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Periodic solutions of nonlinear finite difference systems with time delays

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Abstract

In this paper a coupled system of nonlinear finite difference equations corresponding to a class of periodic-parabolic systems with time delays and with nonlinear boundary conditions in a bounded domain is investigated. Using the method of upper-lower solutions two monotone sequences for the finite difference system are constructed. Existence of maximal and minimal periodic solutions of coupled system of finite difference equations with nonlinear boundary conditions is also discussed. The proof of existence theorem is based on the method of upper-lower solutions and its associated monotone iterations. It is shown that the sequence of iterations converges monotonically to unique solution of the nonlinear finite difference system with time delays under consideration.

Keywords: Periodic solution, periodic parabolic system, finite difference equation, upper and lower solution, quasimonotone nondecreasing function.

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1 Introduction

Many researchers investigated periodic solutions of parabolic boundary value problems. Their work is either on scalar periodic-parabolic boundary value problem [1,2] or to system of reaction-diffusion type of equations with specific reaction functions [3,4]. Much of the work for parabolic initial boundary value problem with time delay [5,6,7] and without time delay [1,2,4,10] is found in literature. The recent work of [12] is on the periodic parabolic system with time delays under linear boundary conditions. Also Pao [8,9] has discussed system of periodic parabolic equations with nonlinear boundary conditions with and without time delays. Recently Pao [7] investigated some numerical aspect of the class of coupled nonlinear systems with time delays. Most of the works in the literature are devoted to the qualitative analysis of the system and dynamics of the system [6]. In this paper we give a treatment to a coupled system of finite difference equations of periodic-parabolic system with time delay and with nonlinear boundary conditions and obtain the results which are motivated by earlier results of Pao [6,7,8].

2 Finite Difference Equations

Consider the system which consists of an arbitrary number of parabolic equations in a bounded domain Ω in $\Re^p$ (p = 1,2,3, . . .) with boundary $\partial\Omega$ and with fixed period T > 0 in the form.

$$(2.1)\qquad \begin{cases} \frac{\partial u^{(l)}}{\partial t} - L^{(l)}u^{(l)} = f^{(l)}(x,t,\mathbf{u},\mathbf{u}_\tau) \ , \quad x\in\Omega\ , \quad t>0 \\ B^{(l)}u^{(l)} = g^{(l)}(x,t,\mathbf{u}), \qquad x\in\partial\Omega \ , \quad t>0 \\ u^{(l)}(x,t) = u^{(l)}(x,t+T), \qquad x\in\Omega, -\tau_l\le t\le 0, \end{cases}$$

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where $\mathbf{u} = \mathbf{u}(x,t) = \big(u^{(1)}(x,t), u^{(2)}(x,t), ..., u^{(N)}(x,t)\big)$

$$\mathbf{u}_\tau = \mathbf{u}_\tau(x,t) = \Big(u^{(1)}(x,t-\tau_1), u^{(2)}(x,t-\tau_2), ..., u^{(N)}(x,t-\tau_N)\Big)$$

for some time delays $\tau_1, \tau_2, ..., \tau_N > 0$ and for each l = 1,2, . . ., N ; $L^{(l)}u^{(l)}$ and $B^{(l)}u^{(l)}$ are given by

$$L^{(l)}u^{(l)} = \nabla\cdot(D^{(l)}\nabla u^{(l)}) + V^{(l)}\cdot\nabla u^{(l)} \quad , \quad B^{(l)}u^{(l)} = \alpha^{(l)}\frac{\partial u^{(l)}}{\partial\nu} + \beta^{(l)}u^{(l)}$$

with $\frac{\partial}{\partial\nu}$ denoting outward normal derivative on $\partial\Omega$. It is assumed that the diffusion coefficient $D^{(l)} = D^{(l)}(x,t) > 0$ and the convection coefficient $V^{(l)} = (V_1^{(l)}, V_2^{(l)}, ..., V_p^{(l)})$ *of* $L^{(l)}$ where $V_\nu^{(l)} = V_\nu^{(l)}(x,t)$ *for* ν = 1,2,. . .,p are continuous on $\overline{D}_T = \overline{\Omega}\times[0,T]$ for every finite T > 0. The coefficients $\alpha^{(l)}$ and $\beta^{(l)} = \beta^{(l)}(x,t)$ *of* $B^{(l)}$ are continuous on $S_T = \partial\Omega\times[0,T]$ with either $\alpha^{(l)} = 0$, $\beta^{(l)} > 0$ (Dirichlet condition) or $\alpha^{(l)} = 1$, $\beta^{(l)} \geq 0$ (Neumann or Robin condition) where $\overline{\Omega} = \Omega\cup\partial\Omega$.

