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Anti-periodic boundary value problems involving nonlinear fractional q-difference equations

Wengui Yang*

Ministry of Public Education, Sanmenxia Polytechnic, Sanmenxia 472200, China.

Abstract

In this paper, we consider a class of anti-periodic boundary value problems involving nonlinear fractional q-difference equations. Some existence and uniqueness results are obtained by applying some standard fixed point theorems. As applications, some examples are presented to illustrate the main results.

Keywords: Fractional q-difference equations, anti-periodic boundary conditions, existence and uniqueness, fixed point theorem.

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1 Introduction

Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes and have recently received considerable attention. For examples and details of anti-periodic boundary conditions, see [2, 3, 4, 5, 7, 10] and the references therein.

The q-difference calculus or quantum calculus is an old subject that was initially developed by Jackson [17, 18], basic definitions and properties of q-difference calculus can be found in the book mentioned in [19].

The fractional q-difference calculus had its origin in the works by Al-Salam [8] and Agarwal [1]. More recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus were made, for example, q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform, q-Taylor's formula, Mittage-Leffler function [9, 22, 23], just to mention some.

Recently, boundary value problems of nonlinear fractional q-difference equations have aroused considerable attention. Many people pay attention to the existence and multiplicity of solutions or positive solutions for boundary value problems of nonlinear fractional q-difference equations by means of some fixed point theorems, such as the Krasnosel'skii fixed-point theorem, the Leggett-Williams fixed-point theorem, and the Schauder fixed-point theorem, For examples, see [11, 12, 20, 21, 26, 27, 28] and the references therein. Graef and Kong [16] investigated the uniqueness, existence, and nonexistence of positive solutions for the boundary value problem with fractional q-derivatives in terms of different ranges of λ. Ahmad et al. [6] studied the following nonlinear fractional q-difference equation with nonlocal boundary conditions by applying some well-known tools of fixed point theory such as Banach contraction principle, Krasnoselskiis fixed point theorem, and the Leray-Schauder nonlinear alternative. Zhao et al. [29] considered some existence results of positive solutions to nonlocal q-integral boundary value problem of nonlinear fractional q-derivatives equation using the generalized Banach contraction principle, the monotone iterative method, and Krasnoselskii's fixed point theorem.

El-Shahed and Hassan [13] studied the existence of positive solutions of the q-difference boundary value problem

$$\begin{cases} -(D_q^2 u)(t) = a(t)f(u(t)), \quad 0 \le t \le 1, \\ \alpha u(0) - \beta D_q u(0) = 0, \quad \gamma u(1) - \delta D_q u(1) = 0. \end{cases}$$

* *E-mail addresses*: yangwg8088@163.com(Wengui Yang)

Ferreira [14] and [15] considered the existence of positive solutions to nonlinear q-difference boundary value problems

$$\begin{cases} -(D_q^\alpha u)(t) = -f(t, u(t)), \quad 0 \le t \le 1, \quad 1 < \alpha \le 2 \\ u(0) = u(1) = 0, \end{cases}$$

and

$$\begin{cases} (D_q^\alpha u)(t) = -f(t, u(t)), \quad 0 \le t \le 1, \quad 2 < \alpha \le 3, \\ u(0) = (D_q u)(0) = 0, \ \ (D_q u)(1) = \beta \ge 0, \end{cases}$$

respectively. By applying a fixed point theorem in cones, sufficient conditions for the existence of nontrivial solutions were enunciated.

In this paper, we investigate the existence and uniqueness results for anti-periodic boundary value problems involving nonlinear fractional q-difference equations given by

$$\begin{cases} ({}^c D_q^\alpha u)(t) = f(t, u(t)), \quad t \in [0,1], \quad 1 < \alpha \le 2, \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1), \end{cases} \tag{1.1}$$

where ${}^c D_q^\alpha$ denotes the Caputo fractional q-derivative of order α, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. Our results are based on some standard fixed point theorems.

2 Preliminaries

For theconvenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory to facilitate analysis of problem (1.1). These details can be found in the recent literature; see [19] and references therein. Let $q \in (0,1)$ and define

$$[a]_q = \frac{q^a - 1}{q - 1}, \quad a \in \mathbb{R}.$$

The q-analogue of the power $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$(a-b)^{(0)} = 1, \ \ (a-b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k), \ \ n \in \mathbb{N}, \ \ a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \ \ x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \ \text{ and } \ (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \text{ and } (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [19]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \quad {}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)},$$

$$\left({}_xD_q \int_0^x f(x,t)d_qt\right)(x) = \int_0^x {}_xD_q f(x,t)d_qt + f(qx,x).$$

We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$ [14].

