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On nonlinear Volterra-Fredholm functional integrodifferential equations with nonlocal condition in Banach spaces

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Abstract

In this paper we study the existence, uniqueness and continuous dependence of solutions of nonlinear Volterra-Fredholm functional integrodifferential equations with nonlocal condition in Banach space by using the Hausdorff's measure of noncompactness and Darbo-Sadovskii fixed point theorem. An application is provided to illustrate the theory.

Keywords: Volterra-Fredholm functional integrodifferential equation, nonlocal condition, Hausdorff's measure of noncompactness, Darbo-Sadovskii fixed point theorem.

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1 Introduction

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = C\big([-r, 0], X\big)$, $0 < r < \infty$, be the Banach space of all continuous functions $x : [-r, 0] \to X$ with the supremum norm

$$\|x\|_C = sup\{\|x(t)\| : -r \le t \le 0\}.$$

We denote the Banach space of all continuous functions $y : [-r, T] \to X$ with the supremum norm

$$\|y\|_B = sup\{\|y(t)\| : -r \le t \le T\}$$

by $B = C\big([-r, T], X\big)$. For any $y \in B$ and $t \in [0, T]$ we denote by y_t the element of $C = C\big([-r, 0], X\big)$ given by $y_t(\theta) = y(t + \theta)$ for $\theta \in [-r, 0]$. Consider the nonlinear Volterra-Fredholm functional integrodifferential equations with nonlocal condition of the type

$$x'(t) + Ax(t) = f\left(t, x_t, \int_0^t a(t, s)h(s, x_s)ds, \int_0^T b(t, s)k(s, x_s)ds\right), \quad t \in [0, T], \tag{1.1}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t), \quad t \in [-r, 0], \tag{1.2}$$

and

$$\frac{d}{dt}[x(t) - w(t, x_t)] + Ax(t) = f\left(t, x_t, \int_0^t a(t, s)h(s, x_s)ds, \int_0^T b(t, s)k(s, x_s)ds\right), \quad t \in [0, T], \tag{1.3}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t), \quad t \in [-r, 0], \tag{1.4}$$

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where $0 < t_1 < ... < t_p \leq T$, $p \in N$, $f : [0, T] \times C \times X \times X \to X$, $a, b : [0, T] \times [0, T] \to \mathbb{R}$, $w, h, k : [0, T] \times C \to X$ are continuous functions, $g : C^p \to C$ is given, ϕ is a given element of C. $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)$ in X.

The study of Cauchy problems with nonlocal conditions is of great significance. It has many applications in physics and other areas of applied mathematics. Q.Dong et al [14] studied the existence of the nonlocal neutral functional differential and integrodifferential equations of the form

$$\begin{aligned} &\frac{d}{dt}[x(t) - g(t, x(t), x_t)] = Ax(t) + f\big(t, x(t), x_t\big), \ \ t \in [0, b], \\ &x_0 = \phi + h(x) \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{dt}[x(t) - g(t, x(t), x_t)] = Ax(t) + \int_0^t K(t, s) f\big(s, x(s), x_s\big), \ \ t \in [0, b], \\ &x_0 = \phi + h(x) \end{aligned}$$

using the Hausdorff's measure of noncompactness. Many authors have investigated the existence, uniqueness and other properties of solutions of the nonlocal Cauchy problems for functional differential equations with delay, see [3, 6, 7, 8] and the references cited therein. Balachandran and Park in [3] established existence, continuous dependence and controllabilty for the functional integrodifferential equation with nonlocal condition of the form

$$\begin{aligned} &\frac{du(t)}{dt} + Au(t) = f\Big(t, u_t, \int_0^t k(t, \tau, u_\tau) d\tau\Big), \ \ t \in [0, a], \\ &u(s) + \big(g(u_{t_1}, ..., u_{t_p})\big)(s) = \phi(s), \ \ \ s \in [-r, 0], \end{aligned}$$

using the Banach fixed point principle. In the present paper we prove the existence, uniqueness, continuous dependence and controllabilty of the mild solutions of the more general nonlocal problems (1.1)-(1.2) and (1.3)-(1.4) using the Hausdorff's measure of noncompactness and the Darbo-Sadovskii's fixed point theorem.

The paper is organised as follows. In section 2 we present the preliminaries. Section 3 deals with the existence of mild solutions of the nonlocal problems (1.1)-(1.2) and (1.3)-(1.4). In section 4 we establish sufficient conditions for the continuous dependence and uniqueness of mild solutions of the nonlocal problems. In section 5 an application is provided to illustrate the theory.

2 Preliminaries

We setforth some preliminaries from [4, 12] and hypotheses that will be used in our further discussions.

The functions a, b being continuous on compact domains, there are constants λ and μ such that

$$\big|a(t, s)\big| \leq \lambda \text{ and } \big|b(t, s)\big| \leq \mu, \ \text{ for } s, t \in [0, T]. \tag{2.1}$$

Since the operator $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)$ in X we have $\|T(t)\| \leq U$ where $U \geq 1$. Let $0 \in \rho(A)$. It is now possible to define the fractional power $A^\alpha, 0 < \alpha \leq 1$ as a closed linear operator on its domain $D(A^\alpha)$. Further, $D(A^\alpha)$ is dense in X and the expression $\|x\|_\alpha = \|A^\alpha x\|$ defines a norm on $D(A^\alpha)$. If X_α is the space $D(A^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$ then X_α is a Banach space and therefore the following Lemma 2.1 obviously holds.

Lemma 2.1. *[12] Let $0 < \alpha \leq \beta \leq 1$. Then the following properties hold:*

(1) X_β is a Banach space and $X_\beta \hookrightarrow X_\alpha$ is continuous.

(2) The function $s \mapsto (A)^\alpha T(s)$ is continuous in the uniform operator topology on $(0, \infty)$ and there exists a positive constant C_α such that $\|A^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}$ for every $t > 0$.

Definition 2.2. *The Hausdorff's measure of noncompactness χ_Y is defined by $\chi_Y(S) = \inf\{r > 0, S$ can be covered by finite number of balls with radii $r\}$ for bounded set S in any Banach space Y.*

Lemma 2.3. *[4] Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, then the following properties are satisfied:*
(1) B is precompact if and only if $\chi_Y(B) = 0$;
(2) $\chi_Y(B) = \chi_Y(\bar{B}) = \chi_Y(convB)$ where $\bar{B}$ and $convB$ mean the closure and convex hull of B respectively;
(3) $\chi_Y(B) \leq \chi_Y(C)$ when $B \subseteq C$;
(4) $\chi_Y(B+C) \leq \chi_Y(B) + \chi_Y(C)$ where $B + C = \{x + y; x \in B, y \in C\}$;
(5) $\chi_Y(B \cup C) \leq \max\{\chi_Y(B), \chi_Y(C)\}$;
(6) $\chi_Y(\lambda B) = |\lambda|\, \chi_Y(B)$ for any $\lambda \in R$;
(7) If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k then $\chi_Z(Q(B)) \leq k\, \chi_Y(B)$ for any bounded set $B \subseteq D(Q)$, where Z is a Banach space;
(8) $\chi_Y(B) = \inf\{d_Y(B,C); C \subseteq Y$ be precompact$\} = \inf\{d_Y(B,C); C \subseteq Y$ be finite valued$\}$, where $d_Y(B,C)$ means the nonsymmetric (or symmetric) Hausdorff distance between B and C in Y;
(9) If $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded, closed nonempty subsets of Y and $\lim_{n\to+\infty} \chi_Y(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.