It is assumed that $f^{(l)}$, $g^{(l)}$ *and* $u^{(l)}$ are continuous functions in their respective domains and $f^{(l)}(\cdot,\mathbf{u},\mathbf{u}_\tau)$, $g^{(l)}(\cdot,\mathbf{u})$ are in general nonlinear in $\mathbf{u}$ and $\mathbf{u}_\tau$; and satisfy the conditions in hypothesis (H_2) of Section 3.

Let $\mathbf{x}_j = (x_{j_1}, x_{j_2}, ..., x_{j_p})$ be an arbitrary mesh point in $\overline{\Omega}$, where j $= (j_1, j_2, ..., j_p)$ is a multiple index with j_ν = 1,2, . . ., M_ν and for each ν = 1,2, . . . , p , M_ν is the total number of mesh points in the x_ν direction. Denote by Ω_p, Λ_p and $Q_0^{(l)}$ the sets of mesh points in $\Omega, \Omega\times(0,\infty)$ *and* $\Omega\times[-\tau_l, 0]$ respectively. Similarly denote by $\partial\Omega_p, S_p$ and $Q_p^{(l)}$ the sets of mesh points in $\partial\Omega, \partial\Omega\times[0,\infty)$ *and* $\overline{\Omega}\times[-\tau_l,\infty)$ respectively. Further let $Q_p = Q_p^{(1)}\times Q_p^{(2)}\times ...\times Q_p^{(N)}$. The set of all mesh points in $\overline{\Omega}$ and $\overline{\Omega}\times[0,\infty)$ are denoted by $\overline{\Omega_p}$ and $\overline{\Lambda}_p$ respectively. It is assumed that, the domain Ω is connected. Let $k_n = t_n - t_{n-1}$ be the time increment and h_ν the spatial increment in the x_ν direction. For each l = 1, 2, . . . , N we choose k_n such that $\tau_l = k_1 + k_2 + ... + k_{s_l}$ for some integer $s_l > 0$.

Define

$$u_{j,n}^{(l)} = u^{(l)}(x_j, t_n) \qquad , \quad \mathbf{u}_{j,n} = (u_{j,n}^{(1)}, u_{j,n}^{(2)}, ..., u_{j,n}^{(N)}),$$
$$u_{j,n-s_l}^{(l)} = u^{(l)}(x_j, t_{n-s_l}) \qquad , \quad \mathbf{u}_{j,n-s} = (u_{j,n-s_1}^{(1)}, u_{j,n-s_2}^{(2)}, ..., u_{j,n-s_N}^{(N)}),$$
$$f^{(l)}(\mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) = f^{(l)}(\mathbf{x}_j, t_n, \mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) \quad , \quad g^{(l)}(\mathbf{u}_{j,n}) = g^{(l)}(\mathbf{x}_j, t_n, \mathbf{u}_{j,n}).$$

Define the standard central difference operators as follows:

$$\triangle_{u_{j,n}}^{(\nu)} = h_\nu^{-2}\left[u(\mathbf{x}_j + h_\nu e_\nu, t_n) - 2u(\mathbf{x}_j, t_n) + u(\mathbf{x}_j - h_\nu e_\nu, t_n)\right]$$
$$\delta_{u_{j,n}}^{(\nu)} = 2h_\nu^{-1}\left[u(\mathbf{x}_j + h_\nu e_\nu, t_n) - u(\mathbf{x}_j - h_\nu e_\nu, t_n)\right]$$

where e_ν is the unit vector in $\Re^p$ with ν^{th} component 1 and zero elsewhere. Approximating the parabolic system in (2.1) by the nonlinear finite difference system, we have

$$(2.2) \qquad \begin{cases} k_n^{-1}(u_{j,n}^{(l)} - u_{j,n-1}^{(l)}) - L^{(l)}u_{j,n}^{(l)} = f^{(l)}(\mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) & in\ \Lambda_p \\ B^{(l)}[u_{j,n}^{(l)}] = g^{(l)}(\mathbf{u}_{j,n}) & on\ S_p, \\ u_{j,n}^{(l)} = u_{j,n+k}^{(l)} & in\ Q_0^{(l)} \ , \ l = 1,2,,...,N \end{cases}$$

where

$$L^{(l)}u_{j,n}^{(l)} = \textstyle\sum_{\nu=1}^{p}\left(D_{j,n}^{(l)}\triangle_{u_{j,n}}^{(\nu)} + (V_{j,n}^{(l)})_\nu\delta^{(\nu)}u_{j,n}^{(l)}\right), D_{j,n}^{(l)} = D^{(l)}(x,t)\ , \ \ (V_{j,n}^{(l)})_\nu = (V^{(l)}(x,t))_\nu$$

$$B^{(l)}[u_{j,n}^{(l)}] = \alpha^{(l)}(\mathbf{x}_j)\,|\mathbf{x}_j - \hat{x_j}|^{-1}\left[u^{(l)}(\mathbf{x}_j, t_n) - u^{(l)}(\hat{x_j}, t_n)\right] + \beta^{(l)}(\mathbf{x}_j, t_n)u^{(l)}(\mathbf{x}_j, t_n),$$

and $\quad u_{j,n+k}^{(l)} = u^{(l)}(\mathbf{x}_j, t_{n+T}),\ T > 0$, In the boundary conditions $\hat{x_j}$ is a suitable point in Ω_p and $|x_j - \hat{x_j}|$ is the distance between x_j and $\hat{x_j}$. We define upper and lower solutions for the discrete problem (2.2) in the following section.