Definition 2.1 ([24]). Let $\alpha \ge 0$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0 f(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)d_qt, \quad \alpha > 0, x \in [0,1].$$

Definition 2.2 ([24]). The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by $D_q^0 f(x) = f(x)$ and

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Definition 2.3 ([24]). The fractional q-derivative of the Caputo type of order $\alpha \ge 0$ is defined by

$$({}^cD_q^\alpha f)(x) = (I_q^{m-\alpha} D_q^m f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 2.1 ([14]). *Let $\alpha, \beta \ge 0$ and f be a function defined on $[0,1]$. Then the next formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = I_q^{\alpha+\beta} f(x)$,

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2.2 ([14]). *Let $\alpha > 0$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then the following equality holds:*

$$(I_q^{\alpha c} D_q^\alpha f)(x) = f(x) - \sum_{k=0}^{m-1} \frac{x^k}{\Gamma_q(k+1)} (D_q^k f)(0),$$

where m is the smallest integer greater than or equal to α.

Lemma 2.3. *For any $y \in C[0,1]$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} ({}^cD_q^\alpha u)(t) = y(t), \quad t \in [0,1], \quad 1 < \alpha \le 2, \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1), \end{cases} \tag{2.2}$$

is given by

$$u(t) = \int_0^1 G(t,qs)y(s)d_qs,$$

where

$$G(t,s)=\begin{cases}\dfrac{2(t-s)^{(\alpha-1)}-(1-s)^{(\alpha-1)}}{2\Gamma_q(\alpha)}+\dfrac{(1-2t)(1-s)^{(\alpha-2)}}{4\Gamma_q(\alpha-1)}, & 0\le s\le t\le 1,\\ -\dfrac{(1-s)^{(\alpha-1)}}{2\Gamma_q(\alpha)}+\dfrac{(1-2t)(1-s)^{(\alpha-2)}}{4\Gamma_q(\alpha-1)}, & 0\le t\le s\le 1.\end{cases} \tag{2.3}$$

Proof. We may apply Lemma 2.1 and Lemma 2.2; we see that

$$u(t)=\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}y(s)d_qs+c_1+c_2t. \tag{2.4}$$

Differentiating both sides of (2.4), we obtain

$$(D_qu)(t)=\frac{1}{\Gamma_q(\alpha-1)}\int_0^t (t-qs)^{(\alpha-2)}y(s)d_qs+c_2.$$

Applying the boundary conditions for the problem (2.2), we find that

$$c_1=\frac{1}{2\Gamma(\alpha)}\int_0^1(1-s)^{(\alpha-1)}y(s)d_qs-\frac{1}{4\Gamma(\alpha-1)}\int_0^1(1-s)^{(\alpha-2)}y(s)d_qs,$$
$$c_2=\frac{1}{2\Gamma_q(\alpha-1)}\int_0^1(1-s)^{(\alpha-2)}y(s)d_qs.$$

Thus, the unique solution of (2.2) is

$$\begin{aligned}u(t) &= \int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}y(s)d_qs-\frac{1}{2}\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}y(s)d_qs-\frac{1-2t}{4}\int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}y(s)d_qs\\ &= \int_0^1 G(t,qs)y(s)d_qs,\end{aligned}$$

where $G(t,s)$ is given by (2.3). This completes the proof. □

3 Main results

In this section, we establish some sufficient conditions for the existence and uniqueness of solutions for boundary value problem (1.1).

Let $\mathbb{C}=C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with the norm defined by $\|u\|=\sup\{|u(t)|,t\in[0,1]\}$.

Now we state some known fixed point theorems which are needed to prove the existence of solutions for (1.1).