Definition 2.4. *The map $Q : W \subseteq Y \to Y$ is said to be a χ_Y-contraction if there exists a positive constant $k < 1$ such that $\chi_Y(Q(S)) \leq k\, \chi_Y(S)$ for any bounded closed subset $S \subseteq W$ where Y is a Banach space.*

The following lemma known as Darbo-Sadovskii fixed point theorem given in [4] plays a crucial role in our subsequent discussions.

Lemma 2.5. *[4] If $W \subseteq Y$ is bounded, closed and convex, the continuous map $Q : W \to W$ is a χ_Y-contraction, then the map Q has atleast one fixed point in W.*

In this paper we use the notations χ and χ_B to denote the Hausdorff's measure of noncompactness of the Banach space X and that of the Banach space $B = C\big([-r, T], X\big)$ respectively.

Lemma 2.6. *[4] If $W \subseteq C([a, b], X)$ is bounded, then*

$$\chi(W(t)) \leq \chi_C(W)$$

for all $t \in [a, b]$, where $W(t) = \{u(t); u \in W\} \subseteq X$. Furthermore if W is equicontinuous on $[a, b]$, then $\chi(W(t))$ is continuous on $[a, b]$ and

$$\chi_C(W) = \sup\{\chi(W(t)), t \in [a, b]\}.$$

Lemma 2.7. *[4] If $W \subseteq C([a, b]; X)$ is bounded and equicontinuous, then $\chi(W(s))$ is continuous and*

$$\chi(\int_a^t W(s)ds) \leq \int_a^t \chi(W(s))ds$$

for all $t \in [a, b]$, where $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$.

Definition 2.8. *The C_0 semigroup $T(t)$ is said to be equicontinuous if $t \to \{T(t)x : x \in S\}$ is equicontinuous for $t > 0$ for all bounded set S in X.*

We know that the analytic semigroup is equicontinuous. The following lemma is obivious.

Lemma 2.9. *If the semigroup $T(t)$ is equicontinuous and $\eta \in L(0, b; R^+)$, then the set $\big\{ \int_0^t T(t-s)u(s)ds, \|u(s)\| \leq \eta(s)$ for a.e. $s \in [0, b]\big\}$ is equicontinuous for $t \in [0, b]$.*

Definition 2.10. *Let $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X. A function $x \in C\big([-r, T], X\big)$ is said to be a mild solution of the nonlocal problem (1.1)-(1.2) if it satisfies the following:*

$$(i)\ \ x(t) = T(t)\big[\phi(0) - \big(g(x_{t_1}, ..., x_{t_p})\big)(0)\big] + \int_0^t T(t-s) f\bigg(s, x_s, \int_0^s a(s,\tau)h(\tau, x_\tau)d\tau, \int_0^T b(s,\tau)k(\tau, x_\tau)d\tau\bigg)ds, t \in [0, T] \tag{2.2}$$

$$(ii)\ \ x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t),\ \ t \in [-r, 0]. \tag{2.3}$$

Definition 2.11. *Let $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X. A function $x \in C\big([-r,T],X\big)$ is said to be a mild solution of the nonlocal problem*(1.3)-(1.4) *if for each $t \in [0,T]$, the function $AT(t-s)w(s,x_s)$, $s \in [0,t)$ is integrable and satisfies the following:*

$$\begin{aligned} &(i)\;\; x(t) = T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0) - w(0,x_0)\big] \\ &+ w(t,x_t) + \int_0^t AT(t-s)w(s,x_s)ds \\ &+ \int_0^t T(t-s)f\Big(s,x_s,\int_0^s a(s,\tau)h(\tau,x_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds, t \in [0,T] \qquad (2.4) \\ &(ii)\;\; x(t) + \big(g(x_{t_1},...,x_{t_p})\big)(t) = \phi(t),\;\; t \in [-r,0]. \qquad (2.5) \end{aligned}$$

We shall make use of the following hypotheses to prove our main results :

(H_1) There exists a continuous function $l : [0,T] \to \mathbb{R}_+ = [0,\infty)$ such that

$$\big\|f(t,\psi,x,y)\big\| \le l(t)\big(\|\psi\|_C + \|x\| + \|y\|\big)$$

for every $t \in [0,T], \psi \in C$ and $x, y \in X$.

(H_2) There exists a continuous function $p : [0,T] \to \mathbb{R}_+$ such that

$$\big\|h(t,\psi)\big\| \le p(t)\,H\big(\|\psi\|_C\big)$$

for every $t \in [0,T], \psi \in C$ where $H : \mathbb{R}_+ \to (0,\infty)$ is a continuous nondecreasing function.

(H_3) There exists a continuous function $q : [0,T] \to \mathbb{R}_+$ such that

$$\big\|k(t,\psi)\big\| \le q(t)\,K\big(\|\psi\|_C\big)$$

for every $t \in [0,T], \psi \in C$ where $K : \mathbb{R}_+ \to (0,\infty)$ is a continuous nondecreasing function.

(H_4) For each $t \in [0,T]$ the function $f(t,.,.,.) : C\times X\times X \to X$ is continuous and for each $(\psi,x,y) \in C\times X\times X$ the function $f(.,\psi,x,y) : [0,T] \to X$ is strongly measurable.

(H_5) For each $t \in [0,T]$ the functions $h(t,.), k(t,.) : C \to X$ are continuous and for each $\psi \in C$ the functions $h(.,\psi), k(.,\psi) : [0,T] \to X$ are strongly measurable.

(H_6) There exists a constant $\rho > 0$ such that

$$\big\|\big(g(u_{t_1},...,u_{t_p})\big)(s) - \big(g(v_{t_1},...,v_{t_p})\big)(s)\big\| \le \rho\,\big\|u-v\big\|_B$$

for $u, v \in B,\;\; s \in [-r,0]$.

(H_7) There exist constant G such that

$$G = \max_{y\in B}\big\|g(y_{t_1},...,y_{t_p})\big\|, \qquad (2.6)$$

(H_8) There exists $0 < \beta < 1$ such that w is $X_\beta-$valued, $A^\beta w(\cdot)$ is continuous and there exist positive constants c_1, c_2 and V such that

$$\|A^\beta w(t,\psi)\| \le c_1\|\psi\| + c_2, \qquad (2.7)$$

$$\|A^\beta\big[w(t,\psi_1) - w(t,\psi_2)\big]\| \le V\|\psi_1 - \psi_2\|_C \qquad (2.8)$$

for $t \in [0,T]$ and $\psi, \psi_1, \psi_2 \in C$.

(H_9) $U_1 + UM^*T\Big\{1 + M^*T\Big[\liminf_{m\to\infty}\Big(\frac{H(m)}{m} + \frac{K(m)}{m}\Big)\Big]\Big\} < 1$

where

$$U_1 = \Big\{\big[U+1\big]\|A^{-\beta}\| + C_{1-\beta}\frac{T^\beta}{\beta}\Big\}\,c_1 \tag{2.9}$$

$$M^* = \sup\{M(t),\ t\in[0,T]\} \quad \text{and} \tag{2.10}$$

$$M(t) = \max\{l(t),\ \lambda p(t),\ \mu q(t)\} \text{ for each } t\in[0,T]. \tag{2.11}$$

(H_{10}) $UM^*T\Big\{1 + M^*T\Big[\liminf_{m\to\infty}\Big(\frac{H(m)}{m} + \frac{K(m)}{m}\Big)\Big]\Big\} < 1$