3 Upper and Lower Solutions

Definition 3.1. *A function $\tilde{\mathbf{u}}_{j,n} \equiv (\tilde{u}^{(1)}_{j,n}, \tilde{u}^{(2)}_{j,n}, ..., \tilde{u}^{(N)}_{j,n})$ in Q_p is called an upper solution of (2.2) if*

$$(3.1) \qquad \begin{cases} k_n^{-1}(\tilde{u}^{(l)}_{j,n} - \tilde{u}^{(l)}_{j,n-1}) - L^{(l)}\tilde{u}^{(l)}_{j,n} \;\geq\; f^{(l)}_{j,n}(\tilde{\boldsymbol{u}}_{j,n}, \tilde{\boldsymbol{u}}_{j,n-s}) \quad in \ \Lambda_p \ , \\ B^{(l)}[\tilde{u}^{(l)}_{j,n}] \;\geq\; g^{(l)}_{j,n}(\tilde{\boldsymbol{u}}_{j,n}) \quad on \ S_p \ , \\ \tilde{u}^{(l)}_{j,n} \;\geq\; \tilde{u}^{(l)}_{j,n+k} \quad in \ Q^{(l)}_0 \ . \end{cases}$$

Similarly $\hat{\mathbf{u}}_{j,n} \equiv (\hat{u}^{(1)}_{j,n}, \hat{u}^{(2)}_{j,n}, ..., \hat{u}^{(N)}_{j,n})$ in Q_p is called a lower solution of (2.2) if it satisfies the inequalities in (3.1) in reverse order.

Suppose $\tilde{\mathbf{u}}_{j,n}, \hat{\mathbf{u}}_{j,n}$ exist and $\tilde{\mathbf{u}}_{j,n} \geq \hat{\mathbf{u}}_{j,n}$.
Define

$$\mathscr{S}^{(1)} = \{\mathbf{u}_{j,n} \in Q_p \ , \ \hat{\mathbf{u}}_{j,n} \leq \mathbf{u}_{j,n} \leq \tilde{\mathbf{u}}_{j,n}\} \ ,$$

$$\mathscr{S}^{(2)} = \{\mathbf{v}_{j,n} \in Q_p \ , \ \hat{\mathbf{u}}_{j,n-s} \leq \mathbf{v}_{j,n-s} \leq \tilde{\mathbf{u}}_{j,n-s}\} \ ,$$

$$\mathscr{S} = \{(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \in Q_p \times Q_p \ ; \ \hat{\mathbf{u}}_{j,n} \leq \mathbf{u}_{j,n} \leq \tilde{\mathbf{u}}_{j,n} \ , \ \hat{\mathbf{u}}_{j,n-s} \leq \mathbf{v}_{j,n-s} \leq \tilde{\mathbf{u}}_{j,n-s}\} \ .$$

Also define

$$\mathbf{f}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) = \Big(f^{(1)}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \ , \ f^{(2)}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \ , \ ... \ , \ f^{(N)}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n})\Big) \ ,$$

$$\mathbf{g}_{j,n}(\mathbf{u}_{j,n}) = \Big(g^{(1)}_{j,n}(\mathbf{u}_{j,n}) \ , \ g^{(2)}_{j,n}(\mathbf{u}_{j,n}) \ , \ ... \ , \ g^{(N)}_{j,n}(\mathbf{u}_{j,n})\Big) \ .$$

Definition 3.2. *A function $\boldsymbol{f}_{j,n}(\boldsymbol{u}_{j,n}, \boldsymbol{v}_{j,n})$ is said to be quasi-monotone nondecreasing in $\mathscr{S}$ if for each* l *and each $(\boldsymbol{u}_{j,n}, \boldsymbol{v}_{j,n}) \in \mathscr{S}$, $f^{(l)}_{j,n}(\boldsymbol{u}_{j,n}, \boldsymbol{v}_{j,n})$ is nondecreasing in $\boldsymbol{u}_{j,n} = (u^{(1)}_{j,n}, u^{(2)}_{j,n} \ , \ ... \ , \ u^{(N)}_{j,n})$ for all $u^{(l)}_{j,n} \neq u^{(m)}_{j,n}$ and nondecreasing in $\boldsymbol{v}_{j,n} = (v^{(1)}_{j,n}, v^{(2)}_{j,n} \ , \ ... \ , \ v^{(N)}_{j,n})$ for all $v^{(m)}_{j,n}$, m = 1,2 , . . . , N .*

We now make the following hypothesis

(H_1) For each $l = 1 \ , \ 2 \ , \ ... \ ,$ N the coefficients $D^{(l)}$, $V^{(l)}$ of $L^{(l)}$ and the functions $f^{(l)}_{j,n}(\cdot)$, $g^{(l)}_{j,n}(\cdot)$ and $\beta(x_j, t_n)$ are all k-periodic in n.

