_i = (C_i - 1)/n_i$ must be very small to increase the accuracy of the computation [5]. Then, if given the first order PT of any shape S at conductivity k denoted by $M(k,S)$ as diagonal matrix of size 3, the desired system of nonlinear equation is obtained by comparing $\hat{f}_1(a,b,c)$, $\hat{f}_2(a,b,c)$ and $\hat{f}_3(a,b,c)$ with the appropriate diagonal element of $M(k,S)$. Finally, by using the original formula (2.7) and (2.8) with (4.14) and rearranged, the system will be in the form

$$m_{ii} + (k-1)\left[m_{ii}\frac{bc}{a^2}\hat{f}_i(a,b,c) - |B|\right] = 0 \tag{4.16}$$

where m_{ii} is the diagonal element of $M(k,S)$ for $i = 1,2,3$.

4.2 Solving system of nonlinear equations

The system of nonlinear equations (4.16) are solved in order to obtain the ellipsoid that has same first order PT with an object S at conductivity k. This ellipsoid must be unique based on formula (2.7) and (2.8). According to [2], at least n nonlinear equations are needed if we want to find a unique solution for n independent variables provided the solution exists. This seems to be true if the system is solved by analytical techniques but our system can only be solved by numerical method. In addition, some authors such as [4] and [7] claims that there is no guarantee to find the unique or all solutions for the system of nonlinear equations by numerical method because of the the difficulties in analyzing the existence and uniqueness of solutions to such system.

Therefore, following the claim by Press et. al [13] that there is no particular 'good method' in solving the system of nonlinear equations, only the standard method in the function $fsolve.m$ of MatLab is used to solve (4.16) with initial value $a = b = c = 0$ in this study. This method is chosen because we believe we can obtain approximately correct unique solutions for the system due to (4.16) satisfies the following criteria :

1. The system has three equations with three unknown variables.
2. All variables are strictly posivite real number.
3. Every equation is continous with respect to every variable.

In this case, condition (1) is obvious, condition (2) is based on the fact that the system is derived based on original definition of the variables in (2.7) and (2.8) while condition (3) is explained in the next lemma.

Lemma 4.3. *Let* $g_i(a,b,c) = m_{ii} + (k-1)\left[m_{ii}\frac{bc}{a^2}\hat{f}_i(a,b,c) - |B|\right]$ *for* $i = 1,2,3$. *Every* $g_i(a,b,c)$ *is continuous for* $a,b,c > 0$.

Proof. Since any other terms in $g_i(a,b,c)$ are just a real-valued constants for all i, it is suffiecient to prove that $g_i(a,b,c)$ continuous by showing $\hat{f}_i(a,b,c)$ is continuous whenever $a,b,c>0$. According to (4.14), $\hat{f}_i(a,b,c)$ is approximated by the summation of $f_i(t_i,a,b,c)$ for some t_i such that $1<t_i<C_i$. The continuity of $g_i(a,b,c)$ for every i whenever $a,b,c>0$ arrives from Lemma 3.2 and by the property of summation between continuous function. □

5 Examples and applications

In order to demonstrate some examples and applications of the previous explanation, the constants in (4.14) are set to be equal such that $C_1=C_2=C_3=50$ and $h_1=h_2=h_3=0.00001$ during every computation throughout this section. Furthermore, all semi axes of every ellipsoid are calculated to only four decimal places. We divide the implementation of the discussed technique to determine the ellipsoid in this paper into three cases and each of them will be discussed next.

5.1 Objects with similar first order PT

An ellipsoid can be constructed for any conducting object so that both of them has similar first order PT as diagonal matrix of size 3. It is assumed that the first order PT of any object is diagonal when the object is centered at the origin. This claim can be numerically seen by using previous proposed method for example in [15]. Thus, based on the first order PT of any other shape, every semi axes of the desired ellipsoid can be computed by solving (4.16). Table 1 shows few common shapes and respective ellipsoids which are obtained by using the method discussed in Section 4 according to the first order PT of every object at the same level of conductivity k.

5.2 Eigenvalues of the first order PT

Sometimes, it is more useful to characterize the first order PT of any object S at conductivity k denoted by $M(k,S)$ by the three eigenvalues of $M(k,S)$ especially when it is difficult to locate the center of the object because of complicated shape. This approach is actually applicable to any shape whether the shape is centered at the origin or not because the eigenvalues preserve regardless the position of the object. As the eigenvalues of the first order PT of ellipsoids in (2.7) are just the diagonal terms, we can then determine an ellipsoid centered at origin from any eigenvalues of $M(k,S)$ by solving (4.16) equal to the corresponds eigenvalues (see Table 2).