(H_{11}) There exists integrable functions $\eta, \eta_1, \eta_2 : [0,T] \to [0,\infty)$ such that for any bounded set $W \subset C([-r,T],\, X)$ and $s\in[0,T]$ we have

$$\begin{aligned}
&\chi\big(T(t-s)\, f\Big(s, W_s, \int_0^s a(s,\tau)h(\tau,W_\tau)d\tau, \int_0^T b(s,\tau)k(\tau,W_\tau)d\tau\Big)\big)\\
&\le \eta(s)\Big(\sup_{-r\le\theta\le 0}\chi\big(W(s+\theta)\big) + \int_0^s \big|a(s,\tau)\big|\,\eta_1(\tau)\sup_{-r\le\theta\le 0}\chi\big(W(\tau+\theta)\big)d\tau\\
&+\int_0^T \big|b(s,\tau)\big|\,\eta_2(\tau)\sup_{-r\le\theta\le 0}\chi\big(W(\tau+\theta)\big)d\tau\Big)
\end{aligned}$$

3 Existence of mild solutions

Theorem 3.1. *Suppose that the hypotheses (H_1)-(H_7), (H_{10}) and (H_{11}) holds. Then the nonlocal problem* (1.1)-(1.2) *has a mild solution x on $[-r,T]$ if*

$$\big\{U\rho + \int_0^T \eta(s)\,\big[1 + \int_0^s \lambda\,\eta_1(\tau)d\tau + \int_0^T \mu\,\eta_2(\tau)d\tau\big]ds\big\} < 1. \tag{3.1}$$

Theorem 3.2. *Suppose that the hypotheses (H_1)-(H_9) and (H_{11}) holds. Then the nonlocal problem* (1.3)-(1.4) *has a mild solution x on $[-r,T]$ if*

$$\big\{\rho_1 + \int_0^T \eta(s)\,\big[1 + \int_0^s \lambda\,\eta_1(\tau)d\tau + \int_0^T \mu\,\eta_2(\tau)d\tau\big]ds\big\} < 1 \tag{3.2}$$

where the constant term

$$\rho_1 = \Big\{U\,\rho + V\,\|A^{-\beta}\|\,\big(U+1\big) + V\,C_{1-\beta}\,\frac{T^\beta}{\beta}\Big\}. \tag{3.3}$$

Proof. The proofs of the Theorems 3.1 - 3.2 resemble one another. Therefore, we give the details of Theorem 3.2 only and the proof of Theorem 3.1 can be completed by closely looking at the proof of the Theorem 3.2 with slight modifications.

We prove the existence of mild solution of nonlinear mixed integrodifferential equations (1.3)-(1.4), by using the Darbo-Sadovskii fixed point theorem and the Hausdorff's measure of noncompactness. Consider the bounded set $B_m = \{y\in B : \|y\|\le m\}$ for each $m\in N$ (the set of all positive integers).
Define an operator $F : B = C\big([-r,T],\, X\big) \to B$ by $F = F_1 + F_2$

$$(F_1x)(t) = \begin{cases} \phi(t) - \big(g(x_{t_1},...,x_{t_p})\big)(t), & -r\le t\le 0\\[2ex] T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0) - w(0,x_0)\big] \\ +w(t,x_t) + \int_0^t AT(t-s)w(s,x_s)ds & 0\le t\le T\end{cases} \tag{3.4}$$

$$(F_2x)(t) = \begin{cases} 0, & -r\le t\le 0\\[2ex] \int_0^t T(t-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau,x_\tau)d\tau, \\ \int_0^T b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds & 0\le t\le T\end{cases} \tag{3.5}$$

Using lemma 2.1, hypothesis (H_8) and the fact that $\|x_s\|_C \le \|x\|_B$ for $s \in (0,t)$ and $x \in B_m$ we have,

$$\begin{aligned} \big\|AT(t-s)w(s,x_s)\big\| &= \big\|A^{1-\beta}T(t-s)A^{\beta}w(s,x_s)\big\| \\ &\le \frac{C_{1-\beta}}{(t-s)^{1-\beta}}(c_1\|x_s\|_C + c_2) \\ &\le \frac{C_{1-\beta}}{(t-s)^{1-\beta}}(c_1\|x\|_B + c_2) \\ &\le \frac{C_{1-\beta}}{(t-s)^{1-\beta}}(c_1 m + c_2). \end{aligned}$$

Using this and the fact that the function $s \to AT(t-s)$ is continuous in the uniform operator topology on $(0,t)$ we conclude that $AT(t-s)w(s,x_s)$ is integrable on $(0,t)$ for every $t \in (0,T]$ and $x \in B_m$. Therefore F is well-defined and with values in B.

From the definition of F it follows that the fixed point of F is the mild solution of the nonlocal problem (1.3)-(1.4). We first show that $F : B \to B$ is continuous. Let $\{u_n\}$ be a sequence of elements of B converging to u in B. Consider the case when $t \in [-r,0]$, then using hypothesis (H_6) we have

$$\begin{aligned} \big\|(Fu_n)(t) - (Fu)(t)\big\| &= \big\|\phi(t) - \big(g(u_{n_{t_1}},...,u_{n_{t_p}})\big)(t) - \phi(t) + \big(g(u_{t_1},...,u_{t_p})\big)(t)\big\| \\ &= \big\|\big(g(u_{t_1},...,u_{t_p})\big)(t) - \big(g(u_{n_{t_1}},...,u_{n_{t_p}})\big)(t)\big\| \\ &\le \rho\,\big\|u-u_n\big\| \to 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.6}$$

Now let $t \in [0,T]$ then using hypotheses (H_4) and (H_5) we have

$$\begin{aligned} &f\Big(t, u_{n_t}, \int_0^t a(t,s)h(s,u_{n_s})ds, \int_0^T b(t,s)k(s,u_{n_s})\Big) \\ &\quad \to f\Big(t, u_t, \int_0^t a(t,s)h(s,u_s)ds, \int_0^T b(t,s)k(s,u_s)\Big). \end{aligned}$$

Using the dominated convergence theorem, hypotheses (H_6), (H_8) and lemma 2.1 we have for $t \in (0,T]$,

$$\begin{aligned} &\big\|(Fu_n)(t) - (Fu)(t)\big\| \\ &= \big\|T(t)\big[\phi(0)\big] - T(t)\big[\big(g(u_{n_{t_1}},...,u_{n_{t_p}})\big)(0)\big] - T(t)\big[w(0,u_{n_0})\big] \\ &+ w(t,u_{n_t}) + \int_0^t AT(t-s)w(s,u_{n_s})ds \\ &+ \int_0^t T(t-s)f\Big(s, u_{n_s}, \int_0^s a(s,\tau)h(\tau,u_{n_\tau})d\tau, \int_0^T b(s,\tau)k(\tau,u_{n_\tau})d\tau\Big)ds \\ &- T(t)\big[\phi(0)\big] + T(t)\big[\big(g(u_{t_1},...,u_{t_p})\big)(0)\big] + T(t)\big[w(0,u_0)\big] \\ &- w(t,u_t) - \int_0^t AT(t-s)w(s,u_s)ds \\ &- \int_0^t T(t-s)f\Big(s, u_s, \int_0^s a(s,\tau)h(\tau,u_\tau)d\tau, \int_0^T b(s,\tau)k(\tau,u_\tau)d\tau\Big)ds\big\| \\ &= \big\|T(t)\big[\big(g(u_{t_1},...,u_{t_p})\big)(0) - \big(g(u_{n_{t_1}},...,u_{n_{t_p}})\big)(0)\big] \\ &+ T(t)\big[w(0,u_0) - w(0,u_{n_0})\big] + w(t,u_{n_t}) - w(t,u_t) \\ &+ \int_0^t AT(t-s)\big[w(s,u_{n_s}) - w(s,u_s)\big]ds \\ &+ \int_0^t T(t-s)\Big[f\Big(s, u_{n_s}, \int_0^s a(s,\tau)h(\tau,u_{n_\tau})d\tau, \int_0^T b(s,\tau)k(\tau,u_{n_\tau})d\tau\Big) \\ &- f\Big(s, u_s, \int_0^s a(s,\tau)h(\tau,u_\tau)d\tau, \int_0^T b(s,\tau)k(\tau,u_\tau)d\tau\Big)\Big]ds\big\| \end{aligned}$$

$$\leq \Big\{V\|A^{-\beta}\|\Big(U\left\|u_0-u_{n0}\right\|_C+\left\|u_{n_t}-u_t\right\|_C\Big)+V\,C_{1-\beta}\left\|u_n-u\right\|_B\frac{t^\beta}{\beta}$$
$$+U\Big[\rho\|u-u_n\|+\int_0^t\Big\|f\Big(s,u_{n_s},\int_0^s a(s,\tau)h(\tau,u_{n_\tau})d\tau,\int_0^T b(s,\tau)k(\tau,u_{n_\tau})d\tau\Big)$$
$$-f\Big(s,u_s,\int_0^s a(s,\tau)h(\tau,u_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,u_\tau)d\tau\Big)\Big\|ds\Big]\Big\}\to 0 \text{ as } n\to\infty. \tag{3.7}$$