(H_2) $\mathbf{f}_{j,n}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n})$ and $\mathbf{g}_{j,n}(\mathbf{u}_{j,n})$ are quasi-monotone nondecreasing C^1 -functions of $(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \in \mathscr{S}$ and $\mathbf{u}_{j,n} \in \mathscr{S}^{(l)}$ respectively.

The hypothesis (H_2) is equivalent to the condition

$\frac{\partial f^{(l)}_{j,n}}{\partial u^{(m)}}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \geq 0$ for $m \neq l$, $\quad \frac{\partial f^{(l)}_{j,n}}{\partial v^{(m)}}(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \geq 0$ for m = 1 ,2 , .., N .

where $(\mathbf{u}_{j,n}, \mathbf{v}_{j,n}) \in \mathscr{S}$.

and $\frac{\partial g^{(l)}_{j,n}}{\partial v^{(m)}}(\mathbf{u}_{j,n} \geq 0$ for $m \neq l$ where $\mathbf{u}_{j,n} \in \mathscr{S}^{(1)}$ for l , m = 1 ,2,.. , N .

The subsets $\mathscr{S}$, $\mathscr{S}^{(1)}$ *and* $\mathscr{S}^{(2)}$ are the sectors between the pairs of upper and lower solutions.

Assume that the quasi-monotone condition in (H_2) holds in the above subsets $\mathscr{S}$ *and* $\mathscr{S}^{(1)}$. Let

$$(3.2)\qquad \begin{cases} \gamma_{j,n}^{(l)} \geq Max\left\{-\frac{\partial f_{j,n}^{(l)}}{\partial \mathbf{u}^{(l)}}(\mathbf{u}_{j,n},\mathbf{v}_{j,n})\ ,\ (j,n)\in\overline{\Lambda}_p\ ,\ (\mathbf{u}_{j,n},\mathbf{v}_{j,n})\in\mathscr{S}\right\} \\ \sigma_{j,n}^{(l)} \geq Max\left\{-\frac{\partial g_{j,n}^{(l)}}{\partial u^{(l)}}(\mathbf{u}_{j,n})\ ,\ (j,n)\in\overline{\Lambda}_p\ ,\ \mathbf{u}_{j,n}\in\mathscr{S}^{(1)}\right\} \end{cases}$$

Define

$$(3.3)\qquad \begin{cases} \mathscr{L}^{(l)}[u_{j,n}^{(l)}] = k_n^{-1}(u_{j,n}^{(l)} - u_{j,n-1}^{(l)}) - L^{(l)}u_{j,n}^{(l)} + \gamma_{j,n}^{(l)}u_{j,n}^{(l)} \\ \mathscr{B}^{(l)}[u_{j,n}^{(l)}] = B^{(l)}[u_{j,n}^{(l)}] + \sigma_{j,n}^{(l)}u_{j,n}^{(l)} \\ F_{j,n}^{(l)}(\mathbf{u}_{j,n},\mathbf{v}_{j,n}) = \gamma_{j,n}^{(l)}u_{j,n}^{(l)} + f_{j,n}^{(l)}(\mathbf{u}_{j,n},\mathbf{v}_{j,n}) \\ G_{j,n}^{(l)}(\mathbf{u}_{j,n}) = \sigma_{j,n}^{(l)}u_{j,n}^{(l)} + g_{j,n}^{(l)}(\mathbf{u}_{j,n})\ ,\ l = 1,2,...,N. \end{cases}$$

By hypothesis (H_2) , $F^{(l)}$ *and* $G^{(l)}$ possess the property,

$$(3.4)\qquad \begin{cases} F_{j,n}^{(l)}(\mathbf{u}_{j,n},\mathbf{v}_{j,n}) \geq F_{j,n}^{(l)}(\mathbf{u}'_{j,n},\mathbf{v}'_{j,n}) \\ \qquad when\ \ (\hat{\mathbf{u}}_{j,n},\hat{\mathbf{v}}_{j,n}) \leq (\mathbf{u}'_{j,n},\mathbf{v}'_{j,n}) \leq (\mathbf{u}_{j,n},\mathbf{v}_{j,n}) \leq (\tilde{\mathbf{u}}_{j,n},\tilde{\mathbf{v}}_{j,n})\ , \\ G_{j,n}^{(l)}(\mathbf{u}_{j,n}) \geq G_{j,n}^{(l)}(\mathbf{u}'_{j,n})\ , \\ \qquad when\ \ \hat{\mathbf{u}}_{j,n} \leq \mathbf{u}'_{j,n} \leq \mathbf{u}_{j,n} \leq \tilde{\mathbf{u}}_{j,n}\ , l = 1,2,...,N. \end{cases}$$

