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Oscillatory properties of third-order quasilinear difference equations

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Abstract

Some new oscillation criteria are obtained for the third-order quasilinear difference equation $\Delta^2\left(p_n\left(\Delta x_n\right)^\alpha\right)-q_n\left(\Delta x_n\right)^\alpha+r_nf(x_n)=0$, $n=0,1,2,...,$ where $\alpha>0$ is the ratio of odd positive integers. The method uses techniques based on Schwarz's inequality. Example is inserted to illustrate the result.

Keywords: Oscillation, third order, quasilinear, difference equation, Schwarz's inequality.

2010 MSC: 39A11.

1 Introduction

The notion of nonlinear difference equations was studied intensively by R.P.Agarwal [1]. Recently there has been a lot of interest in the study of oscillatory behavior of solutions of nonlinear difference equations. Motivated by the references [1]- [28], in this paper, we have considered the oscillatory properties of third-order quasilinear difference equation of the form

$$\Delta^2(p_n\left(\Delta x_n\right)^\alpha)-q_n\left(\Delta x_n\right)^\alpha+r_nf(x_n)=0,n=0,1,2,..., \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$, provided the following conditions are assumed to hold:

(C1) $\alpha>0$ is the ratio of odd positive integer,

(C2) $\{p_n\},\{q_n\},\{r_n\}$ are real positive sequences,

(C3) $f:R\rightarrow R$ is a continuous function and $xf(x)>0$ for all $x\neq 0$,

(C4) there exists a real valued function g such that
$f(u)-f(v)=g(u,v)(u-v)$ for all $u\neq 0, v\neq 0$ and
$g(u,v)\geq L>0\in R$,

(C5) $\sum_{n=M}^{\infty}{p_n}^2<\infty$ for $M\geq 0$,

(C6) $\sum_{n=M}^{\infty}\left(\Delta\left(p_{n+1}\left(\Delta x_{n+1}\right)^\alpha\right)\right)^2<\infty$ for $M\geq 0$,

(C7) $\sum_{n=M}^{\infty}\frac{1}{p_n^{\frac{1}{\alpha}}}=\infty$ for $M\geq 0$,

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(C8) $\sum_{n=M}^{\infty} {q_n}^2 < \infty$ for $M \geq 0$,

(C9) $\sum_{n=M}^{\infty} (n+1)\, r_n = \infty$ for $M \geq 0$,

Our objective here is to proceed further in this direction to obtain the oscillation of all solutions of equation (1.1) which include and generalize some earlier results cited there in references.

By a solution of equation (1.1) we mean a real sequence $\{x_n\}$, $n = 0, 1, 2, ...$, which satisfies equation (1.1) for all $n > n_0$, where $n_0 \geq 0$. We recall that a nontrivial solution of equation (1.1) is said to be oscillatory if for every $M > 0$ there exists an integer $n \geq M$ such that $x_n x_{n+1} \leq 0$; otherwise it is said to be nonoscillatory. Thus, a nonoscillatory solution is either eventually positive or eventually negative.

2 Main Result

In this section, we present some sufficient conditions for the oscillatory properties of all solutions of equation (1.1).

Theorem 2.1. *If the conditions (C1), (C2), (C3), (C4), (C5), (C6), (C7), (C8) and (C9) hold, then every solution of equation (1.1) is oscillatory.*

Proof. Without loss of generality we may assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1) such that $x_n > 0$ for all $n \geq M$, $M \geq 0$ is an integer.
From equation (1.1), we have

$$\Delta\left(p_{n+1}\left(\Delta x_{n+1}\right)^{\alpha}\right) - \Delta\left(p_n\left(\Delta x_n\right)^{\alpha}\right) - q_n\left(\Delta x_n\right)^{\alpha} + r_n f\left(x_n\right) = 0. \tag{2.1}$$

Multiplying equation (2.1) by $\dfrac{n+1}{f\left(x_n\right)}$ and summing from M to $n-1$, we obtain

$$\begin{aligned}\sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} \Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right) - \sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} \Delta\left(p_s\left(\Delta x_s\right)^{\alpha}\right) \\ - \sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} q_s\left(\Delta x_s\right)^{\alpha} + \sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} r_s f\left(x_s\right) = 0.\end{aligned} \tag{2.2}$$

Consider the first summation from equation (2.2),

$$\begin{aligned}\sum_{n=M}^{n-1} \frac{s+1}{f\left(x_s\right)} \Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right) = {} & \frac{n+1}{f\left(x_n\right)} p_{n+1}\left(\Delta x_{n+1}\right)^{\alpha} \\ & - \frac{M+1}{f\left(x_M\right)} p_{M+1}\left(\Delta x_{M+1}\right)^{\alpha} \\ & - \sum_{n=M}^{n-1} p_{s+2}\left(\Delta x_{s+2}\right)^{\alpha} \\ & \quad \left(\frac{f\left(x_s\right)\Delta(s+1) - (s+1)\Delta f\left(x_s\right)}{f\left(x_s\right) f\left(x_{s+1}\right)}\right).\end{aligned}$$