Since$\big\|(Fu_n)-(Fu)\big\|_B=\sup_{t\in[-r,T]}\big\|(Fu_n)(t)-(Fu)(t)\big\|$, inequalities (3.6) and (3.7) imply $Fu_n\to Fu$ in B as $u_n\to u$ in B. Therefore F is continuous.

We shall show that F is a χ_B- contraction on some bounded closed convex subset $B_m\subseteq B=(C[-r,T],\,X)$. And then by using Darbo-Sadovskii's fixed point theorem we get a fixed point of F.

Firstly by using the method of contradiction we obtain a $m\in N$ such that $F_{B_m}\subseteq B_m$. Suppose that for each $m\in N$ there is a $y^m\in B_m$ and $t^m\in[-r,T]$ such that $\|(Fy^m)(t^m)\|>m$. If $t^m\in[-r,0]$ then using hypothesis (H_7) we obtain

$$\begin{aligned} m<\big\|(Fy^m)(t^m)\big\|&\leq\big\|\phi(t^m)\big\|+\big\|\big(g(y^m_{t_1},...,y^m_{t_p})\big)(t^m)\big\|\\ &\leq c+G. \end{aligned}\tag{3.8}$$

where c denotes $\|\phi\|_C$. Also we know that if $\|y^m\|_B\leq m$ then

$$\|y^m_t\|_C\leq m \text{ for all } t\in[0,T] \tag{3.9}$$

Using hypotheses $(H_1)-(H_3)$ and conditions (2.1), (2.6), (2.7), (2.11),(2.10) and (3.9) for the case when $t^m\in[0,T]$ we obtain

$$\begin{aligned} m&<\big\|(Fy^m)(t^m)\big\|\\
&\leq U\,\big[\|\phi(0)\|+\big\|\big(g(y^m_{t_1},...,y^m_{t_p})\big)(0)\big\|\big]+\big\|T(t^m)w(0,y^m_0)\big\|\\
&+\big\|w(t^m,y^m_{t^m})\big\|+\int_0^{t^m}\big\|AT(t^m-s)w(s,y^m_s)\big\|ds\\
&+\int_0^{t^m}U\Big\|f\Big(s,y^m_s,\int_0^s a(s,\tau)h(\tau,y^m_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,y^m_\tau)d\tau\Big)\Big\|ds\\
&\leq\big\|A^{-\beta}\,T(t^m)\,A^\beta\,w(0,y^m_0)\big\|+\big\|A^{-\beta}\,A^\beta\,w(t^m,y^m_{t^m})\big\|\\
&+\int_0^{t^m}\big\|A^{1-\beta}\,T(t^m-s)\big\|\,\big\|A^\beta w(s,y^m_s)\big\|ds+U\Big[c+G\\
&+\int_0^{t^m}\Big\|f\Big(s,y^m_s,\int_0^s a(s,\tau)h(\tau,y^m_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,y^m_\tau)d\tau\Big)\Big\|ds\Big]\\
&\leq U\,\big\|A^{-\beta}\big\|\,(c_1m+c_2)+\big\|A^{-\beta}\big\|\,(c_1m+c_2)\\
&+C_{1-\beta}\,(c_1m+c_2)\int_0^{t^m}(t^m-s)^{\beta-1}ds+U\Big[c+G+\int_0^{t^m}l(s)\\
&\Big(\big\|y^m_s\big\|_C+\int_0^s\big|a(s,\tau)\big|\,\big\|h(\tau,y^m_\tau)\big\|\,d\tau+\int_0^T\big|b(s,\tau)\big|\,\big\|k(\tau,y^m_\tau)\big\|\,d\tau\Big)ds\Big]\\
&\leq[U+1]\,\|A^{-\beta}\,\|\,(c_1m+c_2)+C_{1-\beta}\,(c_1m+c_2)\,\frac{(t^m)^\beta}{\beta}+U\Big[c+G\\
&+\int_0^{t^m}M(s)\Big(m+\int_0^s M(\tau)H(m)d\tau+\int_0^T M(\tau)K(m)d\tau\Big)ds\Big]\\
&\leq\Big\{[U+1]\,\|A^{-\beta}\|+C_{1-\beta}\,\frac{T^\beta}{\beta}\Big\}\,(c_1m+c_2)+U\Big[c+G\\
&+\int_0^T M(s)\Big(m+\int_0^s M(\tau)H(m)d\tau+\int_0^T M(\tau)K(m)d\tau\Big)ds\Big]
\end{aligned}$$

$$\leq \Big\{ \big[U+1\big] \, \|A^{-\beta}\| + C_{1-\beta} \, \frac{T^{\beta}}{\beta} \Big\} \, (c_1 m + c_2) + U \Big[c + G + \int_0^T M^* \Big(m + M^* H(m) T + M^* K(m) T \Big) ds \Big] \tag{3.10}$$

Thus using the fact that $U \geq 1$ we combine (3.8) and (3.10) so that we obtain

$$m < \Big\{ \big[U+1\big] \, \|A^{-\beta}\| + C_{1-\beta} \, \frac{T^{\beta}}{\beta} \Big\} \, (c_1 m + c_2) + U \, [c + G] + U M^* T \Big(m + M^* H(m) T + M^* K(m) T \Big) \tag{3.11}$$

Dividing by m on both sides of (3.11) we obtain

$$1 < \Big\{ \big[U+1\big] \, \|A^{-\beta}\| + C_{1-\beta} \, \frac{T^{\beta}}{\beta} \Big\} \Big(c_1 + \frac{c_2}{m} \Big) + U \Big[\frac{c+G}{m} \Big] + U M^* T \Big(1 + M^* T \, \frac{H(m)}{m} + M^* T \, \frac{K(m)}{m} \Big) \tag{3.12}$$

Now taking lim inf as $m \to \infty$ on both sides of (3.12) we get

$$1 < U_1 + U M^* T \Big\{ 1 + M^* T \Big[\liminf_{m \to \infty} \, \Big(\frac{H(m)}{m} + \frac{K(m)}{m} \Big) \Big] \Big\}.$$

which contradicts the hypothesis (H_9). Thus there is a $m \in N$ such that $F_{B_m} \subseteq B_m$. Hereafter we will consider the restriction of F on this B_m.