Using either $\mathbf{u}_{j,n}^{(0)} = \tilde{\mathbf{u}}_{j,n}$ *or* $\mathbf{u}_{j,n}^{(0)} = \hat{\mathbf{u}}_{j,n}$ as the initial iteration we construct a sequence $\left\{\mathbf{u}_{j,n}^{(m)}\right\} = \left\{(u_{j,n}^{(1)})^m\ ,\ (u_{j,n}^{(2)})^m\ ,\ .\ .\ .\ ,(u_{j,n}^{(N)})^m\right\}$ from the linear discrete system

$$(3.5)\qquad \begin{cases} \mathscr{L}^{(l)}[u_{j,n}^{(l)}]^m = F_{j,n}^{(l)}(\mathbf{u}_{j,n}^{(m-1)},\mathbf{u}_{j,n-s}^{(m-1)}) \quad in\ \ \Lambda_p \\ \mathscr{B}^{(l)}[u_{j,n}^{(l)}]^m = G_{j,n}^{(l)}(\mathbf{u}_{j,n}^{(m-1)}) \qquad on\ \ S_p \\ [u_{j,n}^{(l)}]^m = (u_{j,n+k}^{(l)})^{m-1} \qquad in\ \ Q_0^{(l)}\ , \end{cases}$$

where n = 0 , -1 , -2, . . . , - s_l , $l = 1$,2 , . . . , N , k > 0 , and m = 1 ,2 , . . .

From the above , it is clear that, the sequence $\left\{\mathbf{u}_{j,n}^{(m)}\right\}$ is well defined. Denote this sequence by $\left\{\overline{\mathbf{u}}_{j,n}^{(m)}\right\}$ *if* $\mathbf{u}_{j,n}^{(0)} = \tilde{\mathbf{u}}_{j,n}$ *and by* $\left\{\underline{\mathbf{u}}_{j,n}^{(m)}\right\}$ *if* $\mathbf{u}_{j,n}^{(0)} = \hat{\mathbf{u}}_{j,n}$.

Now we prove the monotone property of these sequences.

Lemma 3.1. *The sequences* $\left\{\overline{\boldsymbol{u}}^{(m)}_{j,n}\right\}$, $\left\{\underline{\boldsymbol{u}}^{(m)}_{j,n}\right\}$ *possess the monotone property,*

$$(3.6) \qquad \hat{\boldsymbol{u}}_{j,n} \le \underline{\boldsymbol{u}}^{(m)}_{j,n} \le \underline{\boldsymbol{u}}^{(m+1)}_{j,n} \le \overline{\boldsymbol{u}}^{(m+1)}_{j,n} \le \overline{\boldsymbol{u}}^{(m)}_{j,n} \le \tilde{\boldsymbol{u}}_{j,n} \ \ on\ Q_p$$

where $m = 1\ ,\ 2\ ,\ .\ .\ .\ .$

Proof. Let $\left[\overline{w}^{(l)}_{j,n}\right]^{(0)} = \left[\overline{u}^{(l)}_{j,n}\right]^{(0)} - \left[\overline{u}^{(l)}_{j,n}\right]^{(1)}$ where $\left[\overline{u}^{(l)}_{j,n}\right]^{(0)} = \tilde{u}^{(l)}_{j,n}$.

By (3.3), (3.5) and (3.1) we have

$$\mathscr{L}^{(l)}\left[\overline{w}^{(l)}_{j,n}\right]^{(0)} = k_n^{-1}(\tilde{u}^{(l)}_{j,n} - \tilde{u}^{(l)}_{j,n-1}) - L^{(l)}\tilde{u}^{(l)}_{j,n} + \gamma^{(l)}_{j,n}\tilde{u}^{(l)}_{j,n} - \left[\gamma^{(l)}_{j,n}\left(\overline{u}^{(l)}_{j,n}\right)^{(0)} + f^{(l)}_{j,n}\left(\overline{\mathbf{u}}^{(0)}_{j,n}, \overline{\mathbf{u}}^{(0)}_{j,n-s}\right)\right]$$

$$= k_n^{-1}(\tilde{u}^{(l)}_{j,n} - \tilde{u}^{(l)}_{j,n-1}) - L^{(l)}\tilde{u}^{(l)}_{j,n} - f^{(l)}_{j,n}(\tilde{\mathbf{u}}^{(l)}_{j,n}\ ,\ \tilde{\mathbf{u}}^{(l)}_{j,n-s}) \ \ge\ 0 \ \ in\ \Lambda_p$$