That is,

$$\begin{aligned}\sum_{n=M}^{n-1}\frac{s+1}{f(x_s)}\Delta\left(p_{s+1}(\Delta x_{s+1})^\alpha\right)=&\frac{n+1}{f(x_n)}p_{n+1}(\Delta x_{n+1})^\alpha\\&-\frac{M+1}{f(x_M)}p_{M+1}(\Delta x_{M+1})^\alpha\\&-\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^\alpha}{f(x_{s+1})}\\&+\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^\alpha(s+1)}{f(x_s)f(x_{s+1})}\\&\quad g(x_{s+2},x_{s+1})\Delta x_{s+1}.\end{aligned}\tag{2.3}$$

Now consider the second summation from equation (2.2) and similarly we obtain

$$\begin{aligned}\sum_{n=M}^{n-1}\frac{s+1}{f(x_s)}\Delta\left(p_s(\Delta x_s)^\alpha\right)=&\frac{n+1}{f(x_n)}p_n(\Delta x_n)^\alpha\\&-\frac{M+1}{f(x_M)}p_M(\Delta x_M)^\alpha\\&-\sum_{n=M}^{n-1}\frac{p_{s+1}(\Delta x_{s+1})^\alpha}{f(x_{s+1})}\\&+\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^{\alpha+1}(s+1)}{f(x_s)f(x_{s+1})}g(x_{s+2},x_{s+1}).\end{aligned}\tag{2.4}$$

Substituting equations (2.3) and (2.4) in equation (2.2), we have

$$\begin{aligned}&\frac{n+1}{f(x_n)}p_{n+1}(\Delta x_{n+1})^\alpha-\frac{M+1}{f(x_M)}p_{M+1}(\Delta x_{M+1})^\alpha-\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^\alpha}{f(x_{s+1})}\\&+\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^\alpha(s+1)}{f(x_s)f(x_{s+1})}g(x_{s+2},x_{s+1})\Delta x_{s+1}+\frac{n+1}{f(x_n)}p_n(\Delta x_n)^\alpha\\&-\frac{M+1}{f(x_M)}p_M(\Delta x_M)^\alpha-\sum_{n=M}^{n-1}\frac{p_{s+1}(\Delta x_{s+1})^\alpha}{f(x_{s+1})}\\&+\sum_{n=M}^{n-1}\frac{p_{s+2}(\Delta x_{s+2})^{\alpha+1}(s+1)}{f(x_s)f(x_{s+1})}g(x_{s+2},x_{s+1})-\sum_{n=M}^{n-1}\frac{s+1}{f(x_s)}q_s(\Delta x_s)^\alpha\\&+\sum_{n=M}^{n-1}(s+1)r_s=0.\end{aligned}$$

That is,

$$\begin{aligned}&\frac{n+1}{f(x_n)}\Delta\left(p_n(\Delta x_n)^\alpha\right)-\sum_{n=M}^{n-1}\frac{1}{f(x_{s+1})}\Delta\left(p_{s+1}(\Delta x_{s+1})^\alpha\right)\\&-\sum_{n=M}^{n-1}\frac{(-s-1)g(x_{s+2},x_{s+1})\Delta x_{s+1}}{f(x_s)f(x_{s+1})}\Delta\left(p_{s+1}(\Delta x_{s+1})^\alpha\right)\\&-\sum_{n=M}^{n-1}\frac{s+1}{f(x_s)}q_s(\Delta x_s)^\alpha\\&\qquad=\frac{M+1}{f(x_M)}\Delta\left(p_M(\Delta x_M)^\alpha\right)-\sum_{n=M}^{n-1}(s+1)r_s.\end{aligned}\tag{2.5}$$

By Schwarz's inequality we obtain the following:

$$\sum_{n=M}^{n-1} \frac{1}{f\left(x_{s+1}\right)} \Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right) \leq \left(\sum_{n=M}^{n-1} \frac{1}{f^{2}\left(x_{s+1}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1}\left(\Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right)\right)^{2}\right)^{\frac{1}{2}}, \tag{2.6}$$

$$\sum_{n=M}^{n-1} \frac{(-s-1) g\left(x_{s+2}, x_{s+1}\right) \Delta x_{s+1}}{f\left(x_{s}\right) f\left(x_{s+1}\right)} \Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right) \leq\left(\sum_{n=M}^{n-1} \frac{(s+1)^{2} g^{2}\left(x_{s+2}, x_{s+1}\right)\left(\Delta x_{s+1}\right)^{2}}{f^{2}\left(x_{s}\right) f^{2}\left(x_{s+1}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1}\left(\Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right)\right)^{2}\right)^{\frac{1}{2}} \tag{2.7}$$

and

$$\sum_{n=M}^{n-1} \frac{s+1}{f\left(x_{s}\right)} q_{s}\left(\Delta x_{s}\right)^{\alpha} \leq\left(\sum_{n=M}^{n-1} \frac{(s+1)^{2}\left(\Delta x_{s+1}\right)^{2}}{f^{2}\left(x_{s}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1} q_{s}^{2}\right)^{\frac{1}{2}}. \tag{2.8}$$

