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A Variant of Jensen's Inequalities

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Abstract

In this paper, we give an estimate from below and from above of a variant of Jensen's Inequalities for convex functions in the discrete and continuous cases.

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1 Introduction and main results

Throughout this note, we write I and $\mathring{I}$ for the intervals $[a,b]$ and (a,b) respectively. A function f is said to be convex on I if $\lambda f(x) + (1-\lambda) f(y) \geqslant f(\lambda x + (1-\lambda) y)$ for all $x, y \in I$ and $0 \leqslant \lambda \leqslant 1$. Conversely, if the inequality always holds in the opposite direction, the function is said to be concave on the interval. A function f that is continuous function on I and twice differentiable on $\mathring{I}$ is convex on I if $f''(x) \geqslant 0$ for all $x \in \mathring{I}$ (concave if the inequality is flipped).

Let $x_1 \leqslant x_2 \leqslant ... \leqslant x_n$ be real numbers and λ_k $(1 \leqslant k \leqslant n)$ be positive weights associated with x_k and whose sum is unity. Then the famous Jensen's discrete and continuous inequalities [2] read:

Theorem A. [2] *If φ is a convex function on an interval containing the x_k, then*

$$\varphi\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leqslant \sum_{k=1}^{n} \lambda_k \varphi(x_k). \tag{1.1}$$

Theorem B. [2] *Let φ be a convex function on $I \subset \mathbb{R}$, let $f : [c,d] \longrightarrow I$ and $p : [c,d] \longrightarrow (0, +\infty)$ be continuous functions on $[c,d]$. Then*

$$\varphi\left(\frac{\int_c^d p(x) f(x)\, dx}{\int_c^d p(x)\, dx}\right) \leqslant \frac{\int_c^d p(x) \varphi(f(x))\, dx}{\int_c^d p(x)\, dx}. \tag{1.2}$$

If φ is strictly convex, then inequality in (1.2) is strict.

In [3], S. M. Malamud gave some complements to the Jensen and Chebyshev inequalities and in [1], I. Budimir, S. S. Dragomir, J. E. Pečarić obtained some results which counterpart Jensen's discret inequality. Recently, A. McD. Mercer [4] studied a variant of the inequality (1.1) and have obtained:

Theorem C. [4] *If φ is a convex function on an interval of positive real numbers containing the x_k, then*

$$\varphi\left(x_1 + x_n - \sum_{k=1}^{n} \lambda_k x_k\right) \leqslant \varphi(x_1) + \varphi(x_n) - \sum_{k=1}^{n} \lambda_k \varphi(x_k). \tag{1.3}$$

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Our purpose in this paper is to give an estimate, from below and from above, of a variant of Jensen's discrete and continuous cases inequalities for convex functions. We obtain the following results:

Theorem 1.1. *Assume that φ is a convex function on I containing the x_k and λ_k $(1 \leqslant k \leqslant n)$ are positive weights associated with x_k and whose sum is unity. Then*

$$2\varphi\left(\frac{a+b}{2}\right) - \sum_{k=1}^{n} \lambda_k \varphi(x_k) \leqslant \varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right)$$
$$\leqslant \varphi(a) + \varphi(b) - \sum_{k=1}^{n} \lambda_k \varphi(x_k). \tag{1.4}$$

If φ is strictly convex, then inequalities in (1.4) *are strict.*

Remark 1.1. *If $[a,b] = [x_1, x_n]$, then the result of Theorem C is given by the right-hand of inequalities* (1.4).

Theorem 1.2. *Let φ be a convex function on $I \subset \mathbb{R}$, let $f : [c,d] \longrightarrow I$ and $p : [c,d] \longrightarrow (0,+\infty)$ be continuous functions on $[c,d]$. Then*

$$2\varphi\left(\frac{a+b}{2}\right) - \frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx} \leqslant \varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$
$$\leqslant \varphi(a) + \varphi(b) - \frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}. \tag{1.5}$$

If φ is strictly convex, then inequalities in (1.5) *are strict.*

Corollary 1.1. *Under the hypotheses of Theorem 1.1, we have*

$$\left|\varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right) + \sum_{k=1}^{n}\lambda_k\varphi(x_k)\right|$$
$$\leqslant \max\left\{2\left|\varphi\left(\frac{a+b}{2}\right)\right|, |\varphi(a)+\varphi(b)|\right\}. \tag{1.6}$$

Corollary 1.2. *Under the hypotheses of Theorem 1.2, we have*

$$\left|\varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right) + \frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}\right|$$
$$\leqslant \max\left\{2\left|\varphi\left(\frac{a+b}{2}\right)\right|, |\varphi(a)+\varphi(b)|\right\}. \tag{1.7}$$