$$\mathscr{B}^{(l)}[\overline{w}^{(l)}_{j,n}]^{(0)} = \left[B^{(l)}[\tilde{u}^{(l)}_{j,n}] + \tilde{u}^{(l)}_{j,n}\sigma^{(l)}_{j,n}\right] - \left[\sigma^{(l)}_{j,n}\left(\overline{u}^{(l)}_{j,n}\right)^{(0)} + g^{(l)}_{j,n}\left(\overline{\mathbf{u}}^{(0)}_{j,n}\right)\right]$$

$$= B^{(l)}[\tilde{u}^{(l)}_{j,n}] - g^{(l)}_{j,n}\left(\tilde{\mathbf{u}}^{(0)}_{j,n}\right) \ \ge\ 0 \ \ on\ S_p$$

$$[\overline{w}^{(l)}_{j,n}]^{(0)} = \tilde{u}^{(l)}_{j,n} - \tilde{u}^{(l)}_{j,n+k} \ \ge\ 0 \ \ in\ Q^{(l)}_0.$$

By positivity Lemma of [11] for finite difference equations of parabolic initial boundary value problem

$$[\overline{w}^{(l)}_{j,n}]^{(0)} \ \ge\ 0 \ \ on\ Q^{(l)}_p$$

Thus $(\overline{u}^{(l)}_{j,n})^{(0)} \ \ge\ [\overline{u}^{(l)}_{j,n}]^{(1)}$ *on* $Q^{(l)}_p$. This yields $\overline{\mathbf{u}}^{(0)}_{j,n} \ \ge\ \overline{\mathbf{u}}^{(1)}_{j,n}$ *on* Q_p.

A similar argument using the property of a lower solution gives $\underline{\mathbf{u}}^{(1)}_{j,n} \ \ge\ \underline{\mathbf{u}}^{(0)}_{j,n}$
Put $[\overline{w}^{(l)}_{j,n}]^{(1)} = [\overline{u}^{(l)}_{j,n}]^{(1)} - [\underline{u}^{(l)}_{j,n}]^{(1)}$. Then by (3.4) and (3.5) , we have

$$\mathscr{L}^{(l)}[\overline{w}^{(l)}_{j,n}]^{(1)} = F^{(l)}_{j,n}\left(\overline{\mathbf{u}}^{(0)}_{j,n}, \overline{\mathbf{u}}^{(0)}_{j,n-s}\right) - F^{(l)}_{j,n}\left(\underline{\mathbf{u}}^{(0)}_{j,n}, \underline{\mathbf{u}}^{(0)}_{j,n-s}\right) \ \ge\ 0 \ \ in\ \Lambda_p$$

$$\mathscr{B}^{(l)}[\overline{w}^{(l)}_{j,n}]^{(1)} = G^{(l)}_{j,n}\left(\overline{\mathbf{u}}^{(0)}_{j,n}\right) - G^{(l)}_{j,n}\left(\underline{\mathbf{u}}^{(0)}_{j,n}\right) \ \ge\ 0 \ \ on\ S_p$$

$$[\overline{w}^{(l)}_{j,n}]^{(1)} = [\overline{u}^{(l)}_{j,n+k}]^{(0)} - [\underline{u}^{(l)}_{j,n+k}]^{(0)} \ \ge\ 0 \ \ on\ Q^{(l)}_0.$$

It follows again from positivity lemma of [11]that $[\overline{w}^{(l)}_{j,n}]^{(1)} \ \ge\ 0.$

i.e. $(\overline{u}^{(l)}_{j,n})^{(1)} \ \ge\ (\underline{u}^{(l)}_{j,n})^{(1)}$ *on* $Q^{(l)}_p$. This gives $\overline{\mathbf{u}}^{(1)}_{j,n} \ \ge\ \underline{\mathbf{u}}^{(1)}_{j,n}$.

The above conclusions show that

$$\overline{\mathbf{u}}^{(0)}_{j,n} \ \ge\ \overline{\mathbf{u}}^{(1)}_{j,n} \ \ge\ \underline{\mathbf{u}}^{(1)}_{j,n} \ \ge\ \underline{\mathbf{u}}^{(0)}_{j,n} \ \ on\ \ Q_p$$

The monotone property (3.6)follows by an induction argument as in [11]. □

It is clear from the monotone property (3.6) that the point-wise limits

$$(3.7) \qquad \lim_{m\to\infty} \overline{\mathbf{u}}^{(m)}_{j,n} = \overline{\mathbf{u}}_{j,n} \text{ and } \lim_{m\to\infty} \underline{\mathbf{u}}^{(m)}_{j,n} = \underline{\mathbf{u}}_{j,n}$$

exist and satisfy the relation

(3.8) $\quad \hat{\mathbf{u}}_{j,n} \leq \underline{\mathbf{u}}_{j,n}^{(m)} \leq \underline{\mathbf{u}}_{j,n}^{(m+1)} \leq \underline{\mathbf{u}}_{j,n} \leq \overline{\mathbf{u}}_{j,n} \leq \overline{\mathbf{u}}_{j,n}^{(m+1)} \leq \overline{\mathbf{u}}_{j,n}^{(m)} \leq \tilde{\mathbf{u}}_{j,n} \;\; on \;\; Q_p$

Now we show that $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ are respectively maximal and minimal k-periodic solutions of (2.2).

