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Hypersphere and the fourth Laplace-Beltrami operator in 4-space

ERHAN GÜLER[*1]

[1] *Faculty of Sciences, Department of Mathematics, Bartın University, 74100 Bartın, Turkey.*

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Abstract. We consider hypersphere $\mathbf{x}(u, v, w)$ in the four dimensional Euclidean space $\mathbb{E}^4$. We compute the fourth Laplace-Beltrami operator of the hypersphere satisfying $\Delta^{IV}\mathbf{x} = \mathcal{A}\mathbf{x}$, where $\mathcal{A} \in Mat\,(4, 4)$.

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1. Introduction

In differential geometry, hyper-surfaces theory have been worked by many mathematicians for a long time. For example, Obata worked [41] certain conditions for a Riemannian manifold to be isometric with a sphere; Takahashi [44] proved that a connected Euclidean submanifold is of 1-type, iff it is either minimal in $\mathbb{E}^m$ or minimal in some hypersphere of $\mathbb{E}^m$; Chern, do Carmo, and Kobayashi [15] gave minimal submanifolds of a sphere with second fundamental form of constant length; Cheng and Yau considered hypersurfaces with constant scalar curvature; Lawson [37] gave minimal submanifolds in his book.

Chen [9–12] studied submanifolds of finite type whose immersion into $\mathbb{E}^m$ (or $\mathbb{E}^m_\nu$) by using a finite number of eigenfunctions of their Laplacian. Some results of 2-type spherical closed submanifolds were given by [6, 7, 10]; Garay researched [25] an extension of Takahashi's theorem in $\mathbb{E}^m$. Chen and Piccinni [13] focused submanifolds with finite type Gauss map in $\mathbb{E}^m$. Dursun [20] considered hypersurfaces with pointwise 1-type Gauss map in $\mathbb{E}^{n+1}$.

[*] **Corresponding author.** Email address: **eguler@bartin.edu.tr** (Erhan GÜLER)
ORCID: https://orcid.org/0000-0003-3264-6239

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In $\mathbb{E}^3$; Takahashi [44] proved that minimal surfaces and spheres are the only surfaces satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Ferrandez, Garay, and Lucas [22] gave that the surfaces satisfying $\Delta H = AH$, $A \in Mat(3,3)$ are either minimal, or an open piece of sphere or of a right circular cylinder; Choi and Kim [17] studied the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind; Garay [24] classified a certain class of finite type surfaces of revolution; Dillen, Pas and Verstraelen [18] focused that the only surfaces satisfying $\Delta r = Ar + B$, $A \in Mat(3,3)$, $B \in Mat(3,1)$ are the minimal surfaces, the spheres and the circular cylinders; Stamatakis and Zoubi [43] obtained surfaces of revolution satisfying $\Delta^{III} x = Ax$; Senoussi and Bekkar [42] introduced helicoidal surfaces M^2 which are of finite type with respect to the fundamental forms I, II and III, i.e., their position vector field $r(u,v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A \in Mat(3,3)$; Kim, Kim and Kim [34] gave Cheng-Yau operator and Gauss map of surfaces of revolution.

In $\mathbb{E}^4$; Moore [39, 40] worked general rotational surfaces; Hasanis and Vlachos [31] considered hypersurfaces with harmonic mean curvature vector field; Cheng and Wan [14] gave complete hypersurfaces with CMC; Kim and Turgay [35] introduced surfaces with L_1-pointwise 1-type Gauss map; Arslan et al [2] worked Vranceanu surface with pointwise 1-type Gauss map; Arslan et al [3] studied generalized rotational surfaces; Aksoyak and Yaylı [32] worked flat rotational surfaces with pointwise 1-type Gauss map; Güler, Magid and Yaylı [29] introduced helicoidal hypersurfaces; Güler, Hacısalihoğlu, and Kim [28] studied Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface; Güler and Turgay [30] focused Cheng-Yau operator and Gauss map of rotational hypersurfaces; Güler [27] found rotational hypersurfaces satisfying $\Delta^I R = AR$, where $A \in Mat(4,4)$. He [26] also studied fundamental form *IV* and curvature formulas of the hypersphere.