Theorem 3.1 ([25]). *Let X be a Banach space. Assume that $T:X\to X$ is a completely continuous operator and the set $V=\{u\in X|u=\mu Tu, 0<\mu<1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.2 ([25]). *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta\in\Omega$ and let $T:\overline{\Omega}\to X$ be a completely continuous operator such that*

$$\|Tu\|\le\|u\|,\quad \forall u\in\partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

We define, in relation to (1.1), an operator $T:\mathbb{C}\to\mathbb{C}$ as follows

$$\begin{aligned}(Tu)(t) &= \int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs-\frac{1}{2}\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs\\ &\quad -\frac{1-2t}{4}\int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs,\quad t\in[0,1].\end{aligned} \tag{3.1}$$

From Lemma 2.3, we observe that the problem (3.1) has a solution if and only if the operator T has a fixed point.

Theorem 3.3. *Assume that there exists a positive constant M such that $|f(t,u)| \leq M$ for $t \in [0,1]$ and $u \in \mathbb{C}$. Then the problem (1.1) has at least one solution.*

Proof. We show, as a first step, that the operator T is completely continuous. Clearly, continuity of the operator T follows from the continuity of f. Let $\Omega \in \mathbb{C}$ be bounded. Then, $u \in \Omega$ together with the assumption $|f(t,u)| \leq M$, we get

$$\begin{aligned} |(Tu)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs \\ &\quad + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs \\ &\leq M\left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs\right) \\ &\leq \frac{M(3\Gamma_q(\alpha)+\Gamma_q(\alpha+1))}{2\Gamma_q(\alpha)\Gamma_q(\alpha+1)} = M_2, \end{aligned}$$

which implies that $\|(Tu)(t)\| \leq M_2$. Furthermore,

$$\begin{aligned} |D_q(Tu)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs \\ &\leq M\left(\int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs\right) \\ &\leq \frac{3M}{2\Gamma_q(\alpha)} = M_3, \end{aligned}$$

Hence, for $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| \leq \int_{t_1}^{t_2} |D_q(Tu)(s)|d_qs \leq M_3(t_2 - t_1).$$

This implies that T is equicontinuous on $[0,1]$. Thus, by the Arzela-Ascoli theorem, the operator $T : \mathbb{C} \to \mathbb{C}$ is completely continuous.

Next, we consider the set $V = \{u \in X | u = \mu Tu, 0 < \mu < 1\}$, and show that the set V is bounded. Let $u \in V$; then $u = \mu Tu$, $0 < \mu < 1$. For any $t \in [0,1]$, we have

$$|u(t)| = \mu|(Tu)(t)| \leq |(Tu)(t)| = M_2.$$

Thus, $\|u\| \leq M_2$ for any $t \in [0,1]$. So, the set V is bounded. Thus, by the conclusion of Theorem 3.1, the operator T has at least one fixed point, which implies that (1.1) has at least one solution. The proof is complete. □

Theorem 3.4. *Let $\lim_{u\to 0} f(t,u)/u = 0$. Then the problem (1.1) has at least one solution.*

Proof. Since $\lim_{u\to 0} f(t,u)/u = 0$, there therefore exists a constant $r > 0$ such that $|f(t,u)| \leq \delta|u|$ for $0 < |u| < r$, where $\delta > 0$ is such that $M_2\delta < 1$.

Define $\Omega = \{u \in \mathbb{C} | \|u\| < r\}$ and take $u \in \mathbb{C}$ such that $\|u\| = r$, that is, $u \in \partial\Omega$. As before, it can be shown that T is completely continuous and $|(Tu)(t)| \leq M_2\delta\|u\|$, which, in view of $M_2\delta < 1$, yields $\|Tu\| \leq \|u\|$, $u \in \partial\Omega$. Therefore, by Theorem 3.2, the operator T has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution. □

Theorem 3.5. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function satisfying*

$$|f(t,u) - f(t,v)| \leq L|u - v|, \quad \forall t \in [0,1], u, v \in \mathbb{R}$$

with

$$L \leq \frac{\Gamma_q(\alpha)\Gamma_q(\alpha+1)}{3\Gamma_q(\alpha)+\Gamma_q(\alpha+1)}.$$

Then the problem (1.1) has a unique solution.