Theorem 3.1. *Let $\hat{\mathbf{u}}_{j,n}$ and $\tilde{\mathbf{u}}_{j,n}$ be ordered lower and upper solutions of (2.2) and let hypothesis $(H_1),(H_2)$ be satisfied. Then the problem (2.2) has a maximal k-periodic solution $\overline{\mathbf{u}}_{j,n}$ and a minimal k-periodic solution $\underline{\mathbf{u}}_{j,n}$ in $\mathscr{S}^{(1)}$. Moreover the sequences $\left\{\overline{\mathbf{u}}_{j,n}^{(m)}\right\}$ and $\left\{\underline{\mathbf{u}}_{j,n}^{(m)}\right\}$ converge monotonically to $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ respectively and satisfy the relation (3.8). If in addition $\overline{\mathbf{u}}_{j,0} = \underline{\mathbf{u}}_{j,0}$ then $\overline{\mathbf{u}}_{j,n} = \underline{\mathbf{u}}_{j,n}$ $(= \mathbf{u}_{j,n}^{*})$, and $\mathbf{u}_{j,n}^{*}$ is the unique solution of (2.2) in $\mathscr{S}^{(1)}$.*

Proof. The sequence $\left\{\mathbf{u}_{j,n}^{(m)}\right\}$ constructed from the linear system (3.5) with initial iteration either upper or lower solution of (2.2) converge to $\overline{\mathbf{u}}_{j,n}$or $\underline{\mathbf{u}}_{j,n}$ according to initial iteration as $\tilde{\mathbf{u}}_{j,n}$ or $\hat{\mathbf{u}}_{j,n}$ respectively and using (3.7) it shows that both $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ satisfy the equations

$$(3.9) \qquad \begin{cases} \mathscr{L}^{(l)}[u_{j,n}^{(l)}] = F_{j,n}^{(l)}(\mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) & in \;\; \Lambda_p \; , \\ \mathscr{B}^{(l)}[u_{j,n}^{(l)}] = G_{j,n}^{(l)}(\mathbf{u}_{j,n},) & on \;\; S_p \; , \\ u_{j,n}^{(l)} = u_{j,n+k}^{(l)} & in \quad Q_0^{(l)} \; . \end{cases}$$

In view of (3.3) $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ are solutions of (2.2).

To show that $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$ are k-periodic solutions we let $w_{j,n}^{(l)} = u_{j,n}^{(l)} - u_{j,n+k}^{(l)}$,

where $u_{j,n}^{(l)}$ stands for either $\overline{u}_{j,n}^{(l)}$ or $\underline{u}_{j,n}^{(l)}$, l =1, 2 , . . . , N.

By Hypothesis (H_1) and mean value theorem, we have

$$\begin{aligned} k_n^{-1}(w_{j,n}^{(l)} - w_{j,n-1}^{(l)}) - L_n^{(l)} w_{j,n}^{(l)} &= k_n^{-1}(u_{j,n}^{(l)} - u_{j,n-1}^{(l)}) - L_n^{(l)} u_{j,n}^{(l)} \\ &\quad - \left[k_n^{-1}(u_{j,n+k}^{(l)} - u_{j,n+k-1}^{(l)}) - L_{n+k}^{(l)} u_{j,n+k}^{(l)}\right] \\ &= f_{j,n}^{(l)}(\mathbf{u}_{j,n}, \mathbf{u}_{j,n-s}) - f_{j,n}^{(l)}(\mathbf{u}_{j,n+k}, \mathbf{u}_{j,n+k-s}) \\ &= \sum_{m=1}^{N} \left(\frac{\partial f_{j,n}^{(l)}}{\partial u^{(m)}}(\xi,\eta)\right) w_{j,n}^{(m)} + \sum_{m=1}^{N} \left(\frac{\partial f_{j,n}^{(l)}}{\partial v^{(m)}}(\xi,\eta)\right) w_{j,n-s}^{(m)} \;\; in \; \Lambda_p \end{aligned}$$

$$(3.10) \qquad \begin{aligned} B^{(l)}[w_{j,n}^{(l)}] &= B_n^{(l)}[u_{j,n}^{(l)}] - B_{n+k}^{(l)}[u_{j,n+k}^{(l)}] \\ &= g_{j,n}^{(l)}(\mathbf{u}_{j,n}) - g_{j,n}^{(l)}(\mathbf{u}_{j,n+k}) \\ &= \sum_{m=1}^{N} \left(\frac{\partial g_{j,n}^{(l)}}{\partial u^{(m)}}(\xi')\right) w_{j,n}^{(m)} \qquad \text{on} \quad S_p \; , \end{aligned}$$

and $\quad w_{j,n}^{(l)} = u_{j,n}^{(l)} - u_{j,n+k}^{(l)} = 0 \qquad$ in $\quad Q_0^{(l)}$, l = 1,2, ..., N .