In [5], S. Simić have obtained an upper global bound without a differentiability restriction on f. Namely, he proved the following:

Theorem D. [5] *If φ is a convex function on I containing the x_k and λ_k $(1 \leqslant k \leqslant n)$ are positive weights associated with x_k and whose sum is unity, then*

$$\sum_{k=1}^{n}\lambda_k\varphi(x_k) - \varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right) \leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right). \tag{1.8}$$

In the following, we improve this result by proving:

Theorem 1.3. *If φ is a convex function on I containing the x_k and λ_k $(1 \leqslant k \leqslant n)$ are positive weights associated with x_k and whose sum is unity, then*

$$0 \leqslant \left| \sum_{k=1}^{n} \lambda_{\sigma(k)} \varphi\left(a+b-x_k\right) - \varphi\left(a+b-\sum_{k=1}^{n} \lambda_k x_k\right) \right.$$
$$\left. + \sum_{k=1}^{n} \lambda_{\sigma(k)} \varphi\left(x_k\right) - \varphi\left(\sum_{k=1}^{n} \lambda_k x_k\right) \right| \leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right) \tag{1.9}$$

holds for all permutation $\sigma(k)$ of $\{1, 2, ..., n\}$.

Theorem 1.4. *Let φ be a convex function on $I \subset \mathbb{R}$, let $f : [c, d] \longrightarrow I$ and $p : [c, d] \longrightarrow (0, +\infty)$ be continuous functions on $[c, d]$. Then*

$$0 \leqslant \frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx} - \varphi\left(\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$
$$\leqslant \frac{\int_c^d p(x)\varphi(a+b-f(x))\,dx}{\int_c^d p(x)\,dx} - \varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$
$$+\frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx} - \varphi\left(\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right) \leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right). \tag{1.10}$$

Corollary 1.3. *If φ is a convex function on $I \subset \mathbb{R}$, $f : [0, 1] \longrightarrow I$ is a continuous function on $[0, 1]$, then*

$$0 \leqslant \int_0^1 \varphi(f(x))\,dx - \varphi\left(\int_0^1 f(x)\,dx\right)$$
$$\leqslant \varphi\left(a+b-\int_0^1 f(x)\,dx\right) - \int_0^1 \varphi(a+b-f(x))\,dx + \int_0^1 \varphi(f(x))\,dx$$
$$-\varphi\left(\int_0^1 f(x)\,dx\right) \leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right). \tag{1.11}$$

Corollary 1.4. *If φ is a convex function on I containing the x_k and λ_k $(1 \leqslant k \leqslant n)$ are positive weights associated with x_k and whose sum is unity, then*

$$0 \leqslant \sum_{k=1}^{n} \lambda_k \varphi(x_k) - \varphi\left(\sum_{k=1}^{n} \lambda_k x_k\right)$$
$$\leqslant \sum_{k=1}^{n} \lambda_k \varphi(a+b-x_k) - \varphi\left(a+b-\sum_{k=1}^{n} \lambda_k x_k\right) + \sum_{k=1}^{n} \lambda_k \varphi(x_k) - \varphi\left(\sum_{k=1}^{n} \lambda_k x_k\right)$$
$$\leqslant \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right). \tag{1.12}$$

Remark 1.2. *If φ is a concave function, then the above inequalities are opposite.*

2 Lemma

Towards proving these theorems we shall need the following lemma.

Lemma 2.1. *Let φ be convex function on $I = [a, b]$. Then, we have*

$$2\varphi\left(\frac{a+b}{2}\right) \leqslant \varphi(a+b-x) + \varphi(x) \leqslant \varphi(a) + \varphi(b). \tag{2.1}$$

Proof. Let φ be a convex function on I. Then, we have

$$\varphi\left(\frac{a+b}{2}\right)=\varphi(\frac{a+b-x+x}{2})\leqslant\frac{1}{2}\left(\varphi(a+b-x)+\varphi(x)\right). \tag{2.2}$$

If we choose $x=\lambda a+(1-\lambda)\,b$ $(0\leqslant\lambda\leqslant 1)$ in (2.2), then we obtain

$$\begin{aligned}&\frac{1}{2}\left(\varphi(a+b-x)+\varphi(x)\right)\\ &=\frac{1}{2}\left(\varphi(a+b-(\lambda a+(1-\lambda)\,b))+\varphi(\lambda a+(1-\lambda)\,b)\right)\\ &=\frac{1}{2}\left(\varphi\left(\lambda b+(1-\lambda)\,a\right)+\varphi\left(\lambda a+(1-\lambda)\,b\right)\right).\end{aligned} \tag{2.3}$$