Now we show that F_1 is Lipschitz continuous. Let $x, y \in B_m$ then using hypothesis (H_6) we have for $t \in [-r, 0]$

$$\begin{aligned} \big\|(F_1 x)(t) - (F_1 y)(t)\big\| &\leq \big\| \big(g(x_{t_1}, ..., x_{t_p})\big)(t) - \big(g(y_{t_1}, ..., y_{t_p})\big)(t) \big\| \\ &\leq \rho \big\|x - y\big\|_B \end{aligned} \tag{3.13}$$

Now using hypothesis (H_6), condition (2.8) and lemma 2.1 for $t \in [0, T]$ we have

$$\begin{aligned} \big\|(F_1 x)(t) - (F_1 y)(t)\big\| &= \big\|T(t)\big[\phi(0)\big] - T(t)\big[\big(g(x_{t_1}, ..., x_{t_p})\big)(0)\big] - T(t)\big[w(0, x_0)\big] \\ &+ w(t, x_t) + \int_0^t AT(t-s) w(s, x_s) ds \\ &- T(t)\big[\phi(0)\big] + T(t)\big[\big(g(y_{t_1}, ..., y_{t_p})\big)(0)\big] + T(t)\big[w(0, y_0)\big] \\ &- w(t, y_t) - \int_0^t AT(t-s) w(s, y_s) ds \big\| \\ &= \big\|T(t)\big[\big(g(y_{t_1}, ..., y_{t_p})\big)(0) - \big(g(x_{t_1}, ..., x_{t_p})\big)(0)\big] \\ &+ T(t)\big[w(0, y_0) - w(0, x_0)\big] + w(t, x_t) - w(t, y_t) \\ &+ \int_0^t AT(t-s)\big[w(s, x_s) - w(s, y_s)\big] ds \big\| \\ &\leq U \, \rho \, \big\|y - x\big\| + \big\|A^{-\beta} \, T(t) \, A^{\beta} \, \big[w(0, y_0) - w(0, x_0)\big]\big\| \\ &+ \big\|A^{-\beta} \, A^{\beta} \, \big[w(t, x_t) - w(t, y_t)\big]\big\| \\ &+ \int_0^t \big\|A^{1-\beta} \, T(t-s) \, A^{\beta} \, \big[w(s, x_s) - w(s, y_s)\big]\big\| \, ds \\ &\leq U \, \rho \, \big\|y - x\big\| + V \, \big\|A^{-\beta}\big\| \, \Big[U \big\|y_0 - x_0\big\|_C + \big\|x_t - y_t\big\|_C \Big] \\ &+ V \, C_{1-\beta} \, \big\|x - y\big\|_B \, \int_0^t (t-s)^{\beta - 1} ds \\ &\leq U \, \rho \, \big\|y - x\big\|_B + V \, \big\|A^{-\beta}\big\| \, (U + 1) \, \big\|y - x\big\|_B \\ &+ V \, C_{1-\beta} \, \big\|x - y\big\|_B \, \frac{t^{\beta}}{\beta} \\ &\leq \Big\{ U \, \rho + V \, \big\|A^{-\beta}\big\| \, (U + 1) + V \, C_{1-\beta} \, \frac{T^{\beta}}{\beta} \Big\} \big\|x - y\big\|_B \end{aligned} \tag{3.14}$$

Since $U \geq 1$ and $\rho > 0$ we have $U\rho\|x-y\|_B \geq \rho\|x-y\|_B$ and so in view of (3.13) and (3.14) we obtain

$$\|(F_1x)(t)-(F_1y)(t)\| \leq \left\{U\,\rho + V\,\|A^{-\beta}\|(U+1) + V\,C_{1-\beta}\,\frac{T^\beta}{\beta}\right\}\|x-y\|_B.$$

for all $t \in [-r,T]$ and $x,y \in B_m$. Consequently using (3.3) we get

$$\|(F_1x)-(F_1y)\| \leq \rho_1\,\|x-y\|_B$$

Thus F_1 is Lipschitzian with Lipschitz constant ρ_1. Hence using lemma 2.3(7) we now have

$$\chi_B(F_1W) \leq \rho_1\chi_B(W) \tag{3.15}$$

for any bounded set $W \subseteq B_m$.

Further let W be any bounded subset of B_m. We first show that F_2W is bounded. Let $y \in W \subseteq B_m$ then $\|y\|_B \leq m$ and so $\|y_t\|_C \leq m,\ t \in [0,T]$. Let $t \in [-r,0]$ and $y \in B_m$ then from the definition of F_2 we have

$$\|(F_2y)(t)\| = 0$$

Now for $t \in [0,T]$ and $y \in B_m$ we get

$$\begin{aligned}
\|(F_2y)(t)\| &\leq \int_0^t U\Big\|f\Big(s,y_s,\int_0^s a(s,\tau)h(\tau,y_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big)\Big\|ds\\
&\leq U\Big[\int_0^t l(s)\Big(\|y_s\|_C + \int_0^s |a(s,\tau)|\,\|h(\tau,y_\tau)\|d\tau\\
&+ \int_0^T |b(s,\tau)|\,\|k(\tau,y_\tau)\|d\tau\Big)ds\Big]\\
&\leq U\int_0^t M(s)\Big(m + \int_0^s M(\tau)H(m)d\tau + \int_0^T M(\tau)K(m)d\tau\Big)ds\\
&\leq U\int_0^t M^*\Big(m + M^*H(m)\,s + M^*K(m)\,T\Big)ds\\
&\leq U\,T\,M^*\Big(m + \frac{TM^*H(m)}{2} + T\,M^*K(m)\Big)
\end{aligned} \tag{3.16}$$

The R.H.S. of the inequality (3.16) being constant we conclude that the set $\{(F_2y)(t) : y \in W, -r \leq t \leq T\}$ is bounded in X and hence F_2W is bounded in B . Now we prove that F_2W is equicontinuous. For this let $y \in W,\ s_1,s_2 \in [-r,T]$ and consider the following cases :

Case:1 Suppose $0 \leq s_1 \leq s_2 \leq T$ then using hypothesis $(H_1)-(H_3)$ and conditions (2.11), (2.10) and (3.9), we get

$$\begin{aligned}
&\|(F_2y)(s_2)-(F_2y)(s_1)\|\\
&= \Big\|\int_0^{s_2} T(s_2-s)f\Big(s,y_s,\int_0^s a(s,\tau)h(\tau,y_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big)ds\\
&- \int_0^{s_1} T(s_1-s)f\Big(s,y_s,\int_0^s a(s,\tau)h(\tau,y_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big)ds\Big\|\\
&\leq \int_0^{s_1} \|T(s_2-s)-T(s_1-s)\|\\
&\Big[l(s)\Big(\|y_s\|_C + \Big\|\int_0^s a(s,\tau)h(\tau,y_\tau)d\tau\Big\| + \Big\|\int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big\|\Big)\Big]ds\\
&+ \int_{s_1}^{s_2} \|T(s_2-s)\|\\
&\Big[l(s)\Big(\|y_s\|_C + \Big\|\int_0^s a(s,\tau)h(\tau,y_\tau)d\tau\Big\| + \Big\|\int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big\|\Big)\Big]ds
\end{aligned}$$

$$\leq \int_0^{s_1} \big\|[T(s_2-s)-T(s_1-s)]\big\|$$
$$\Big[M(s)\Big(m+\int_0^s M(\tau)H(m)d\tau+\int_0^T M(\tau)K(m)d\tau\Big)\Big]ds$$
$$+U\int_{s_1}^{s_2} M(s)\Big(m+\int_0^s M(\tau)H(m)d\tau+\int_0^T M(\tau)K(m)d\tau\Big)ds$$
$$\leq \int_0^{s_1} \big\|[T(s_2-s)-T(s_1-s)]\big\|\Big[M^*\Big(m+M^*H(m)s+M^*K(m)T\Big)\Big]ds$$
$$+U\int_{s_1}^{s_2} M^*\Big(m+M^*H(m)\int_0^s d\tau+M^*K(m)\int_0^T d\tau\Big)ds$$
$$\leq \gamma\int_0^{s_1} \big\|T(s_2-s)-T(s_1-s)\big\|ds+U\gamma\big|s_2-s_1\big|\Big\}$$
$$\to 0 \ \text{ as } s_2\to s_1,$$

where $\gamma=\Big[M^*\Big(m+M^*H(m)T+M^*K(m)T\Big)\Big]$. The compactness of T(t) for $t>0$ implies the continuity in the uniform operator topology. Therefore the right hand side of above equation tends to zero as $s_2\to s_1$.