In Minkowski 4-space $\mathbb{E}^4_1$; Ganchev and Milousheva [23] indicated analogue of surfaces of [39, 40]; Arvanitoyeorgos, Kaimakamais, and Magid [5] studied that if the mean curvature vector field of M^3_1 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M^3_1 has CMC; Arslan and Milousheva considered meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map; Turgay introduced some classifications of Lorentzian surfaces with finite type Gauss map; Dursun and Turgay gave space-like surfaces in with pointwise 1-type Gauss map. Aksoyak and Yaylı [33] obtained general rotational surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4_2$. Bektaş, Canfes, and Dursun [8] worked surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map in $\mathbb{E}^5_2$.

We consider hypersphere in the four dimensional Euclidean space $\mathbb{E}^4$. In Section 2, we give some notions of four space. We give curvature formulas of any hypersurface in Section 3. Finally, we define hypersphere in Section 4. We compute hypersphere satisfying $\Delta^{IV}\mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix $\mathcal{A}$ in the last section.

2. Preliminaries

In this section, we give some of basic facts and definitions, then describe notations used in this paper. Let $\mathbb{E}^m$ denote the Euclidean m-space with the canonical Euclidean metric tensor given by $\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$. Consider an m-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote the Levi-Civita connections of $\mathbb{E}^m$ and M by $\widetilde{\nabla}$ and ∇, respectively. We shall use letters X, Y, Z, W (resp., ξ, η) to denote vectors fields tangent (resp., normal) to M. The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$
$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \tag{2.2}$$

where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

For each $\xi \in T_p^{\perp} M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M$ at

$p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X,Y,)Z,W \rangle = \langle h(Y,Z), h(X,W) \rangle - \langle h(X,Z), h(Y,W) \rangle, \tag{2.3}$$
$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.4}$$

where $R,\ R^D$ are the curvature tensors associated with connections ∇ and D, respectively, and $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Hypersurfaces of Euclidean space

Now, let M be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $\mathbf{S}$ its shape operator (i.e. Weingarten map) and x its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of consisting of principal directions of M corresponding from the principal curvature k_i for $i = 1, 2, \ldots n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Then the first structural equation of Cartan is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i,j = 1,2,\ldots,n, \tag{2.5}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. We denote the Levi-Civita connection of M and $\mathbb{E}^{n+1}$ by ∇ and $\widetilde{\nabla}$, respectively. Then, from the Codazzi equation (2.3), we have

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \tag{2.6}$$
$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \tag{2.7}$$

for distinct $i, j, l = 1, 2, \ldots, n$.

We put $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$

We use following notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By the definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. We call the function s_k as the k-th mean curvature of M. We would like to note that functions $H = \frac{1}{n} s_1$ and $K = s_n$ are called the mean curvature and Gauss-Kronecker curvature of M, respectively. In particular, M is said to be j-minimal if $s_j \equiv 0$ on M.

In $\mathbb{E}^{n+1}$, to find the i-th curvature formulas $\mathfrak{C}_i$ (Curvature formulas sometimes are represented as mean curvature H_i, and sometimes as Gaussian curvature K_i by different writers, such as [1] and [36]. We will call it just i-th curvature $\mathfrak{C}_i$ in this paper.), where $i = 0, ..., n$, firstly, we use the characteristic polynomial of $\mathbf{S}$:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \tag{2.8}$$

where $i = 0, ..., n$, I_n denotes the identity matrix of order n. Then, we get curvature formulas $\binom{n}{i} \mathfrak{C}_i = s_i$. That is, $\binom{n}{0} \mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1} \mathfrak{C}_1 = s_1, \ldots, \binom{n}{n} \mathfrak{C}_n = s_n = K$.

k-th fundamental form of M is defined by $I\left(\mathbf{S}^{k-1}(X), Y\right) = \left\langle \mathbf{S}^{k-1}(X), Y\right\rangle$. So, we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i I\left(\mathbf{S}^{n-i}(X), Y\right) = 0. \tag{2.9}$$

In particular, one can get classical result $\mathfrak{C}_0 III - 2\mathfrak{C}_1 II + \mathfrak{C}_2 I = 0$ of surface theory for $n = 2$. See [36] for details.