where $\xi = \xi_{j,n}$, $\xi' = \xi'_{j,n}$ and $\eta = \eta_{j,n}$ are some intermediate values in $\mathscr{S}^{(1)}$ and $\mathscr{S}^{(2)}$ respectively and

$$(3.11) \qquad w_{j,n-s}^{(l)} = u_{j,n-s_l}^{(l)} - u_{j,n+k-s_l}^{(l)}.$$

Let $s = min\{s_l \; , \; s_l > 0 \;\; , \; l = 1,2,...,N\} \; > \; 0$ and consider the system (3.10) in the domain $\Lambda_s = \Omega_p \times$ [0 , s] . From (3.10) and (3.11) we have $w_{j,n-s}^{(l)} = 0$ on Λ_s .

This implies that

$$(3.12)\qquad \begin{cases} k_n^{-1}(w_{j,n}^{(l)} - w_{j,n-1}^{(l)}) - L_n^{(l)} w_{j,n}^{(l)} = \sum\limits_{m=1}^{N} b_{j,n}^{lm} w_{j,n}^{(m)} & in\ \Lambda_s\ , \\ B^{(l)}\left[w_{j,n}^{(l)}\right] = \sum\limits_{m=1}^{N} c_{j,n}^{lm} w_{j,n}^{(m)} & on\ S_s\ , \\ w_{j,0}^{(l)} = 0 & in\ \Omega_p\ , \end{cases}$$

where

$$S_s = \partial\Omega_p \times [0, s]\ ,$$

$$b_{j,n}^{lm} = \frac{\partial f_{j,n}^{(l)}}{\partial u^{(m)}}(\xi_{j,n}, \eta_{j,n})\ ,$$

and $c_{j,n}^{lm} = \frac{\partial g_{j,n}^{(l)}}{\partial u^{(m)}}(\xi'_{j,n})$.

From the hypothesis (H_2) it is clear that $b^{lm} \geq 0$ and $c^{lm} \geq 0$ on Λ_s when $m \neq l$.
By Lemma 10.9.1 of [10] we obtain $w_{j,n}^{(l)} = 0$ *on* $\overline{\Lambda}_s = \overline{\Omega}_p \times [0, s]$. This shows that $w_{j,n-s}^{(l)} = 0$ *on* $\overline{\Lambda}_{2s} = \overline{\Omega}_p \times [0, 2s]$ and so $w_{j,n}^{(l)}$ satisfies the equations in (3.12), in the domain Λ_{2s}. It follows again from Lemma 10.9.1 of [10] that $w_{j,n}^{(l)} = 0$ *on* $\overline{\Lambda}_{2s}$.

A continuation of the similar argument shows that $w_{j,n}^{(l)} = 0$ *on* $\overline{\Omega}_p \times [0, Ms]$ for every positive integer M. This proves the periodic property $\mathbf{u}_{j,n} = \mathbf{u}_{j,n+k}$ *on* Q_p.

Since by definition every k-periodic solution $\mathbf{u}_{j,n}^*$ of (2.2) is an upper solution as well as a lower solution, the consideration of $\left(\mathbf{u}_{j,n}^*, \hat{\mathbf{u}}_{j,n}\right)$ and $\left(\tilde{\mathbf{u}}_{j,n}, \mathbf{u}_{j,n}^*\right)$ as the pair of upper and lower solutions in the above argument, leads to the relation $\underline{\mathbf{u}}_{j,n} \leq \mathbf{u}_{j,n}^* \leq \overline{\mathbf{u}}_{j,n}$ *on* Q_p. This ensures the maximal and minimal property of $\overline{\mathbf{u}}_{j,n}$ and $\underline{\mathbf{u}}_{j,n}$. Finally if $\overline{\mathbf{u}}_{j,0} = \underline{\mathbf{u}}_{j,0}$ $(\equiv \mathbf{u}_{j,0})$ then by considering problem (2.2) as an initial boundary value problem with condition $\mathbf{u}_{j,0} = \mathbf{u}_j$, the standard existence-uniqueness theorem for finite difference system of initial boundary value problem of parabolic type ensures that $\overline{\mathbf{u}}_{j,n} = \underline{\mathbf{u}}_{j,n}$ on Q_p. This completes the proof. □

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