Case:2 Suppose $-r\leq s_1\leq 0\leq s_2\leq T$ then we get

$$\big\|(F_2y)(s_2)-(F_2y)(s_1)\big\|$$
$$=\Big\|\int_0^{s_2} T(s_2-s)f\Big(s,y_s,\int_0^s a(s,\tau)h(\tau,y_\tau)d\tau,\int_0^T b(s,\tau)k(\tau,y_\tau)d\tau\Big)ds\Big\|$$

Now proceeding as in Case 1 for the integral on the right hand side of above inequality we further obtain

$$\big\|(F_2y)(s_2)-(F_2y)(s_1)\big\|\leq U\gamma\big|s_2-s_1\big|\to 0 \ \text{ as } s_2\to 0_+ \text{ and } s_1\to 0_-.$$

Case:3 Suppose $-r\leq s_1\leq s_2\leq 0$. In this case we have

$$\big\|(F_2y)(s_2)-(F_2y)(s_1)\big\|=0 \tag{3.17}$$

Thus cases (1)-(3) imply that $\|(F_2y)(s_2)-(F_2y)(s_1)\|\to 0$ as $s_1\to s_2$, for all $s_1,s_2\in[-r,T]$. Thus we conclude that F_2W is an equicontinuous family of functions.

Further for a bounded subset W of B_m we define the notations $W(t)=\{x(t);x\in W\}\subseteq X$ and $W_t=\{x_t;x\in W\}\subseteq C\big([-r,0],X\big)$. Now using lemma 2.3, lemma 2.6-2.7, lemma 2.9 and hypothesis (H_{11}) we obtain

$$\chi_B(F_2W)=\sup_{-r\leq t\leq T}\chi(F_2W(t))$$
$$=\sup_{0\leq t\leq T}\chi(F_2W(t))$$
$$=\sup_{0\leq t\leq T}\chi(\int_0^t T(t-s)f\Big(s,W_s,\int_0^s a(s,\tau)h(\tau,W_\tau)d\tau,$$
$$\int_0^T b(s,\tau)k(\tau,W_\tau)d\tau\Big)ds)$$
$$=\sup_{0\leq t\leq T}\int_0^t \chi\big(T(t-s)f\Big(s,W_s,\int_0^s a(s,\tau)h(\tau,W_\tau)d\tau,$$
$$\int_0^T b(s,\tau)k(\tau,W_\tau)d\tau\Big)\big)ds$$
$$\leq \sup_{0\leq t\leq T}\int_0^t \eta(s)\Big(\sup_{-r\leq\theta\leq 0}\chi(W(s+\theta))$$
$$+\int_0^s \big|a(s,\tau)\big|\,\eta_1(\tau)\sup_{-r\leq\theta\leq 0}\chi(W(\tau+\theta))d\tau$$
$$+\int_0^T \big|b(s,\tau)\big|\,\eta_2(\tau)\sup_{-r\leq\theta\leq 0}\chi(W(\tau+\theta))d\tau\Big)ds$$

$$
\begin{aligned}
&\leq \sup_{0\leq t\leq T}\int_0^t \eta(s)\Big(\sup_{s-r\leq s+\theta\leq s}\chi(W(s+\theta))\\
&+\int_0^s \lambda\,\eta_1(\tau)\sup_{\tau-r\leq\tau+\theta\leq\tau}\chi(W(\tau+\theta))d\tau\\
&+\int_0^T \mu\,\eta_2(\tau)\sup_{\tau-r\leq\tau+\theta\leq\tau}\chi(W(\tau+\theta))d\tau\Big)ds\\
&=\sup_{0\leq t\leq T}\int_0^t \eta(s)\Big(\sup_{s-r\leq s_1\leq s}\chi(W(s_1))+\int_0^s \lambda\,\eta_1(\tau)\sup_{\tau-r\leq\tau_1\leq\tau}\chi(W(\tau_1))d\tau\\
&+\int_0^T \mu\,\eta_2(\tau)\sup_{\tau-r\leq\tau_1\leq\tau}\chi(W(\tau_1))d\tau\Big)ds\\
&\leq \sup_{0\leq t\leq T}\int_0^t \eta(s)\Big(\sup_{-r\leq s_1\leq T}\chi(W(s_1))+\int_0^s \lambda\,\eta_1(\tau)\sup_{-r\leq\tau_1\leq T}\chi(W(\tau_1))d\tau\\
&+\int_0^T \mu\,\eta_2(\tau)\sup_{-r\leq\tau_1\leq T}\chi(W(\tau_1))d\tau\Big)ds\\
&\leq \sup_{0\leq t\leq T}\int_0^t\Big(\chi_B(W)+\int_0^s\lambda\,\eta_1(\tau)\chi_B(W)d\tau+\int_0^T\mu\,\eta_2(\tau)\chi_B(W)d\tau\Big)ds\\
&\leq \chi_B(W)\sup_{0\leq t\leq T}\int_0^t\eta(s)\Big(1+\int_0^s\lambda\,\eta_1(\tau)d\tau+\int_0^T\mu\,\eta_2(\tau)d\tau\Big)ds\\
&\leq \chi_B(W)\int_0^T\eta(s)\Big(1+\int_0^s\lambda\,\eta_1(\tau)d\tau+\int_0^T\mu\,\eta_2(\tau)d\tau\Big)ds \qquad (3.18)
\end{aligned}
$$

Therefore using (3.2), (3.15) and (3.18) we obtain

$$
\begin{aligned}
\chi_B(FW) &\leq \chi_B(F_1W)+\chi_B(F_2W)\\
&\leq\Big(\rho_1+\int_0^T\eta(s)\big[1+\int_0^s\lambda\,\eta_1(\tau)d\tau+\int_0^T\mu\,\eta_2(\tau)d\tau\big]ds\Big)\chi_B(W)\\
&<\chi_B(W) \qquad (3.19)
\end{aligned}
$$

for any bounded subset W of B_m.

Hence F is a χ_B- contraction. Now applying lemma 2.5 we get a fixed point x of F in B_m. This x is a mild solution of (1.3)-(1.4). The proof of the theorem is complete. □

4 Continuous dependence of mild solution

Theorem 4.1. *Suppose that the functions f, g, h, k, w satisfy the hypotheses (H_1)-(H_9) and (H_{11}). Also suppose that*
(H_{12}) *there exist a constant N such that*

$$\big\|f(t,x_t,z_1,z_2)-f(t,y_t,z_3,z_4)\big\|\leq N\big[\big\|x_t-y_t\big\|_C+\big\|z_1-z_3\big\|+\big\|z_2-z_4\big\|\big] \qquad (4.1)$$

for $t\in[0,T]$, $x,y\in B$ and $z_1,z_2,z_3,z_4\in X$.

(H_{13}) *there exist a constant P such that*

$$\big\|h(t,x_t)-h(t,y_t)\big\|\leq P\big\|x_t-y_t\big\|_C \qquad (4.2)$$

for $t\in[0,T]$ and $x_t,y_t\in C$.