For a Euclidean submanifold $x\colon M \longrightarrow \mathbb{E}^m$, the immersion (M, x) is called *finite type,* if x can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of (M, x), i.e. $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \ldots, x_k$ non-constant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. If λ_i are different, M is called *k-type*. See [10] for details.

2.2. Rotational hypersurfaces

We obtain a rotational hypersurface (rot-hypface for short) in Euclidean 4-space. Before we proceed, we would like to note that the definition of rot-hypfaces in Riemannian space forms were defined in [19]. A rot-hypface $M \subset \mathbb{E}^{n+1}$ generated by a curve $\mathcal{C}$ around an axis $\mathcal{C}$ that does not meet $\mathcal{C}$ is obtained by taking the orbit of $\mathcal{C}$ under those orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed (See [19, Remark 2.3]).

Throughout the paper, we identify a vector (a, b, c, d) with its transpose. Consider the case $n = 3$, and let $\mathcal{C}$ be the curve parametrized by

$$\gamma(w) = (\xi(w), 0, 0, \varphi(w)), \tag{2.10}$$

where ξ, φ are differentiable functions. If $\mathfrak{r}$ is the x_4-axis, then an orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed has the form

$$\mathbf{Z}(v, w) = \begin{pmatrix} \cos u \cos v & -\sin u & -\cos u \sin v & 0 \\ \sin u \cos v & \cos u & -\sin u \sin v & 0 \\ \sin v & 0 & \cos v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u, v \in \mathbb{R}.$$

Therefore, the parametrization of the rot-hypface generated by a curve $\mathcal{C}$ around an axis $\mathfrak{r}$ is given by $\mathbf{x}(u, v, w) = \mathbf{Z}(u, v)\gamma(w)$.

Definition 2.1. *Let $\mathbf{x} = \mathbf{x}(u, v, w)$ be an immersion from $M^3 \subset \mathbb{E}^3$ to $\mathbb{E}^4$. In 4-space, inner product is given by*

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4,$$

and triple vector product is defined by

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix},$$

where $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$.

Definition 2.2. *Definition For a hypface $\mathbf{x}$ in 4-space, we have*

$$(g_{ij})_{3\times 3}, \ (h_{ij})_{3\times 3}, \ (t_{ij})_{3\times 3}, \tag{2.11}$$

where (g_{ij}), (h_{ij}), and (t_{ij}) are the first, second, and the third fundamental form matrices (or I,II, and III), respectively, where $g_{11} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$, $g_{12} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle$, $g_{13} = \langle \mathbf{x}_u, \mathbf{x}_w \rangle$, $g_{22} = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$, $g_{23} = \langle \mathbf{x}_v, \mathbf{x}_w \rangle$, $g_{33} =$

$\langle \mathbf{x}_w, \mathbf{x}_w \rangle$, $h_{11} = \langle \mathbf{x}_{uu}, e \rangle$, $h_{12} = \langle \mathbf{x}_{uv}, e \rangle$, $h_{13} = \langle \mathbf{x}_{uw}, e \rangle$, $h_{22} = \langle \mathbf{x}_{vv}, e \rangle$, $h_{23} = \langle \mathbf{x}_{vw}, e \rangle$, $h_{33} = \langle \mathbf{x}_{ww}, e \rangle$, $e_{11} = \langle e_u, e_u \rangle$, $e_{12} = \langle e_u, e_v \rangle$, $e_{13} = \langle e_u, e_w \rangle$, $e_{22} = \langle e_v, e_v \rangle$, $e_{23} = \langle e_v, e_w \rangle$, $e_{33} = \langle e_w, e_w \rangle$. *Here,*

$$e = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\|} \tag{2.12}$$

is unit normal (i.e. the Gauss map) of hypface $\mathbf{x}$*.*

Product matrices $(g_{ij})^{-1} \cdot (h_{ij})$ gives the matrix of the shape operator $\mathbf{S}$ of hypface $\mathbf{x}$ in 4-space. See [28–30] for details.