By using the convexity of φ, we get

$$\frac{1}{2}\left(\varphi\left(\lambda b+(1-\lambda)\,a\right)+\varphi\left(\lambda a+(1-\lambda)\,b\right)\right)\leqslant\frac{1}{2}(\varphi(a)+\varphi(b)). \tag{2.4}$$

Thus, by (2.2), (2.3) and (2.4) , we obtain

$$\varphi\left(\frac{b+a}{2}\right)\leqslant\frac{1}{2}\left(\varphi(a+b-x)+\varphi(x)\right)\leqslant\frac{1}{2}(\varphi(a)+\varphi(b)). \tag{2.4}$$

3 Proof of Theorems

Proof of Theorem 1.1. Let φ be a convex function and let λ_k $(0\leqslant k\leqslant n)$ be positive weights associated with x_k and whose sum is unity. Then, by using inequality (1.1) , we have

$$\begin{aligned}\varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right)&=\varphi\left(\sum_{k=1}^{n}\lambda_k\left(a+b\right)-\sum_{k=1}^{n}\lambda_k x_k\right)\\ &=\varphi\left(\sum_{k=1}^{n}\lambda_k\left(a+b-x_k\right)\right)\leqslant\sum_{k=1}^{n}\lambda_k\varphi\left(a+b-x_k\right).\end{aligned} \tag{3.1}$$

By Lemma 2.1, we get

$$\begin{aligned}\varphi\left(\sum_{k=1}^{n}\lambda_k\left(a+b-x_k\right)\right)&\leqslant\sum_{k=1}^{n}\lambda_k\left(\varphi\left(a\right)+\varphi\left(b\right)-\varphi\left(x_k\right)\right)\\ &=\varphi\left(a\right)+\varphi\left(b\right)-\sum_{k=1}^{n}\lambda_k\varphi\left(x_k\right).\end{aligned} \tag{3.2}$$

From (3.1) and (3.2) , we obtain

$$\varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right)\leqslant\varphi\left(a\right)+\varphi\left(b\right)-\sum_{k=1}^{n}\lambda_k\varphi\left(x_k\right),$$

which is the right-hand of inequalities in (1.4). Now, using Lemma 2.1, we obtain

$$2\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right)\leqslant\varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right). \tag{3.3}$$

Since φ is a convex function, then from (3.3) and inequality (1.1), we deduce that

$$\begin{aligned}2\varphi\left(\frac{a+b}{2}\right)-\sum_{k=1}^{n}\lambda_k\varphi\left(x_k\right)&\leqslant 2\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right)\\ &\leqslant\varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right),\end{aligned}$$

which is the left-hand of inequalities in (1.4). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let φ be a convex function. Then, by using inequality (1.2), we have

$$\varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)}\right)=\varphi\left(\frac{\int_c^d p(x)(a+b-f(x))\,dx}{\int_c^d p(x)}\right)$$
$$\leqslant\frac{\int_c^d p(x)\varphi(a+b-f(x))\,dx}{\int_c^d p(x)\,dx}. \tag{3.4}$$

By Lemma 2.1, we get

$$\frac{\int_c^d p(x)\varphi(a+b-f(x))\,dx}{\int_c^d p(x)\,dx}\leqslant\frac{\int_c^d p(x)(\varphi(a)+\varphi(b)-\varphi(f(x)))\,dx}{\int_c^d p(x)\,dx}$$
$$=\varphi(a)+\varphi(b)-\frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}. \tag{3.5}$$

From (3.4) and (3.5) , we obtain

$$\varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)}\right)\leqslant\varphi(a)+\varphi(b)-\frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx},$$

which is the right-hand inequalities in (1.5). Using now Lemma 2.1, we obtain

$$2\varphi\left(\frac{a+b}{2}\right)\leqslant\varphi\left(\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right)+\varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right). \tag{3.6}$$

This implies

$$2\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right)\leqslant\varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right). \tag{3.7}$$

Since φ is a convex function, then from (3.7) and inequality (1.2), we deduce that

$$2\varphi\left(\frac{a+b}{2}\right)-\frac{\int_c^d p(x)\varphi(f(x))\,dx}{\int_c^d p(x)\,dx}\leqslant 2\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$
$$\leqslant\varphi\left(a+b-\frac{\int_c^d p(x) f(x)\,dx}{\int_c^d p(x)\,dx}\right).$$

The left-hand of inequalities in (1.5) is proved. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. By using Lemma 2.1, we obtain for all $x_k \in I$

$$2\varphi\left(\frac{a+b}{2}\right)\leqslant\varphi(a+b-x_k)+\varphi(x_k)\leqslant\varphi(a)+\varphi(b). \tag{3.8}$$