Proof. Defining $\sup_{t\in[0,1]}|f(t,0)| = K < \infty$ and selecting

$$r \geq \frac{K(3\Gamma_q(\alpha)+\Gamma_q(\alpha+1))}{\Gamma_q(\alpha)\Gamma_q(\alpha+1)},$$

we show that $TB_r \subset B_r$, where $B_r = \{u \in \mathbb{C} : \|u\| \leq r\}$. For $u \in B_r$, we have

$$\begin{aligned}
&|(Tu)(t)| \\
\leq\ & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs \\
& + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs \\
\leq\ & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}(|f(s,u(s))-f(s,0)|+|f(s,0)|)d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}(|f(s,u(s)) \\
& -f(s,0)|+|f(s,0)|)d_qs + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}(|f(s,u(s))-f(s,0)|+|f(s,0)|)d_qs \\
\leq\ & (Lr+K)\left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs\right) \\
\leq\ & (Lr+K)\frac{3\Gamma_q(\alpha)+\Gamma_q(\alpha+1)}{2\Gamma_q(\alpha)\Gamma_q(\alpha+1)} \leq r.
\end{aligned}$$

Taking the maximum over the interval $[0,1]$, we get $\|(Tu)(t)\| \leq r$. Now, for $u, v \in \mathbb{C}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
&\|(Tu)(t)-(Tv)(t)\| \\
\leq\ & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))-f(s,v(s))|d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))-f(s,v(s))|d_qs \\
& + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))-f(s,v(s))|d_qs \\
\leq\ & L\|u-v\|\left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs + \frac{|1-2t|}{4}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs\right) \\
\leq\ & \frac{L(3\Gamma_q(\alpha)+\Gamma_q(\alpha+1))}{2\Gamma_q(\alpha)\Gamma_q(\alpha+1)}\|u-v\| = \Lambda_{L,\alpha}\|u-v\|,
\end{aligned}$$

where

$$\Lambda_{L,\alpha} = \frac{L(3\Gamma_q(\alpha)+\Gamma_q(\alpha+1))}{2\Gamma_q(\alpha)\Gamma_q(\alpha+1)},$$

which depends only on the parameters involved in the problem. As $\Lambda_{L,\alpha} < 1$, T is therefore a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (the Banach fixed point theorem). □

4 Some examples

Example 4.1. Consider the anti-periodic fractional q-difference boundary value problem

$$\begin{cases} ({}^cD_q^\alpha u)(t) = \dfrac{e^{-\cos^2 u(t)}[5+\cos 2t + 4\ln(5+2\sin^2 u(t))]}{2+\sin^2 u(t)}, \quad t\in[0,1], \quad 1<\alpha\leq 2, \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1). \end{cases} \tag{4.1}$$

Clearly, $M = 3 + 2\ln 7$, and the hypothesis of Theorem 3.3 holds. Therefore, the conclusion of Theorem 3.3 implies that the problem (4.1) has at least one solution.

Example 4.2. Consider the anti-periodic fractional q-difference boundary value problem

$$\begin{cases} ({}^cD_q^\alpha u)(t) = (16 + u^3(t))^{\frac{1}{2}} + 2(t^2+1)(\tan u(t) - u(t)) - 4, \quad t \in [0,1], \quad 1 < \alpha \leq 2, \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1). \end{cases} \tag{4.2}$$

It can easily be verified that all the assumptions of Theorem 3.4 holds. Consequently, the conclusion of Theorem 3.4 implies that the problem (4.2) has at least one solution.

Example 4.3. Consider the anti-periodic fractional q-difference boundary value problem

$$\begin{cases} ({}^cD_q^\alpha u)(t) = \dfrac{e^{-\pi t}|u(t)|}{(5 + e^{-\pi t})(1 + |u(t)|)}, \quad t \in [0,1], \\ u(0) = -u(1), \quad (Du)(0) = -(Du)(1), \end{cases} \tag{4.3}$$

where $\alpha = 1.5$ and $q = 0.5$. Let

$$f(t,u) = \frac{e^{-\pi t}|u|}{(5 + e^{-\pi t})(1 + |u|)}.$$

Clearly, $L = 1/5$ as $|f(t,u) - f(t,v)| \leq 1/5|u - v|$. Further,

$$\frac{L(3\Gamma_q(\alpha) + \Gamma_q(\alpha+1))}{\Gamma_q(\alpha)\Gamma_q(\alpha+1)} = \frac{3\Gamma_{0.5}(1.5) + \Gamma_{0.5}(2.5)}{5\Gamma_{0.5}(1.5)\Gamma_{0.5}(2.5)} \approx 0.721135 < 1.$$

Thus, all the assumptions of Theorem 3.5 are satisfied. Therefore, the conclusion of Theorem 3.5 implies that the problem (4.3) has a unique solution.

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