3. i-th Curvatures

To compute the i-th mean curvature formula $\mathfrak{C}_i$, where $i = 0, .., 3$, we use characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$:

$$P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda I_3) = 0.$$

Then, obtain $\mathfrak{C}_0 = 1$ (by definition), $\binom{3}{1}\mathfrak{C}_1 = \binom{3}{1}H = -\frac{b}{a}$, $\binom{3}{2}\mathfrak{C}_2 = \frac{c}{a}$, $\binom{3}{3}\mathfrak{C}_3 = K = -\frac{d}{a}$.

Therefore, we find i-th curvature folmulas depends on the coefficients of the fundamental forms (g_{ij}) and (h_{ij}) in 4-space. See [26] for details.

Theorem 3.1. *Any hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^4_2$ *has following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{\left\{\begin{array}{c} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})g_{33} + (g_{11}g_{22} - g_{12}^2)h_{33} \\ -2(g_{13}h_{13}g_{22} - g_{23}h_{13}g_{12} - g_{13}h_{23}g_{12} \\ +g_{11}g_{23}h_{23} - g_{13}g_{23}h_{12}) - g_{23}^2h_{11} - g_{13}^2h_{22} \end{array}\right\}}{3\left[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2\right]}, \tag{3.1}$$

$$\mathfrak{C}_2 = \frac{\left\{\begin{array}{c} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})\,h_{33} + \left(h_{11}h_{22} - g_{12}^2\right)g_{33} \\ -2(g_{13}h_{13}h_{22} - g_{23}h_{13}h_{12} - g_{13}h_{23}h_{12} \\ +g_{23}h_{23}h_{11} - h_{13}h_{23}g_{12}) - g_{11}h_{23}^2 - g_{22}h_{13}^2 \end{array}\right\}}{3\left[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2\right]}, \tag{3.2}$$

$$\mathfrak{C}_3 = \frac{\left(h_{11}h_{22} - h_{12}^2\right)h_{33} - h_{11}h_{23}^2 + 2h_{12}h_{13}h_{23} - h_{22}h_{13}^2}{(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2}. \tag{3.3}$$

Proof. See [26] for details. ■

A hypersurface $\mathbf{x}$ in $\mathbb{E}^4$ is i-minimal, when $\mathfrak{C}_i = 0$ identically on $\mathbf{x}$.

4. Hypersphere

In this section, we define hypersphere, then find its differential geometric properties in $\mathbb{E}^4$.

For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, and let ℓ be a straight line in Π.

Definition. A rotational hypersurface in $\mathbb{E}^4$ is called hypersphere, when a profile curve

$$\gamma(w) = (r\cos w, 0, 0, r\sin w)$$

rotates around a axis $\ell = (0, 0, 0, 1)$ for hyperradius $r > 0$.

So, in 4-space, the hypersphere which is spanned by the vector ℓ, is as follows

$$\mathbf{x}(u, v, w) = Z(u, v)\gamma(w). \tag{4.1}$$

Therefore, more clear form of (4.1) is as follows

$$\mathbf{x}(u,v,w)=\begin{pmatrix} r\cos u\cos v\cos w\\ r\sin u\cos v\cos w\\ r\sin v\cos w\\ r\sin w\end{pmatrix}, \tag{4.2}$$

where $r>0$ and $0\leq u,v,w\leq 2\pi$. When $w=0$, we have a sphere in $\mathbb{E}^4$.

Next, we will obtain the Gauss map and the curvatures of the hypersphere (4.2). The first quantities of (4.2) are as follows

$$(g_{ij})=\text{diag}\left(r^2\cos^2 v\cos^2 w,\ r^2\cos^2 w,\ r^2\right). \tag{4.3}$$

We have $\det(g_{ij})=r^6\cos^2 v\cos^4 w$. Using (2.12), we get the Gauss map of the hypersphere (4.2) as follows

$$e=\begin{pmatrix} \cos u\cos v\cos w\\ \sin u\cos v\cos w\\ \sin v\cos w\\ \sin w\end{pmatrix}. \tag{4.4}$$

Using the second differentials of (4.2) with respect to u,v,w, and the Gauss map (4.4) of the hypersphere (4.2), we have the second quantities as follows

$$(h_{ij})=\text{diag}\left(-r\cos^2 v\cos^2 w,\ -r\cos^2 w,\ -r\right). \tag{4.5}$$