5.3 First order PT for several ellipsoids

Suppose that we want to investigate several ellipsoids where their first order PT are related. Let the first order PT of the ellipsoid A, $(x/2)^2+(y/2)^2+(z)^2=1$ at conductivity 2.75 be

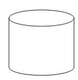
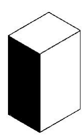
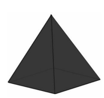
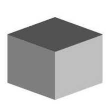
$$M_A=\begin{bmatrix}20.74 & 0.00 & 0.00\\ 0.00 & 20.74 & 0.00\\ 0.00 & 0.00 & 15.25\end{bmatrix}.$$

If M_A satisfies $M_AM_B=I$ where $I=M_C$ is the identity matrix, B and C at the same conductivity with A can be obtained by solving (4.16) equal to $M_B=M_A^{-1}$ and I respectively where they are obtained as the ellipsoid with semi principal axes 0.1738, 0.1738, 0.3782 and sphere of radius 0.6.

6 Discussion and conclusion

A system of nonlinear equation has being developed and solved in this study to determine an ellipsoid that share similar mathematical description with other object which is the first order PT. A few mathematical properties are discussed to provide framework for this purpose. By achieving this, an ellipsoid can be constructed from the same material with the appropriated object as the conductivity for both are equally fixed to deeper investigate other relevant properties for both objects.

Furthermore, the trapezoidal rule from numerical integration technique of finite interval is used to develop the system of nonlinear equations from the analytical formula of the first order PT for the ellipsoid. These

Object	k	First Order PT	Ellipsoid $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$
 **Cylinder** $d = 3$, $h = 3$	5×10^{-5}	$\begin{bmatrix} -33.81 & 0.00 & 0.00 \\ 0.00 & -33.81 & 0.00 \\ 0.00 & 0.00 & -33.53 \end{bmatrix}$	$a = 1.7427$ $b = 1.7427$ $c = 1.7671$
 **Hemisphere** $d = 3$	1×10^{-2}	$\begin{bmatrix} -9.70 & 0.00 & 0.00 \\ 0.00 & -9.70 & 0.00 \\ 0.00 & 0.00 & -15.41 \end{bmatrix}$	$a = 1.5235$ $b = 1.5235$ $c = 0.7703$
 **Cuboid** $2 \times 4 \times 1$	1.5	$\begin{bmatrix} 27.95 & 0.00 & 0.00 \\ 0.00 & 29.92 & 0.00 \\ 0.00 & 0.00 & 25.03 \end{bmatrix}$	$a = 2.5182$ $b = 4.3458$ $c = 1.3990$
 **Pyramid** $2 \times 2 \times 2$	500	$\begin{bmatrix} 12.85 & 0.00 & 0.00 \\ 0.00 & 12.85 & 0.00 \\ 0.00 & 0.00 & 8.51 \end{bmatrix}$	$a = 1.0773$ $b = 1.0773$ $c = 0.7576$
 **Cube** $2 \times 2 \times 2$	10000	$\begin{bmatrix} 28.90 & 0.00 & 0.00 \\ 0.00 & 28.90 & 0.00 \\ 0.00 & 0.00 & 28.90 \end{bmatrix}$	$a = 1.3201$ $b = 1.3201$ $c = 1.3201$

note : d = diameter, h = height

Table 1: Ellipsoid and Object with Similar First Order PT

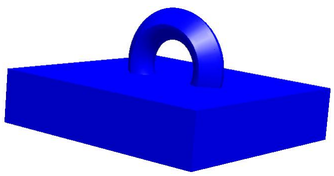
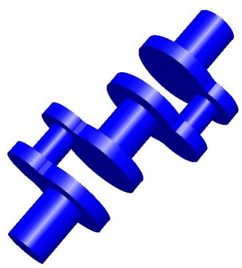
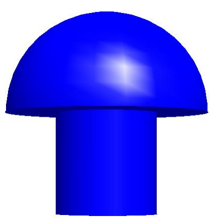
Object	k	Eigenvalues of First Order PT	Ellipsoid $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$
 **half-Ring on Cuboid**	1.5	56.50 56.50 46.62	$a = 4.4860$ $b = 4.4860$ $c = 1.4607$
 **Shaft**	500	30.90×10^5 12.42×10^5 8.40×10^5	$a = 84.8042$ $b = 41.8165$ $c = 29.2011$
 **Mushroom**	10000	87.42 79.00 79.00	$a = 1.9735$ $b = 1.8144$ $c = 1.8144$

Table 2: The Similar Eigenvalues of First Order PT

nonlinear equations are derived step by step while the convergence of each equation is also discussed here. This method is choosen since we want to develop the simplest set of equations with the hope that they can be easily and directly solved by any existing method.

Finally, by solving the developed system of nonlinear equations, some examples of ellipsoids which are determined from the first order PT for related object are also given. These examples are categorized into three different situations and can be further explored for relevant applications in the future. In addition, the first order PT for every ellipsoid determined in Section 5.1 will give the similar first order PT for the other related objects while ellipsoids from Section 5.2 will give the same eigenvalues of the first order PT for the other related objects accurate to three decimal places if both of them are calculated back by using formula (2.7).

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