(H_{14}) *there exist a constant Q such that*

$$\big\|k(t,x_t)-k(t,y_t)\big\|\leq Q\big\|x_t-y_t\big\|_C \qquad (4.3)$$

for $t \in [0,T]$ *and* $x_t, y_t \in C$.

Then for each $\phi_1, \phi_2 \in C$ *and for the corresponding mild solutions* x_1, x_2 *of the problems*

$$\frac{d}{dt}[x(t) - w(t,x_t)] + Ax(t) = f\Big(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^T b(t,s)k(s,x_s)ds\Big), \; t \in [0,T], \tag{4.4}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi_i(t), \quad t \in [-r,0], \quad (i = 1,2), \tag{4.5}$$

the following inequality

$$\|x_1 - x_2\|_B \le U\|\phi_1 - \phi_2\|_C + \Big[U\rho + (U+1)V\|A^{-\beta}\| + V\,C_{1-\beta}\frac{T^\beta}{\beta} + UN\{T + (\lambda P + \mu Q)T^2\}\Big]\|x_1 - x_2\|_B \tag{4.6}$$

holds and if

$$U^* = \Big[U\rho + (U+1)V\|A^{-\beta}\| + V\,C_{1-\beta}\frac{T^\beta}{\beta} + UN\{T + (\lambda P + \mu Q)T^2\}\|\Big] < 1 \tag{4.7}$$

then

$$\|x_1 - x_2\|_B \le \frac{U}{1 - U^*}\|\phi_1 - \phi_2\|_C \tag{4.8}$$

Proof. Let $\phi_i\,(i = 1,2)$ be arbitrary functions of C and let $x_i\,(i = 1,2)$ be mild solutions of the nonlocal problems (1.3)-(1.4). Then using the hypothesis (H_6) and the fact that $U \ge 1$ we have for $t \in [-r,0]$,

$$\begin{aligned}
\|x_1(t) - x_2(t)\| &= \|\phi_1(t) - \big(g((x_1)_{t_1}, ..., (x_1)_{t_p})\big)(t) - \phi_2(t) + \big(g((x_2)_{t_1}, ..., (x_2)_{t_p})\big)(t)\| \\
&\le \|\phi_1(t) - \phi_2(t)\| \\
&\quad + \|\big(g((x_2)_{t_1}, ..., (x_2)_{t_p})\big)(t) - \big(g((x_1)_{t_1}, ..., (x_1)_{t_p})\big)(t)\| \\
&\le \|\phi_1 - \phi_2\|_C + \rho\|x_2 - x_1\|_B \\
&\le U\|\phi_1 - \phi_2\|_C + U\rho\|x_2 - x_1\|_B.
\end{aligned} \tag{4.9}$$

Now using hypotheses $(H_6), (H_8), (H_{12}) - (H_{14})$, lemma 2.1 and condition (2.1) we have for $t \in [0,T]$,

$$\begin{aligned}
\|x_1(t) - x_2(t)\| \le\; & \|T(t)\big[\phi_1(0) - \phi_2(0)\big]\| \\
& + \|T(t)\big[\big(g((x_2)_{t_1}, ..., (x_2)_{t_p})\big)(0) - \big(g((x_1)_{t_1}, ..., (x_1)_{t_p})\big)(0)\big]\| \\
& + \|T(t)\big[w(0,(x_2)_0) - w(0,(x_1)_0)\big]\| + \|w(t,(x_1)_t) - w(t,(x_2)_t)\| \\
& + \|\int_0^t AT(t-s)\big[w(s,(x_1)_s) - w(s,(x_2)_s)\big]ds\| \\
& + \|\int_0^t T(t-s)\Big[f\Big(s,(x_1)_s, \int_0^s a(s,\tau)h(\tau,(x_1)_\tau)d\tau, \\
& \int_0^T b(s,\tau)k(\tau,(x_1)_\tau)d\tau\Big) - f\Big(s,(x_2)_s, \int_0^s a(s,\tau)h(\tau,(x_2)_\tau)d\tau, \\
& \int_0^T b(s,\tau)k(\tau,(x_2)_\tau)d\tau\Big)\Big]ds\|
\end{aligned}$$

$$
\begin{aligned}
&\leq U\|\phi_1-\phi_2\|_C + U\rho\|x_2-x_1\|_B \\
&+ U\|A^{-\beta}A^{\beta}[w(0,(x_2)_0)-w(0,(x_1)_0)] \\
&+ \|A^{-\beta}A^{\beta}[w(t,(x_1)_t)-w(t,(x_2)_t)]\| \\
&+ \int_0^t \|A^{1-\beta}T(t-s)\|\|A^{\beta}[w(s,(x_1)_s)-w(s,(x_2)_s)]\|ds \\
&+ U\int_0^t N\Big[\|(x_1)_s-(x_2)_s\|_C + \|\int_0^s a(s,\tau)h(\tau,(x_1)_\tau)d\tau \\
&- \int_0^s a(s,\tau)h(\tau,(x_2)_\tau)d\tau\| \\
&+ \|\int_0^T b(s,\tau)k(\tau,(x_1)_\tau)d\tau - \int_0^T b(s,\tau)k(\tau,(x_2)_\tau)d\tau\|\Big]ds \\
&\leq U\|\phi_1-\phi_2\|_C + U\rho\|x_2-x_1\|_B + UV\|A^{-\beta}\|\|x_2-x_1\|_B \\
&+ V\|A^{-\beta}\|\,\|x_2-x_1\|_B + VC_{1-\beta}\frac{t^{\beta}}{\beta}\|x_2-x_1\|_B \\
&+ UN\int_0^t\Big[\|x_1-x_2\|_B + \lambda P\int_0^s \|(x_1)_\tau)-(x_2)_\tau)\|_C d\tau \\
&+ \mu Q\int_0^T \|(x_1)_\tau)-(x_2)_\tau)\|_C d\tau\Big]ds \\
&\leq U\|\phi_1-\phi_2\|_C + \big[U\rho + (U+1)V\|A^{-\beta}\| + VC_{1-\beta}\frac{T^{\beta}}{\beta} \\
&+ UN\{T+(\lambda P+\mu Q)T^2\}\big]\|x_2-x_1\|_B. \qquad (4.10)
\end{aligned}
$$

Thus in view of inequality (4.9) and (4.10) we get

$$
\begin{aligned}
\|x_1(t)-x_2(t)\| \leq U\|\phi_1-\phi_2\|_C &+ \big[U\rho+(U+1)V\|A^{-\beta}\| + VC_{1-\beta}\frac{T^{\beta}}{\beta} \\
&+ UN\{T+(\lambda P+\mu Q)T^2\}\big]\|x_1-x_2\|_B, \qquad t\in[-r,T] \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
\|x_1-x_2\|_B \leq U\|\phi_1-\phi_2\|_C &+ \big[U\rho+(U+1)V\|A^{-\beta}\| + VC_{1-\beta}\frac{T^{\beta}}{\beta} \\
&+ UN\{T+(\lambda P+\mu Q)T^2\}\big]\|x_1-x_2\|_B. \qquad (4.12)
\end{aligned}
$$

Using (4.7) we get

$$\|x_1-x_2\|_B \leq \frac{U}{1-U^*}\|\phi_1-\phi_2\|_C.$$

Hence the proof is complete. □

Remark 4.2. *We remark that the uniqueness of the solution of the nonlocal problem* (1.3)-(1.4) *follows from the above continuous dependence theorem.*

5 Application

As an application of the Theorem 3.1, we consider the system (1.1)-(1.2) with control parameter

$$x^{'}(t) + Ax(t) = Ez(t) + f\Big(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^{\zeta} b(t,s)k(s,x_s)ds\Big),\ \ t\in[0,\zeta], \qquad (5.1)$$

$$x(t) + \big(g(x_{t_1},...,x_{t_p})\big)(t) = \phi(t),\ \ t\in[-r,0], \qquad (5.2)$$

where E is a bounded linear operator from a Banach space Z to X and $z \in L^2([0,\zeta], Z)$. In this case the mild solution is given by

$$(i)\ \ x(t) = T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big] + \int_0^t T(t-s)Ez(s)ds$$
$$+ \int_0^t T(t-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau,x_\tau)d\tau, \int_0^\zeta b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds, t\in[0,\zeta] \tag{5.3}$$
$$(ii)\ \ x(t) + \big(g(x_{t_1},...,x_{t_p})\big)(t) = \phi(t),\ \ t\in[-r,0]. \tag{5.4}$$

We say that the system (5.1)-(5.2) is controllable to the origin if for any given initial function $\phi \in C$ there exists a control $z \in L^2([0,\zeta], Z)$ such that the mild solution $x(t)$ of (5.1)-(5.2) satisfies $x(\zeta) = 0$.