So, we get $\det(h_{ij})=-r^3\cos^2 v\cos^4 w$. Using $(g_{ij})^{-1}\cdot(h_{ij})$, we calculate the shape operator matrix of the hypersphere (4.2): $\mathbf{S}=-\frac{1}{r}I_3$. Differentiating (4.4) with respect to u,v,w, we find the third quantities as follows

$$(t_{ij})=\text{diag}\left(\cos^2 v\cos^2 w,\ \cos^2 w,\ 1\right). \tag{4.6}$$

Here, $\det(t_{ij})=\cos^2 v\cos^4 w$. Computing (3.1), (3.2) and (3.3), with (4.3), (4.5), respectively, we find the curvatures of the hypersphere (4.2) as follows ($\mathfrak{C}_0=1$ by definition)

$$\mathfrak{C}_1=-\frac{1}{r},\ \mathfrak{C}_2=\frac{1}{r^2},\ \mathfrak{C}_3=-\frac{1}{r^3}.$$

Using $(f_{ij})=(t_{ij})\cdot\mathbf{S}=(h_{ij})\cdot\mathbf{S^2}=(g_{ij})\cdot\mathbf{S}^3$, we obtain the fourth fundamental form matrix $(f_{ij})_{3\times 3}$ of hypersphere (4.2) as follows

$$(f_{ij})=\text{diag}\left(-\tfrac{1}{r}\cos^2 v\cos^2 w,\ -\tfrac{1}{r}\cos^2 w,\ -\tfrac{1}{r}\right). \tag{4.7}$$

See [26] for details.

5. Hypersphere Satisfying $\Delta^{IV}\mathbf{x}=\mathcal{A}\mathbf{x}$

In this section, we give the fourth Laplace-Beltrami operator of a smooth function, then calculate it using hypersphere.

The inverse of the fourth fundamental form matrix $IV=(f_{ij})$ of any hypersurface is as follows

$$\frac{1}{f}\begin{pmatrix} f_{22}f_{33}-f_{23}f_{32} & -(f_{12}f_{33}-f_{13}f_{32}) & f_{12}f_{23}-f_{13}f_{22}\\ -(f_{21}f_{33}-f_{31}f_{23}) & f_{11}f_{33}-f_{13}f_{31} & -(f_{11}f_{23}-f_{21}f_{13})\\ f_{21}f_{32}-f_{22}f_{31} & -(f_{11}f_{32}-f_{12}f_{31}) & f_{11}f_{22}-f_{12}f_{21}\end{pmatrix},$$

where

$$\begin{aligned} f&=\det(f_{ij})\\ &=f_{11}f_{22}f_{33}-f_{11}f_{23}f_{32}+f_{12}f_{31}f_{23}-f_{12}f_{21}f_{33}+f_{21}f_{13}f_{32}-f_{13}f_{22}f_{31}.\end{aligned}$$

Definition 5.1. *The fourth Laplace-Beltrami operator of a smooth function* $\phi = \phi(x^1, x^2, x^3) \mid_{\mathbf{D}} (\mathbf{D} \subset \mathbb{R}^3)$ *of class* C^3 *with respect to the fourth fundamental form of a hypersurface* $\mathbf{M}$ *is the operator* Δ^{IV} *which is defined by as follows*

$$\Delta^{IV}\phi = \frac{1}{|f|^{1/2}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i}\left(|f|^{1/2} f^{ij} \frac{\partial \phi}{\partial x^j}\right). \tag{5.1}$$

where $\left(f^{ij}\right) = (f_{kl})^{-1}$ *and* $f = \det(f_{ij})$.