Multiplying (3.8) by $\lambda_{\sigma(k)}$ and adding, we get

$$2\varphi\left(\frac{a+b}{2}\right)\leqslant\sum_{k=1}^n\lambda_{\sigma(k)}\varphi(a+b-x_k)+\sum_{k=1}^n\lambda_{\sigma(k)}\varphi(x_k)\leqslant\varphi(a)+\varphi(b). \tag{3.9}$$

On other hand by Lemma 2.1, we have

$$2\varphi\left(\frac{a+b}{2}\right)\leqslant\varphi\left(a+b-\sum_{k=1}^n\lambda_k x_k\right)+\varphi\left(\sum_{k=1}^n\lambda_k x_k\right)\leqslant\varphi(a)+\varphi(b).$$

This implies

$$-\left(\varphi(a)+\varphi(b)\right) \leqslant -\varphi\left(a+b-\sum_{k=1}^{n}\lambda_k x_k\right) - \varphi\left(\sum_{k=1}^{n}\lambda_k x_k\right)$$

$$\leqslant -2\varphi\left(\frac{a+b}{2}\right). \tag{3.10}$$

By addition from (3.9) and (3.10), we get our result.

Proof of Theorem 1.4. By using Lemma 2.1, we obtain for all $f(x) \in I$

$$2\varphi\left(\frac{a+b}{2}\right) \leqslant \varphi\left(a+b-f(x)\right) + \varphi\left(f(x)\right) \leqslant \varphi(a)+\varphi(b). \tag{3.11}$$

Multiplying (3.11) by $p(x)$ and integrating over $[c,d]$, we get

$$2\varphi\left(\frac{a+b}{2}\right) \leqslant \frac{\int_c^d p(x)\,\varphi\left(a+b-f(x)\right)dx}{\int_c^d p(x)\,dx} + \frac{\int_c^d p(x)\,\varphi\left(f(x)\right)dx}{\int_c^d p(x)\,dx}$$

$$\leqslant \varphi(a)+\varphi(b). \tag{3.12}$$

On other hand by Lemma 2.1, we have

$$2\varphi\left(\frac{a+b}{2}\right) \leqslant \varphi\left(a+b-\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right) + \varphi\left(\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$

$$\leqslant \varphi(a)+\varphi(b). \tag{3.13}$$

This implies

$$-\left(\varphi(a)+\varphi(b)\right) \leqslant -\varphi\left(a+b-\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right) - \varphi\left(\frac{\int_c^d p(x)\,f(x)\,dx}{\int_c^d p(x)\,dx}\right)$$

$$\leqslant -2\varphi\left(\frac{a+b}{2}\right). \tag{3.14}$$

By addition from (3.13) and (3.14), we get our result.

4 Applications

Let $x_k \in [a,b]$ $(b > a > 0)$, $\lambda_k \in [0,1]$ such that $\sum_{k=1}^{n}\lambda_k = 1$. Then, by Theorem 1.1 and Theorem 1.3 for $\varphi(x) = -\ln x$, we obtain respectively

$$\sqrt{ab} \leqslant \sqrt{\frac{A'G+AG'}{2}} \leqslant \frac{a+b}{2}$$

and

$$1 \leqslant \sqrt{\frac{A}{G}\frac{A'}{G'}} \leqslant \frac{\frac{a+b}{2}}{\sqrt{ab}},$$

where $A=\sum_{k=1}^{n}\lambda_k x_k$, $G=\prod_{k=1}^{n}x_k^{\lambda_k}$, $A'=a+b-\sum_{k=1}^{n}\lambda_k x_k$ and $G'=\prod_{k=1}^{n}(a+b-x_k)^{\lambda_k}$.

References

[1] I. Budimir, S. S. Dragomir, J. E. Pečarić, *Further reverse results for Jensen's discrete inequality and applications in information theory*, JIPAM. J. Inequal. Pure Appl. Math. 2 (2001), no. 1, Article 5, 14 pp.

[2] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and new inequalities in analysis. Mathematics and its Applications* (East European Series), 61. Kluwer Academic Publishers Group, Dordrecht, 1993.

[3] S. M. Malamud, *Some complements to the Jensen and Chebyshev inequalities and a problem of W. Walter*, Proc. Amer. Math. Soc. 129 (2001), no. 9, 2671–2678.

[4] A. McD. Mercer, *A Variant of Jensen's inequality,* JIPAM. J. Inequal. Pure Appl. Math. 4 (2003), no. 4, Article 73, 2 pp.

[5] S. Simić, *Jensen's inequality and new entropy bounds*, Appl. Math. Lett. 22 (2009), no. 8, 1262–1265.

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