In view of the above inequalities (2.6), (2.7) and (2.8), the summations in (2.5) are bounded.Therefore, equation (2.5) becomes

$$\begin{aligned} &\frac{n+1}{f\left(x_{n}\right)} \Delta\left(p_{n}\left(\Delta x_{n}\right)^{\alpha}\right)-\left(\sum_{n=M}^{n-1} \frac{1}{f^{2}\left(x_{s+1}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1}\left(\Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right)\right)^{2}\right)^{\frac{1}{2}} \\ &-\left(\sum_{n=M}^{n-1} \frac{(s+1)^{2} g^{2}\left(x_{s+2}, x_{s+1}\right)\left(\Delta x_{s+1}\right)^{2}}{f^{2}\left(x_{s}\right) f^{2}\left(x_{s+1}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1}\left(\Delta\left(p_{s+1}\left(\Delta x_{s+1}\right)^{\alpha}\right)\right)^{2}\right)^{\frac{1}{2}} \\ &-\left(\sum_{n=M}^{n-1} \frac{(s+1)^{2}\left(\Delta x_{s+1}\right)^{2}}{f^{2}\left(x_{s}\right)}\right)^{\frac{1}{2}}\left(\sum_{n=M}^{n-1} q_{s}^{2}\right)^{\frac{1}{2}} \\ &\leq \frac{M+1}{f\left(x_{M}\right)} \Delta\left(p_{M}\left(\Delta x_{M}\right)^{\alpha}\right)-\sum_{n=M}^{n-1}(s+1) r_{s}. \end{aligned} \tag{2.9}$$

In view of the conditions (C1),(C2),(C3),(C4),(C5),(C6),(C8),(C9) and from the above inequality (2.9), we obtain

$$\frac{n+1}{f\left(x_{n}\right)} \Delta\left(p_{n}\left(\Delta x_{n}\right)^{\alpha}\right) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Hence there exists an integer $M_1 > 0$ such that

$$\Delta\left(p_{n}\left(\Delta x_{n}\right)^{\alpha}\right)<0 \text{ for } n \geq M_{1}.$$

Summing the above inequality from M_1 to $n-1$, we have

$$p_{n}^{\frac{1}{\alpha}} \Delta x_{n}<p_{M_{1}}^{\frac{1}{\alpha}} \Delta x_{M_{1}}. \tag{2.10}$$

Hence there exists a real number $K > 0$ such that $p_{M_1}^{\frac{1}{\alpha}} \Delta x_{M_1} < -K$.
Therefore, from equation (2.10), we have

$$p_{n}^{\frac{1}{\alpha}} \Delta x_{n}<-K.$$

$$\text{i.e.,} \quad \Delta x_{n}<-\frac{K}{p_{n}^{\frac{1}{\alpha}}}.$$

Summing the above inequality from M_1 to $n-1$, we have

$$x_{n}<x_{M_{1}}-K \sum_{n=M_{1}}^{n-1} \frac{1}{p_{n}^{\frac{1}{\alpha}}}. \tag{2.11}$$

In view of the condition (C7), from the above inequality (2.11) we find that $x_n \to -\infty$ as $n \to \infty$. This is a contradiction to the fact that $x_n > 0$ for all $n \geq M \geq 0$.

The proof is similar to the case when $x_n < 0$ for all $n \geq M$, $M \geq 0$ is an integer.

Hence the theorem is completely proved. □

3 Example

Example 3.1. *Consider the difference equation*

$$\Delta^2\left(\frac{1}{n}\left(\Delta x_n\right)^3\right) - \frac{4\left(2n^2+4n+1\right)}{n\left(n+1\right)\left(n+2\right)}\left(\Delta x_n\right)^3 + \frac{8\left(2n^2+4n+1\right)}{n\left(n+1\right)\left(n+2\right)}x_n = 0,\ n > 0. \tag{3.1}$$

$$\left[\text{Here } p_n = \frac{1}{n},\ q_n = \frac{4\left(2n^2+4n+1\right)}{n\left(n+1\right)\left(n+2\right)},\ r_n = \frac{8\left(2n^2+4n+1\right)}{n\left(n+1\right)\left(n+2\right)},\ \text{and } f\left(x_n\right) = x_n\right]$$

All conditions of Theorem (2.1) are satisfied.
Hence all solutions of equation (3.1) are oscillatory.
In fact, $\{x_n\} = \{(-1)^n\}$ *is such a solution of equation (3.1).*

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