Clearly, we can write (5.1) as follows

$$\frac{1}{|f|^{1/2}} \left\{ \begin{array}{c} \frac{\partial}{\partial x^1}\left(|f|^{1/2} f^{11} \frac{\partial \phi}{\partial x^1}\right) - \frac{\partial}{\partial x^1}\left(|f|^{1/2} t^{12} \frac{\partial \phi}{\partial x^2}\right) + \frac{\partial}{\partial x^1}\left(|f|^{1/2} t^{13} \frac{\partial \phi}{\partial x^3}\right) \\ -\frac{\partial}{\partial x^2}\left(|f|^{1/2} f^{21} \frac{\partial \phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(|f|^{1/2} t^{22} \frac{\partial \phi}{\partial x^2}\right) - \frac{\partial}{\partial x^2}\left(|f|^{1/2} t^{23} \frac{\partial \phi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^3}\left(|f|^{1/2} f^{31} \frac{\partial \phi}{\partial x^1}\right) - \frac{\partial}{\partial x^3}\left(|f|^{1/2} t^{32} \frac{\partial \phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(|f|^{1/2} t^{33} \frac{\partial \phi}{\partial x^3}\right) \end{array} \right\}. \tag{5.2}$$

Hence, using a smooth function $\phi = \phi(u, v, w)$**,** we re-write (5.2) as follows

$$\frac{1}{|f|^{1/2}} \left\{ \begin{array}{c} \frac{\partial}{\partial x^1}\left(\frac{(f_{22}f_{33}-f_{23}f_{32})\phi_u-(f_{13}f_{32}-f_{12}f_{33})\phi_v+(f_{12}f_{23}-f_{13}f_{22})\phi_w}{|f|^{1/2}}\right) \\ -\frac{\partial}{\partial x^2}\left(\frac{(f_{12}f_{33}-f_{13}f_{32})\phi_u-(f_{11}f_{33}-f_{13}f_{31})\phi_v+(f_{21}f_{13}-f_{11}f_{23})\phi_w}{|f|^{1/2}}\right) \\ +\frac{\partial}{\partial x^3}\left(\frac{(f_{12}f_{23}-f_{13}f_{22})\phi_u-(f_{21}f_{13}-f_{11}f_{23})\phi_v+(f_{11}f_{22}-f_{12}f_{21})\phi_w}{|f|^{1/2}}\right) \end{array} \right\}. \tag{5.3}$$

Therefore, the fourth Laplace-Beltrami operator of the hypersphere (4.2) is given by

$$\Delta^{IV}\mathbf{x} = \frac{1}{|f|^{1/2}} \left\{ \frac{\partial}{\partial u}\left(\frac{f_{22}f_{33}\mathbf{x}_u}{|f|^{1/2}}\right) + \frac{\partial}{\partial v}\left(\frac{f_{11}f_{33}\mathbf{x}_v}{|f|^{1/2}}\right) + \frac{\partial}{\partial w}\left(\frac{f_{11}f_{22}\mathbf{x}_w}{|f|^{1/2}}\right) \right\}, \tag{5.4}$$

Getting more clear form of the fourth Laplace-Beltrami operator $\Delta^{IV}\mathbf{x}$ of the hypersphere (4.2)**,** we use (4.7) and (5.4). Differentiating $\frac{f_{22}f_{33}}{|f|^{1/2}}\mathbf{x}_u$, $\frac{f_{11}f_{33}}{|f|^{1/2}}\mathbf{x}_v$, $\frac{f_{11}f_{22}}{|f|^{1/2}}\mathbf{x}_w$, with respect to u, v, w, respectively, and substituting them into (5.4), we get following relations between the fourth Laplace-Beltrami operator, Gauss map, and the curvatures of the hypersphere (4.2).

Corollary 5.2. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.2). *Then* $\mathbf{x}$ *has*

$$\Delta^{IV}\mathbf{x} = -3r^2 e,$$

where e *is the Gauss map of the hypersphere* $\mathbf{x}$.

Corollary 5.3. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.2). *Then* $\mathbf{x}$ *has* $\Delta^{IV}\mathbf{x} = \mathcal{A}\mathbf{x}$, *where*

$$\mathcal{A} = -3r\mathfrak{C}_0 I_4 = 3r^2\mathfrak{C}_1 I_4 = -3r^3\mathfrak{C}_2 I_4 = 3r^4\mathfrak{C}_3 I_4,$$

$\mathcal{A} \in Mat(4, 4)$, *and* I_4 *is identity matrix.*

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References

[1] ALIAS, L.J., GÜRBÜZ, N.: An extension of Takashi theorem for the linearized operators of the highest order mean curvatures, *Geom. Dedicata*, **121**(2006), 113–127.