Note:We note that in this section the interval $[0,T]$ is replaced by $[0,\zeta]$ for notational convenience.
To derive the result we need the following additional hypotheses:

(H_{15}) The linear operator Ψ from $L^2([0,\zeta], Z)$ into X, defined by

$$\Psi z = \int_0^\zeta T(\zeta - s)Ez(s)ds$$

has an inverse operator Ψ^{-1} which takes values in $\frac{L^2([0,\zeta],Z)}{ker\Psi}$ such that the operator $E\Psi^{-1}$ is bounded.

(H_{16}) $UM^*\zeta\big[1 + U\,\|E\Psi^{-}\|\zeta\big]\Big\{1 + M^*\zeta\Big[\liminf_{m\to\infty}\Big(\frac{H(m)}{m} + \frac{K(m)}{m}\Big)\Big]\Big\} < 1$

Theorem 5.1. *If the hypotheses (H_1)-(H_7), (H_{11})-$(H_{14})$and (H_{15})-(H_{16}) are satisfied, then the system* (5.1) *with* (5.2) *is controllable if*

$$\big\{U\rho_2 + \int_0^\zeta \eta(s)\,\big[1 + \int_0^s \lambda\,\eta_1(\tau)d\tau + \int_0^\zeta \mu\,\eta_2(\tau)d\tau\big]ds\big\} < 1. \tag{5.5}$$

where the constant term

$$\rho_2 = U\rho + U\,\|E\Psi^{-1}\|\,\zeta\,\big[U\rho + \zeta UN\big\{1 + \lambda P\zeta + \mu Q\zeta\big\}\big]. \tag{5.6}$$

Proof. Using hypothesis (H_{15}) for an arbitrary function $x(\cdot)$, define the control

$$z(t) = -\,\Psi^{-1}\Big[T(\zeta)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big]$$
$$+ \int_0^\zeta T(\zeta-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau,x_\tau)d\tau, \int_0^\zeta b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds\Big](t) \tag{5.7}$$

for $t\in[0,\zeta]$. Using this control define an operator Γ as

$$(\Gamma x)(t) = \begin{cases} T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big] + \int_0^t T(t-s)Ez(s)ds \\ + \int_0^t T(t-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau,x_\tau)d\tau, \\ \int_0^\zeta b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds & 0\le t\le\zeta \\ \\ \phi(t) - \big(g(x_{t_1},...,x_{t_p})\big)(t), & -r\le t\le 0 \end{cases} \tag{5.8}$$

Substituting $z(s)$ in (5.8), we get

$$(\Gamma x)(t) = \begin{cases} T(t)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big] \\ - \int_0^t T(t-s)E\Psi^{-1}\Big[T(\zeta)\big[\phi(0) - \big(g(x_{t_1},...,x_{t_p})\big)(0)\big] + \int_0^\zeta T(\zeta-\eta)\cdot \\ f\Big(\eta, x_\eta, \int_0^\eta a(\eta,\tau)h(\tau,x_\tau)d\tau, \int_0^\zeta b(\eta,\tau)k(\tau,x_\tau)d\tau\Big)d\eta\Big](s)ds \\ + \int_0^t T(t-s)f\Big(s, x_s, \int_0^s a(s,\tau)h(\tau,x_\tau)d\tau, \\ \int_0^\zeta b(s,\tau)k(\tau,x_\tau)d\tau\Big)ds & 0\le t\le\zeta \\ \\ \phi(t) - \big(g(x_{t_1},...,x_{t_p})\big)(t), & -r\le t\le 0 \end{cases} \tag{5.9}$$

Clearly $(\Gamma x)(\zeta) = 0$, which means that the control z steers the system from the initial function ϕ to the origin in time ζ if we can obtain a fixed point of the operator Γ. The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted. Thus the system (5.1) with (5.2) is controllable. □

References

[1] D. Bahuguna, and S. Agarwal, Approximations of solutions to neutral functional differential equations with nonlocal history conditions, *J. Math. Anal. Appl.*, 317(2006), 583-602.

[2] K. Balachandran, and R. Sakthivel, Existence of solutions of neutral functional integrodifferential equation in Banach space, *Proc. Indian Acad. Sci. Math. Sci.,* 109(1999), 325-332.

[3] K. Balachandran, and J.Y. Park, 2001. Existence of a mild solution of a functional integrodifferential equation with nonlocal condition, *Bull. Korean Math. Soc.*, 38(1)(2001), 175-182.

[4] J. Banas, and K. Goebel, *Measure of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Math. Dekker, New York, Vol 60, 1980.

[5] M. Benchohra, and S.K. Ntouyas, Nonlocal Cauchy problems for neutral functional Differential and integrodifferential inclusions, *J. Math. Anal. Appl.*, 258(2001), 573-590.

[6] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Appl. Math. Stoc.Anal.*, 162(1991), 494-505.

[7] L. Byszewski, and H. Akca, On a mild solution of a semilinear functional-differential evolution nonlocal problem, *J. Appl. Math. Stoc.Anal.*, 10(1997), 265-271.

[8] L. Byszewiski, and H. Akca, Existence of solutions of a semilinear functional-differential evolution nonlocal problem, *Nonlinear Analysis*, 34(1998), 65-72.

[9] X. Fu X, and K. Ezzinbi, Existence of solutions for neutral functional differential evolution equations with nonlocal conditions, *Nonlinear Anal.*, 54(2003), 215-227.

[10] E. Hernandez, and H.R. Henriquez, Existence results for partial neutral functional differential equations with unbounded delay, *J. Math. Anal. Appl.*, 221(1998), 452-475.

[11] E. Hernandez, Existence results for a class of semi-linear evolution equations, *Electron. J. Differential Equations*, Vol 2001, pp. 1-14.

[12] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[13] D. Qixiang, F. Zhenbin, and Gang Li, Semilinear differential equations with nonlocal conditions in Banach spaces, *International J. Nonlinear Science*, 2(3)(2006), 131-139.

[14] D. Qixiang, F. Zhenbin, and Gang Li, Existence of solutions to nonlocal neutral functional differential and integrodifferential equations, *International J. Nonlinear Science*, 5(2)(2008), 140-151.

[15] Y. Runping, and Z. Guowei, Neutral functional differential equations of second order with infinite delays, *Electronic Journal of Differential Equations*, Vol 2010, No.36, pp. 1-12.

[16] X. Xue, Nonlinear differential equations with nonlocal conditions in Banach spaces, *Nonlinear Anal.*, 63(2005), 575-586.

[17] X. Xue, Existence of solutions for semilinear nonlocal Cauchy problems in Banach spaces,*Electronic Journal of Differential Equations*, 64(2005), 1-7.

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