[2] ARSLAN, K., BAYRAM, B.K., BULCA, B., KIM, Y.H., MURATHAN, C., ÖZTÜRK, G.: Vranceanu surface in $\mathbb{E}^4$ with pointwise 1-type Gauss map, *Indian J. Pure Appl. Math.,* **42**(1)(2011), 41–51.

[3] ARSLAN, K., BAYRAM, B.K., BULCA, B., ÖZTÜRK, G.: Generalized rotation surfaces in $\mathbb{E}^4$, *Results in Math.,* **61**(3)(2012), 315–327.

[4] ARSLAN, K., MILOUSHEVA, V.: Meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map in Minkowski 4-space, *Taiwanese J. Math.,* **20**(2)(2016), 311–332.

[5] Arvanitoyeorgos, A., KAIMAKAMIS, G., MAGID, M.: Lorentz hypersurfaces in $\mathbb{E}_1^4$ satisfying $\Delta H = \alpha H$, *Illinois J. Math.,* **53**(2)(2009), 581–590.

[6] BARROS, M., CHEN, B.Y.: Stationary 2-type surfaces in a hypersphere, *J. Math. Soc. Japan*, **39**(4)(1987), 627–648.

[7] BARROS, M., Garay, O.J.: 2-type surfaces in S^3, *Geom. Dedicata*, **24**(3)(1987), 329–336.

[8] BEKTAŞ, B.; CANFES, E.Ö; DURSUN, U.: Classification of surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map, *Math. Nachr.,* **290**(16)(2017), 2512–2523.

[9] CHEN, B.Y.: On submanifolds of finite type, *Soochow J. Math.,* **9**(1983), 65–81.

[10] CHEN, B.Y.: *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, Singapore, 1984.

[11] CHEN, B.Y.: *Finite Type Submanifolds and Generalizations*, University of Rome, 1985.

[12] CHEN, B.Y.: Finite type submanifolds in pseudo-Euclidean spaces and applications, *Kodai Math. J.,* **8**(3)(1985), 358–374.

[13] CHEN, B.Y., PICCINNI, P.: Submanifolds with finite type Gauss map, *Bull. Austral. Math. Soc.,* **35**(1987), 161–186.

[14] CHENG, Q.M., WAN, Q.R.: Complete hypersurfaces of $\mathbb{R}^4$ with constant mean curvature, *Monatsh. Math.,* **118**(1994), 171–204.

[15] CHERN, S.S., DO CARMO, M., KOBAYASHI, S.: *Minimal Submanifolds of a Sphere with Second Fundamental Form of Constant Length*, Functional Analysis and Related Fields, Springer, Berlin, 1970.

[16] CHENG, S.Y., YAU, S.T.: Hypersurfaces with constant scalar curvature, *Math. Ann.,* **225**(1977), 195–204.

[17] CHOI, M., KIM, Y.H.: Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map, *Bull. Korean Math. Soc.,* **38**(2001), 753–761.

[18] DILLEN, F., PAS, J., VERSTRAELEN, L.: On surfaces of finite type in Euclidean 3-space, *Kodai Math. J.,* **13**(1990), 10–21.

[19] DO CARMO, M., DAJCZER, M.: Rotation Hypersurfaces in Spaces of Constant Curvature, *Trans. Amer. Math. Soc.,* **277**(1983), 685–709.

[20] DURSUN, U.: Hypersurfaces with pointwise 1-type Gauss map, *Taiwanese J. Math.,* **11**(5)(2007), 1407–1416.

[21] DURSUN, U., TURGAY, N.C.: Space-like surfaces in Minkowski space $\mathbb{E}_1^4$ with pointwise 1-type Gauss map, *Ukrainian Math. J.,* **71**(1)(2019), 64–80.

[22] FERRANDEZ, A., GARAY, O.J., LUCAS, P.: On a certain class of conformally at Euclidean hypersurfaces. In Global Analysis and Global Differential Geometry; Springer: Berlin, Germany 48–54, 1990.

[23] GANCHEV, G., MILOUSHEVA, V.: General rotational surfaces in the 4-dimensional Minkowski space, *Turkish J. Math.,* **38**(2014), 883–895.

[24] GARAY, O.J.: On a certain class of finite type surfaces of revolution, *Kodai Math. J.,* **11**(1988), 25–31.

[25] GARAY, O.J.: An extension of Takahashi's theorem, *Geom. Dedicata*, **34**(1990), 105–112.

[26] GÜLER, E.: Fundamental form *IV* and curvature formulas of the hypersphere, *Malaya J. Mat.,* **8**(4)(2020), 2008–2011.

[27] GÜLER, E.: Rotational hypersurfaces satisfying $\Delta^I R = AR$ in the four-dimensional Euclidean space, *J. Polytech.,* **24**(2)(2021), 517-520.

[28] GÜLER, E., HACISALIHOĞLU, H.H., KIM, Y.H.: The Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface in 4-space, *Symmetry*, **10**(9)(2018), 1–12.

[29] GÜLER, E., MAGID, M., YAYLI, Y.: Laplace-Beltrami operator of a helicoidal hypersurface in four-space, *J. Geom. Symmetry Phys.,* **41**(2016), 77–95.

[30] GÜLER, E., TURGAY, N.C.: Cheng-Yau operator and Gauss map of rotational hypersurfaces in 4-space, *Mediterr. J. Math.,* **16**(3)(2019), 1–16.

[31] HASANIS, TH., VLACHOS, TH.: Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field, *Math. Nachr.,* **172**(1995), 145–169.

[32] KAHRAMAN AKSOYAK, F., YAYLI, Y.: Flat rotational surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4$, *Honam Math. J.,* **38**(2)(2016), 305–316.

[33] KAHRAMAN AKSOYAK, F., YAYLI, Y.: General rotational surfaces with pointwise 1-type Gauss map in pseudo-Euclidean space $\mathbb{E}_2^4$, *Indian J. Pure Appl. Math.,* **46**(1)(2015), 107–118.

[34] KIM, D.S., KIM, J.R., KIM, Y.H.: Cheng-Yau operator and Gauss map of surfaces of revolution, *Bull. Malays. Math. Sci. Soc.,* **39**(4)(2016), 1319–1327.

[35] KIM, Y.H., TURGAY, N.C.: Surfaces in $\mathbb{E}^4$ with L_1-pointwise 1-type Gauss map, *Bull. Korean Math. Soc.,* **50**(3)(2013), 935–949.

[36] KÜHNEL, W.: *Differential Geometry, Curves-surfaces-manifolds*, Third ed. Translated from the 2013 German ed. AMS, Providence, RI, 2015.

[37] LAWSON, H.B.: *Lectures on Minimal Submanifolds*, 2nd ed. Mathematics Lecture Series 9; Publish or Perish, Inc., Wilmington, DE, USA, 1980.

[38] LEVI-CIVITA, T.: Famiglie di superficie isoparametriche nellordinario spacio euclideo, *Rend. Acad. Lincei.,* **26**(1937), 355–362.

[39] MOORE, C.: Surfaces of rotation in a space of four dimensions, *Ann. Math.,* **21**(1919), 81–93.

[40] MOORE, C.: Rotation surfaces of constant curvature in space of four dimensions, *Bull. Amer. Math. Soc.,* **26**(1920), 454–460.

[41] OBATA, M.: Certain conditions for a Riemannian manifold to be isometric with a sphere, *J. Math. Soc. Jpn.,* **14**(1962), 333–340.

[42] SENOUSSI, B., BEKKAR, M.: Helicoidal surfaces with $\Delta^J r = Ar$ in 3-dimensional Euclidean space, *Stud. Univ. Babeş-Bolyai Math.,* **60**(3)(2015), 437–448.

[43] STAMATAKIS, S., ZOUBI, H.: Surfaces of revolution satisfying $\Delta^{III} x = Ax$, *J. Geom. Graph,* **14**(2)(2010), 181–186.

[44] TAKAHASHI, T.: Minimal immersions of Riemannian manifolds, *J. Math. Soc. Jpn.,* **18**(1966), 380–385.

[45] TURGAY, N.C.: Some classifications of Lorentzian surfaces with finite type Gauss map in the Minkowski 4-space, *J. Aust. Math. Soc.,* **99**(3)(2015), 415–427.

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